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LOCALLY FUNCTIONALLY COUNTABLE SUBALGEBRA
OF $\mathcal{R}(L)$

M. Elyasi, A. A. Estaji, and M. Robat Sarpoushi

Abstract. Let $L_c(X) = \{f \in C(X) : C_f = X\}$, where $C_f$ is the union of all open subsets $U \subseteq X$ such that $|f(U)| \leq \aleph_0$. In this paper, we present a pointfree topology version of $L_c(X)$, named $R_{l\ell c}(L)$. We observe that $R_{l\ell c}(L)$ enjoys most of the important properties shared by $R(L)$ and $R_c(L)$, where $R_c(L)$ is the pointfree version of all continuous functions of $C(X)$ with countable image. The interrelation between $R(L)$, $R_{l\ell c}(L)$, and $R_c(L)$ is examined. We show that $L_c(X) \cong R_{l\ell c}(\mathcal{O}(X))$ for any space $X$. Frames $L$ for which $R_{l\ell c}(L) = R_c(L)$ are characterized.

1. Introduction

In this paper, all spaces are assumed to be Tychonoff, all frames are completely regular, and all rings are commutative with an identity element.

The notation $C(X)$ denotes the ring of all real-valued continuous functions on a topological space $X$ (see [12]). Let $C_c(X)$ (resp. $C^f(X)$) denote the ring of all continuous functions of $C(X)$ with the countable (resp. finite) image. The ring $C_c(X)$ was introduced and studied in [10]. This subalgebra has more attendance recently; see, for example, [1, 4, 11, 14, 17, 18]. In [16], the authors introduced and studied the ring $R_c(L)$ as the pointfree topology version of $C_c(X)$ (see also [6, 8, 9]). By $L_c(X)$, we mean the ring of all continuous functions that $C_f$ is dense in $X$ for $f \in C(X)$, where $C_f = \bigcup\{U : U \in \mathcal{O}(X) \text{ and } |f(U)| \leq \aleph_0\}$; see [15]. Note that $C_c(X)$ is the largest subring of $C(X)$ whose elements have the countable image and that the subring $L_c(X)$ of $C(X)$ lies between $C_c(X)$ and $C(X)$. This motivates us to introduce this subring in a pointfree topology, named, $R_{l\ell c}(L)$.

A brief outline of this paper is as follows. In Section 2, we review, some definitions and results of frames and continuous functions.

In Section 3, we present a new subring of $R(L)$ that contains $R_c(L)$. We define $R_{l\ell c}(L)$ the set of all $\alpha \in R(L)$ such that $(C_\alpha)^* = \bot$, where $C_\alpha$ is the join of all elements $a \in L$ with $\alpha|_a \in R_c(\downarrow a)$ (see Definition 3.1). We show that $R_{l\ell c}(L)$ is a subring of $R(L)$. We observe that $R_{l\ell c}(L)$ enjoys most of the important properties.

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that are shared by $\mathcal{R}(L)$ and $\mathcal{R}_c(L)$. Next, we introduce other subrings of $\mathcal{R}(L)$ (see Definition 3.18) and study their relations with $\mathcal{R}(L)$, $\mathcal{R}_c(L)$, and $\mathcal{R}_{lc}(L)$ (see Proposition 3.21).

In Section 4, we prove the equality of $\mathcal{R}_{lc}(L)$ and $\mathcal{R}(L)$ under certain conditions (see Propositions 4.3 and 4.7). Analogous to the main objective of research in the context $\mathcal{R}(L)$, we will try to study some useful facts about $\mathcal{R}_{lc}(L)$ and algebraic properties of $\mathcal{R}_{lc}(L)$ (see Proposition 4.13).

In the final section, we study the constant functions that are obtained from the restriction of a frame map $\alpha \in \mathcal{R}(L)$ to the codomain $M$ for every sublocale $M$ of $L$, and we denote $\mathcal{R}_{(M,\text{constant})}(L)$ to be the set of all $\alpha \in \mathcal{R}(L)$ such that $\alpha^{|M} \in \mathcal{R}^1(M)$. A relation between $\mathcal{R}_{(M,\text{constant})}(L)$ and $\mathcal{R}_c(L)$ is investigated.

2. Preliminaries

2.1. Functionally and locally functionally countable subalgebra of $C(X)$. We know $L_c(X) = \{f \in C(X): \overline{C_f} = X\}$, where $C_f = \bigcup\{U: U \in \mathcal{D}(X) \text{ and } |f(U)| \leq \aleph_0\}$. In [15], it was proved that $L_c(X)$ is a subalgebra as well as a sublattice of $C(X)$ containing $C_c(X)$, and this subring is called the locally functionally countable subalgebra of $C(X)$. The properties of the subalgebra $L_c(X)$ were mentioned in [15]. Similar to the above definition, $L_F(X)$ and $L_1(X)$ are the locally functionally finite and constant, respectively.

2.2. Frames and their homomorphism. Our notation and terminology for frames and locales will be that of [13] and [19]. We shall not discourse at length upon the rudiments of pointfree topology here, however, we recall some basic notion.

A frame (or locale) is a complete lattice $L$ in which the infinite distributive law

$$a \land \bigvee S = \bigvee\{a \land s: s \in S\}$$

holds for all $a \in L$ and $S \subseteq L$. We denote by $\perp$ and $\top$, respectively, the bottom and the top elements of $L$. The frame of open subsets of a topological space $X$ is denoted by $\mathcal{D}(X)$. An element $p \neq \top$ is a prime in a frame $L$ if $x \land y \leq p$ implies that $x \leq p$ or $y \leq p$. The set of all prime elements of $L$ is denoted by $\Sigma L$.

Every frame is a complete Heyting algebra with the Heyting implication given by

$$a \rightarrow b = \bigvee\{x \in L: a \land x \leq b\}.$$  

The pseudocomplement of $a \in L$ is the element $a^* = a \rightarrow \perp = \bigvee\{x \in L: x \land a = \perp\}$. If $a \lor a^* = \top$, then $a$ is said to be complemented.

Recall from [3] (see also [2]) that the frame of reals $\mathcal{L}(\mathbb{R})$ is obtained by taking the ordered pairs $(p, q)$ of rational numbers as generators and imposing the following relations:

(R1) $(p, q) \land (r, s) = (p \lor r, q \land s)$.
(R2) $(p, q) \lor (r, s) = (p, s)$ whenever $p \leq r < q \leq s$.
(R3) $(p, q) = \bigvee\{(r, s): p < r < s < q\}$.
(R4) $\top = \bigvee\{(p, q): p, q \in \mathbb{Q}\}$.
For every \( p, q \in \mathbb{Q} \), put
\[
\langle p, q \rangle := \{ x \in \mathbb{Q} : p < x < q \} \quad \text{and} \quad \downarrow p, q := \{ x \in \mathbb{R} : p < x < q \}.
\]

Corresponding to every operation \( \circ : \mathbb{Q}^2 \to \mathbb{Q} \) (in particular \( \circ \in \{+, \cdot, \wedge, \vee\} \) we define an operation on \( R(L) \), denoted by the same symbol \( \circ \), by
\[
\alpha \circ \beta (p, q) = \bigvee \{ \alpha(r, s) \wedge \beta(u, w) : \langle r, s \rangle \circ \langle u, w \rangle \subseteq \langle p, q \rangle \},
\]
where \( \langle r, s \rangle \circ \langle u, w \rangle \subseteq \langle p, q \rangle \) means that for each \( r < x < s \) and \( u < y < w \), we have \( p < x \circ y < q \). For every \( r \in \mathbb{R} \), define the constant frame map \( r : R(L) \to \mathbb{R} \) by
\[
r(p, q) = \top, \text{ whenever } p < r < q, \text{ and otherwise } r(p, q) = \bot.
\]

An element \( \alpha \) of \( R(L) \) is said to be bounded if there exist \( p, q \in \mathbb{Q} \) such that \( \alpha(p, q) = \top \). The set of all bounded elements of \( R(L) \) is denoted by \( R^b(L) \), which is a sub-f-ring of \( R(L) \). The cozero map is the map \( \text{coz} : R(L) \to L \), defined by
\[
\text{coz}(\alpha) = \bigvee \{ \alpha(p, 0) \vee \alpha(0, q) : p, q \in \mathbb{Q} \}.
\]

A cozero element of \( L \) is an element of the form \( \text{coz}(\alpha) \) for some \( \alpha \in R(L) \) (see [3]). The cozero part of \( L \), denoted by \( \text{Coz}(L) \), is the set of all cozero elements. It is well known that \( L \) is completely regular if and only if \( \text{Coz}(L) \) generates \( L \). The homomorphism \( \tau : \mathcal{L}(\mathbb{R}) \to \mathcal{O}(\mathbb{R}) \) given by \( (p, q) \mapsto \downarrow p, q \) is an isomorphism (see [3] Proposition 2).

For a topology space \( X \) and every \( A \subseteq X \) and \( f \in C(X) \), we have \( (f|_A)^{-1} : \mathcal{O}(\mathbb{R}) \to \mathcal{O}(A) \) with \( (f|_A)^{-1}(U) = f^{-1}(U) \cap A \) for every \( U \in \mathcal{O}(\mathbb{R}) \). Also, for every \( \alpha \in R(L) \) and every \( a \in L \), we have \( \alpha|_a : \mathcal{L}(\mathbb{R}) \to \downarrow a \) with \( \alpha|_a(p, q) = \alpha(p, q) \wedge a \).

An element \( \alpha \in R(L) \) is said to have the pointfree countable image if there is a countable subset \( S \) of \( \mathbb{R} \) with \( \alpha \upharpoonright S \) (we say \( \alpha \) overlap of \( S \)), where \( \alpha \upharpoonright S \) means that \( \tau(u) \cap S = \tau(v) \cap S \) implies \( \alpha(u) = \alpha(v) \) for any \( u, v \in \mathcal{L}(\mathbb{R}) \). In [10], it is shown that for any \( \alpha \in R(L) \) and any \( S \subseteq \mathbb{R} \), the following statements are equivalent:

1. \( \alpha \upharpoonright S \),
2. \( \tau(p, q) \cap S = \tau(v) \cap S \) implies \( \alpha(p, q) = \alpha(v) \), for any \( v \in \mathcal{L}(\mathbb{R}) \) and any \( p, q \in \mathbb{Q} \), and
3. \( \tau(p, q) \cap S \subseteq \tau(v) \cap S \) implies \( \alpha(p, q) \leq \alpha(v) \), for any \( v \in \mathcal{L}(\mathbb{R}) \) and any \( p, q \in \mathbb{Q} \).

For any frame \( L \), we put
\[
R_c(L) := \{ \alpha \in R(L) : \alpha \text{ has the pointfree countable image} \}.
\]

For any completely regular frame \( L \), the set \( R_c(L) \) is a sub-f-ring of \( R(L) \). The ring \( R_c(L) \) is introduced as the pointfree version of \( C_c(X) \) (see [10]). Also, \( R^F(L) \) is the pointfree version of \( CF(X) \). We denote the set of all constant functions of \( R(L) \) by \( R^1(L) \).

2.3. Sublocales. A sublocale of a locale \( L \) is a subset \( S \subseteq L \) such that
(i) for every \( A \subseteq S \), \( \bigwedge A \subseteq S \), and
(ii) for every \( a \in L \) and \( s \in S \), \( a \rightarrow s \in S \).
The lattice of all sublocales of \( L \) is denoted by \( \mathcal{S}(L) \). The meet in this lattice is intersection. The join of any collection \( \{ S_i : i \in I \} \subseteq \mathcal{S}(L) \) is given by
\[
\bigvee_i S_i = \left\{ M : M \subseteq \bigcup_i S_i \right\}.
\]
The lattice \( \mathcal{S}(L) \), partially ordered by inclusion, is a *coframe*. The smallest sublocale of \( L \) is \( \mathcal{O} = \{ \top \} \), which is called the *void* sublocale. Indeed the largest is \( L \).

### 3. The subalgebra \( \mathcal{R}_{lc}(L) \) of \( \mathcal{R}(L) \)

In this section, we introduce the pointfree topology version of the ring \( L_{c}(X) \). We begin with the following definition.

**Definition 3.1.** For every \( \alpha \in \mathcal{R}(L) \), we put
\[
\mathcal{C}_\alpha = \{ a \in L : \alpha|_a \in \mathcal{R}_c(\downarrow a) \}
\]
and
\[
C_{\alpha} = \bigvee \mathcal{C}_\alpha.
\]

We say that an element \( \alpha \) of \( \mathcal{R}(L) \) has the *pointfree locally countable image* if \((C_{\alpha})^* = \perp\). We put
\[
\mathcal{R}_{lc}(L) := \{ \alpha \in \mathcal{R}(L) : \alpha \text{ has the pointfree locally countable image} \}.
\]

Also, \( \mathcal{R}_{lc}(L) \) is called the pointfree locally functionally countable image subring of \( \mathcal{R}(L) \).

We show that this definition is a conservative extension for continuous functions on topological spaces. Throughout this article, for \( f \in C(X) \) and the isomorphism \( \tau : \mathcal{L}(\mathbb{R}) \rightarrow \mathcal{O}(\mathbb{R}) \), the frame map \( f^{-1} \circ \tau : \mathcal{L}(\mathbb{R}) \rightarrow \mathcal{O}(X) \) is denoted by \( f_\tau \).

**Proposition 3.2.** If \( f \in C(X) \), then \( f \in L_{c}(X) \) if and only if \( f_\tau \in \mathcal{R}_{lc}(\mathcal{O}(X)) \).

Recall from [3] that for any space \( X \), there is a one-one onto map \( \text{Frm} (\mathcal{L}(\mathbb{R}), \mathcal{O}(X)) \rightarrow \text{Top}(X, \mathcal{R}) \) given by the correspondence \( \varphi \mapsto \tilde{\varphi} \) such that
\[
p < \tilde{\varphi}(x) < q \quad \text{if and only if} \quad x \in \varphi(p, q)
\]
whenever \( p < q \) in \( \mathbb{Q} \) (also, see [4]). This means that \( \mathcal{R}(\mathcal{O}X) \cong C(X) \) for any topological space \( X \). Here, we give a counterpart of this result.

**Proposition 3.3.** For any space \( X \), \( \mathcal{R}_{lc}(\mathcal{O}(X)) \cong L_{c}(X) \).

**Proof.** We define \( \theta : L_{c}(X) \rightarrow \mathcal{R}_{lc}(\mathcal{O}(X)) \) by \( \theta(g) = g_\tau \). By Proposition 3.2, \( \theta \) is well defined and injective. Let \( \alpha \in \mathcal{R}_{lc}(\mathcal{O}(X)) \). Then \( \alpha \circ \tau^{-1} : \mathcal{O}(\mathbb{R}) \rightarrow \mathcal{O}(X) \) is a frame map, and hence by [5] Theorem 1, there exists a unique continuous function \( f : X \rightarrow \mathbb{R} \), such that \( f^{-1} = \alpha \circ \tau^{-1} \). Therefore, \( \theta(f) = f^{-1} \circ \tau = \alpha \). Now let \( p, q \in \mathbb{Q} \) and let \( U \in \mathcal{O}(X) \). Then
\[
(f^{-1}|_U)(\tau(p, q)) = f^{-1}(\downarrow p, q\downarrow) \land U = \alpha(\tau^{-1}(\downarrow p, q\downarrow)) \land U = \alpha(p, q) \land U = \alpha|_U(p, q).
\]
Therefore, \((f^{-1}|_U) \circ \tau = \alpha|_U\). Now, since \(f|_U\) has the countable image, then \((f^{-1}|_U)^{-1} \circ \tau = \alpha|_U\) has a pointfree countable image ([15, Proposition 3.11]). Therefore, \(C_\alpha = C_f\) and hence \(f \in L_c(X)\). □

**Lemma 3.4.** For every \(\alpha \in \mathcal{R}(L)\) and every \(a \in L\), if \(\alpha \in \mathcal{R}_c(L)\), then \(\alpha|_a \in \mathcal{R}_c(\downarrow a)\).

**Proof.** It is evident. □

By this lemma, it manifests that \(\mathcal{R}_F(L) \subseteq \mathcal{R}_c(L) \subseteq \mathcal{R}_{\ell c}(L) \subseteq \mathcal{R}(L)\).

**Remark 3.5.** Note that the equality between these objects may not necessarily hold. For example, let the basic neighborhood of \(x\) be the set \(\{x\}\), for each point \(x \geq \sqrt{2}\) and for the rest of the real numbers (i.e., \(x < \sqrt{2}\)), let the basic neighborhoods be the usual open intervals containing \(x\). This is a topology \(\tau\) on \(\mathbb{R}\) and in this case, we put \(X = \mathbb{R}\). Clearly, \(X\) is a completely regular Hausdorff space, which is finer than the usual topology of \(\mathbb{R}\). Consider the function \(f: X \to \mathbb{R}\) defined by \(f(x) = x\) for \(x \geq \sqrt{2}\) and \(f(x) = \sqrt{2}\), otherwise, so we have \(f \in L_c(X) \setminus C_c(X)\) (for more details, see [15]). Proposition 3.3 implies \(f_\tau \in \mathcal{R}_{\ell c}(\mathcal{O}(X)) \setminus \mathcal{R}_c(\mathcal{O}(X))\) (because \(C_c(X) \cong \mathcal{R}_c(\mathcal{O}(X))\), by [6, Lemma 3.16]). Now, consider the identity function \(\text{id}_\tau: X \to \mathbb{R}\), which is continuous. Then \(\text{id}_\tau \in C(X) \setminus L_c(X)\). It follows from Proposition 3.3 that \(\text{id}_\tau \in \mathcal{R}(\mathcal{O}(X)) \setminus \mathcal{R}_{\ell c}(\mathcal{O}(X))\).

We need the following lemmas to show that \(\mathcal{R}_{\ell c}(L)\) is a sub-\(f\)-ring and \(\mathcal{R}\)-subalgebra of \(\mathcal{R}(L)\). The proof is routine, so we omit it.

**Lemma 3.6.** Let \(\alpha, \beta \in \mathcal{R}(L)\) and \(a, b \in L\) be given. Then the following statements hold:

1. \(\Delta \in \{+,-,\wedge,\lor\}, \text{ then } \alpha \Delta \beta|_a = \alpha|_a \Delta \beta|_a\).
2. \(\alpha|_b \in \mathcal{R}_c(\downarrow b), \text{ then } \alpha|_a \in \mathcal{R}_c(\downarrow a)\).

**Lemma 3.7.** If \(\alpha, \beta \in \mathcal{R}(L)\), then \(C_\alpha \Delta \beta \geq C_\alpha \wedge C_\beta\) for every \(\Delta \in \{+,-,\wedge,\lor\}\).

**Proof.** Let \(a, b \in L\) such that \(\alpha|_a \in \mathcal{R}_c(\downarrow a)\) and \(\beta|_b \in \mathcal{R}_c(\downarrow b)\). By part (2) of Lemma 3.6 we have \(\alpha|_{a \wedge b}, \beta|_{a \wedge b} \in \mathcal{R}_c(\downarrow a \wedge b)\). Now, by part (1) of Lemma 3.6 we have

\[
\alpha \Delta \beta|_{a \wedge b} = \alpha|_{a \wedge b} \Delta \beta|_{a \wedge b} \in \mathcal{R}_c(\downarrow (a \wedge b))
\]

for every \(\Delta \in \{+,-,\wedge,\lor\}\). Therefore,

\[
C_\alpha \wedge C_\beta = \bigvee \{a \wedge b: a, b \in L, \alpha|_a \in \mathcal{R}_c(\downarrow a), \beta|_b \in \mathcal{R}_c(\downarrow b)\} \\
\leq \bigvee \{c \in L: \alpha \Delta \beta|_c \in \mathcal{R}_c(\downarrow c)\} \\
= C_\alpha \Delta \beta,
\]

for every \(\Delta \in \{+,-,\wedge,\lor\}\). □

It is evident that for every \(0 \neq r \in \mathbb{R}\), \(\alpha|_a \triangleright S\) if and only if \(r\alpha|_a \triangleright \{rx: x \in S\}\) for every \(\alpha \in \mathcal{R}(L)\) and every \(a \in L\). By this fact and Lemma 3.7 the following proposition holds.
Proposition 3.8. It follows that $\mathcal{R}_{lc}(L)$ is a sub-f-ring and an $\mathbb{R}$-subalgebra of $\mathcal{R}(L)$.

Remark 3.9. Recall that $|\alpha| = \alpha \lor (-\alpha)$ for every $\alpha \in \mathcal{R}(L)$. For every $p, q \in \mathbb{Q}$, we have

$$|\alpha|(p, q) = |\alpha|(p, -) \land |\alpha|(\cdot, q) = \begin{cases} \bot & \text{if } q \leq 0, \\ -\alpha(p, q) \lor \alpha(p, q) & \text{if } p \geq 0, \\ \alpha(-q, q) & \text{if } p < 0 < q. \end{cases}$$

Now, let us state the results in relation to the absolute value function and the rings $\mathcal{R}_{c}(L)$ and $\mathcal{R}_{lc}(L)$.

Proposition 3.10. If $S$ is a subset of $\mathbb{R}$ and $|\alpha| \triangleright S$, then $|\alpha| \triangleright S \cap [0, \infty)$ for every $\alpha \in \mathcal{R}(L)$.

Proof. Put $S_1 := S \cap [0, \infty)$. Let $(p, q), \nu \in L(\mathbb{R})$ with $\tau(p, q) \cap S_1 \subseteq \tau(\nu) \cap S_1$ be given. We show that $|\alpha|(p, q) \leq |\alpha|(\nu)$ by considering several cases. Therefore, $|\alpha| \triangleright S_1$.

First case. If $p \geq 0$, then $\tau(p, q) \cap S = \tau(p, q) \cap S_1 \subseteq \tau(\nu) \cap S_1 = \tau(\nu) \cap S$, which follows that $|\alpha|(p, q) \leq |\alpha|(\nu)$.

Second case. If $q \leq 0$, then $\bot = |\alpha|(p, q) \leq |\alpha|(\nu)$.

Third case. If $0 \in \tau(p, q) \cap S_1$, then $0 \in \tau(\nu) \cap S_1$. Therefore, there exists an element $n \in \mathbb{N}$ such that $\tau(\frac{-1}{n}, \frac{1}{n}) \subseteq \tau(\nu) \cap \tau(p, q)$. On the other hand, we have

$$\tau(p, q) = \tau\left(p, \frac{-1}{n+1}\right) \cup \tau\left(\frac{-1}{n}, \frac{1}{n}\right) \cup \tau\left(\frac{1}{n+1}, q\right),$$

and so $|\alpha|(p, q) = |\alpha|(p, \frac{-1}{n+1}) \lor |\alpha|(\frac{-1}{n}, \frac{1}{n}) \lor |\alpha|(\frac{1}{n+1}, q)$.

- Since $\tau(\frac{-1}{n}, \frac{1}{n}) \subseteq \tau(\nu)$, then $|\alpha|(\frac{-1}{n}, \frac{1}{n}) \leq |\alpha|(\nu)$.
- Since $\tau(p, \frac{-1}{n+1}) \cap S_1 \subseteq \tau(p, q) \cap S \subseteq \tau(\nu) \cap S_1$, case (2) implies $|\alpha|(p, \frac{-1}{n+1}) \leq |\alpha|(\nu)$.
- Since $\tau(\frac{1}{n+1}, q) \cap S_1 \subseteq \tau(p, q) \cap S \subseteq \tau(\nu) \cap S_1$, case (1) implies $|\alpha|(\frac{1}{n+1}, q) \leq |\alpha|(\nu)$.

Therefore, $|\alpha|(p, q) \leq |\alpha|(\nu)$.

Fourth case. If $0 \in \tau(p, q)$ and $0 \notin S_1$, then $0 \notin S$. Since $((-\infty, 0) \cup (0, \infty)) \cap S = \mathbb{R} \cap S$, then $|\alpha|((-0) \lor (0, -)) = |\alpha| (\top) = \top$. Therefore

$$|\alpha|(p, q) = |\alpha|(p, q) \land \top = |\alpha|(p, q) \land (|\alpha|(-0) \lor |\alpha|(0, -)) = |\alpha|(p, -0) \lor |\alpha|(p, q) \land (0, -)$$

$$= |\alpha|(p, 0) \lor |\alpha|(0, q).$$

- Since $\tau(p, 0) \cap S_1 \subseteq \tau(p, q) \cap S_1 \subseteq \tau(\nu) \cap S_1$, case (2) implies $|\alpha|(p, 0) \leq |\alpha|(\nu)$.
- Since $\tau(0, q) \cap S_1 \subseteq \tau(p, q) \cap S \subseteq \tau(\nu) \cap S_1$, case (1) implies $|\alpha|(0, q) \leq |\alpha|(\nu)$.

So, given the above relations, it follows that $|\alpha|(p, q) \leq |\alpha|(\nu)$. \qed
Proposition 3.11. If $S \subseteq \lfloor 0, \infty \rfloor$ and $|\alpha| \downarrow S$, then $\alpha \downarrow S \cup \{-x: x \in S\}$ for every $\alpha \in \mathcal{R}(L)$.

Proof. Put $S_1 := S \cup \{-x: x \in S\}$, and let $(p, q), v \in \mathcal{L}(\mathbb{R})$ with $\tau(p, q) \cap S_1 \subseteq \tau(v) \cap S_1$ be given. For every $v \in \mathcal{L}(\mathbb{R})$, let $v^+ = \tau^{-1}(\tau(v) \cap (0, \infty))$ and $v^- = \tau^{-1}(\tau(v) \cap (-\infty, 0))$. We show that $|\alpha|(p, q) \leq |\alpha|(v)$ by considering several cases. Therefore, $|\alpha| \downarrow S_1$.

First case. If $p \geq 0$, then
\[
\tau(p, q) \cap S = \tau(p, q) \cap S_1 = \tau(p, q) \cap S_1 \cap (0, \infty) \subseteq \tau(v) \cap S_1 \cap (0, \infty) = \tau(v) \cap S \cap (0, \infty) = \tau(v^+) \cap S.
\]
Hence $|\alpha|(p, q) \leq |\alpha|(v^+)$. Therefore, Remark 3.9 implies
\[
\alpha(p, q) = \alpha(p, q) \land \left( -\alpha(p, q) \lor \alpha(p, q) \right) = \alpha(p, q) \land |\alpha|(p, q) \leq \alpha(p, q) \land |\alpha|(v^+) = \alpha(p, q) \land \alpha(v^+) \leq \alpha(v^+) \leq \alpha(v).
\]
Second case. If $q \leq 0$, then by Remark 3.9
\[
\alpha(p, q) = \alpha(p, q) \land \left( -\alpha(p, q) \lor \alpha(p, q) \right) = \alpha(p, q) \land |\alpha|(p, q) = \alpha(p, q) \land \bot \leq \alpha(v) .
\]
The proofs of parts (3) and (4) are similar to those in parts (3) and (4) of the previous proposition. \hfill \square

Proposition 3.12. For every $\alpha \in \mathcal{R}(L)$, $\alpha \in \mathcal{R}_c(L)$ if and only if $|\alpha| \in \mathcal{R}_c(L)$.

Proof. Necessary. If $\alpha \in \mathcal{R}_c(L)$, then $(-\alpha) \in \mathcal{R}_c(L)$ and hence $|\alpha| = \alpha \lor (-\alpha) \in \mathcal{R}_c(L)$, because $\mathcal{R}_c(L)$ is an $f$-ring.

Sufficiency. By Propositions 3.10 and 3.11 it is clear. \hfill \square

For the proof of the next lemma, see [6, Lemma 3.7].

Lemma 3.13. Let $\alpha$ be a unit element of $\mathcal{R}(L)$. Then $\alpha \in \mathcal{R}_c(L)$ if and only if $\alpha^{-1} \in \mathcal{R}_c(L)$.

The previous propositions lead to the next result.

Proposition 3.14. For every $\alpha, \beta \in \mathcal{R}(L)$, the following statements hold:
Corollary 3.16. It follows that $\mathcal{R}_{lc}(L)$ is a sublattice of $\mathcal{R}(L)$.

Corollary 3.17. For every $\alpha \in \mathcal{R}_{lc}(L)$, $\text{coz}(\alpha) = \top$ if and only if $\alpha$ is a unit element in $\mathcal{R}_{lc}(L)$.

Here, we introduce another subring of $\mathcal{R}(L)$.
Definition 3.18. For every \( \alpha \in \mathcal{R}(L) \), we put
\[
\mathcal{F}_\alpha = \{ a \in L : \alpha |_a \in \mathcal{R}^F(\downarrow a) \} \quad \text{and} \quad F_\alpha = \bigvee \mathcal{F}_\alpha.
\]

An element \( \alpha \) of \( \mathcal{R}(L) \) has the pointfree locally finite image if \( (F_\alpha)^* = \bot \). We define
\[
\mathcal{R}_\ell^F(L) := \{ \alpha \in \mathcal{R}(L) : \alpha \text{ has the pointfree locally finite image} \}.
\]

Also, for every \( \alpha \in \mathcal{R}(L) \), we put
\[
l_\alpha = \{ a \in L : \alpha |_a \in \mathcal{R}^1(\downarrow a) \} \quad \text{and} \quad 1_\alpha = \bigvee l_\alpha.
\]

An element \( \alpha \) of \( \mathcal{R}(L) \) has the pointfree locally constant image if \( (1_\alpha)^* = \bot \). We define
\[
\mathcal{R}_\ell^1(L) := \{ \alpha \in \mathcal{R}(L) : \alpha \text{ has the pointfree locally constant image} \}.
\]

One can easily see that \( \mathcal{R}^1(L) \cong \mathbb{R} \).

Remark 3.19. Similar to Proposition 3.3, we can see that \( C_\ell^F(X) \cong \mathcal{R}_\ell^F(\mathcal{O}(X)) \) and \( C_\ell^1(X) \cong \mathcal{R}_\ell^1(\mathcal{O}(X)) \) for any space \( X \). We note that Proposition 3.8 and Corollary 3.16 are also valid for \( \mathcal{R}_\ell^F(L) \) and \( \mathcal{R}_\ell^1(L) \).

Proposition 3.20. For any frame \( L \), \( \mathcal{R}^F(L) \subseteq \mathcal{R}_\ell^F(L) \) and \( \mathcal{R}^1(L) \subseteq \mathcal{R}_\ell^1(L) \).

Proof. Let \( \alpha \in \mathcal{R}^F(L) \) be given. Then there exists a finite subset \( S \) of \( \mathbb{R} \) such that \( \alpha \blacktriangleright S \). Suppose that \( a \in L \) and that \( u, v \in \mathcal{L}(\mathbb{R}) \) such that \( \tau(u) \cap S = \tau(v) \cap S \). Since \( \alpha \blacktriangleright S \), we have \( \alpha(u) = \alpha(v) \), and so \( \alpha(u) \wedge a = \alpha(v) \wedge a \), which implies that \( \alpha |_a(u) = \alpha |_a(v) \). Thus, \( \alpha |_a \in \mathcal{R}^F(\downarrow a) \), and so \( (F_\alpha)^* = (\bigvee L)^* = (\top)^* = \bot \). Hence, \( \alpha |_a \in \mathcal{R}_\ell^F(\downarrow a) \).

For every \( r \in \mathbb{R} \), in [16], it is shown that \( \alpha = r \) if and only if \( \alpha \blacktriangleright \{ r \} \). By this fact, we end this section with the next result.

Proposition 3.21. For any frame \( L \), we have \( \mathcal{R}_\ell^1(L) \subseteq \mathcal{R}_\ell^F(L) \subseteq \mathcal{R}_{dc}(L) \subseteq \mathcal{R}(L) \).

Proof. Let \( \alpha \in \mathcal{R}_\ell^1(L) \) and let \( a \in l_\alpha \). Then \( \alpha |_a \in \mathcal{R}^1(\downarrow a) \). Hence, \( \alpha |_a = r \) for some \( r \in \mathbb{R} \), which implies that \( \alpha |_a \blacktriangleright \{ r \} \). This shows that \( \alpha |_a \in \mathcal{R}_\ell^F(\downarrow a) \) and so \( a \in \mathcal{F}_\alpha \). Therefore, \( l_\alpha \subseteq \mathcal{F}_\alpha \). By the assumptions, we have \( (F_\alpha)^* \leq (1_\alpha)^* = \bot \), which shows that \( \alpha \in \mathcal{R}_\ell^F(L) \). The inclusion \( \mathcal{R}_\ell^F(L) \subseteq \mathcal{R}_{dc}(L) \) is clear, because every finite set is countable.

4. \( \mathcal{R}_{dc}(L) \) versus \( \mathcal{R}(L) \) and \( \mathcal{R}_c(L) \)

We are interested in characterization frames \( L \) for which \( \mathcal{R}_{dc}(L) = \mathcal{R}(L) \). First, we give some definitions and notations.

Definition 4.1. Let \( L \) be a lattice. Then the element \( \bot < p \in L \) is called a particle if \( p \leq \bigvee_i a_i \), whenever \( \bigvee_i a_i \) exists, implies \( p \leq a_i \) for some \( i \).

For any frame \( L \), we put \( P(L) := \{ p \in L : p \text{ is a particle of } L \} \).

Lemma 4.2 ([20]). We have \( \mathcal{R}(2) \cong \mathbb{R} \), where \( 2 = \{ \bot, \top \} \).
Remark 4.3. Recall from [15, Proposition 2.11] that if \((X, \mathfrak{D}(X))\) is a completely regular and Hausdorff topology space such that \(I(X)\), the set of isolated points of \(X\), is dense in \(X\), then \(L_1(X) = L_F(X) = L_c(X) = C(X)\). Now, we study this result in frames. Also, note that \(U \in \mathfrak{D}(X)\) is a particle if and only if \(|U| = 1\); therefore \(\overline{I(X)} = X\) if and only if \(\left( \bigvee P(\mathfrak{D}(X)) \right)^* = \perp\).

Proposition 4.4. Let \(L\) be a Boolean algebra and let \(\left( \bigvee P(L) \right)^* = \perp\). Then
\[
\mathcal{R}_f^L(L) = \mathcal{R}_c^L(L) = \mathcal{R}_c(L) = \mathcal{R}(L).
\]

Proof. By Proposition [3.21] we have \(\mathcal{R}_f^L(L) \subseteq \mathcal{R}_c^L(L) \subseteq \mathcal{R}_c(L) \subseteq \mathcal{R}(L)\).

Conversely, it is enough to show that \(\mathcal{R}(L) \subseteq \mathcal{R}_f^L(L)\). Let \(\alpha \in \mathcal{R}(L)\) be given. If \(p\) is a particle element, then \(p\) is an atom and by Lemma 4.2 we have \(\mathcal{R}(\downarrow p) = \mathcal{R}(2) \cong \mathcal{R}\). Therefore, \(\alpha|_p \in \mathcal{R}(\downarrow p) \cong \mathcal{R}\) implies that there is an element \(r \in \mathcal{R}\) such that \(\alpha|_p = r\), which more implies \(p \in \iota_\alpha\). This shows that \(P(L) \subseteq \iota_\alpha\) and so \((\bigvee \iota_\alpha)^* \subseteq \left( \bigvee P(L) \right)^* = \perp\). Therefore, \(\alpha \in \mathcal{R}_f^L(L)\).

Here, we give a condition that is \(\mathcal{R}_c(L) = \mathcal{R}(L)\).

Proposition 4.5. For every frame \(L\), \(\mathcal{R}_c(L) = \mathcal{R}(L)\) if and only if for every \(\alpha \in \mathcal{R}(L)\) and every \(\perp \neq a \in L\), there exists an element \(b \neq \perp\) such that \(b \leq a\) and \(\alpha|_b \in \mathcal{R}_c(b)\).

Proof. Necessity. Assume that \(\mathcal{R}_c(L) = \mathcal{R}(L)\), that \(\alpha \in \mathcal{R}(L)\), and that \(\perp \neq a \in L\). Then \((C_\alpha)^* = \perp\) and we conclude \(a \land c_\alpha \neq \perp\). Therefore, there exists an element \(x \in C_\alpha\) such that \(x \land a \neq \perp\). Now, Lemma 3.6 implies \(\alpha|_{(x \land a)} \in \mathcal{R}_c(\downarrow (x \land a))\).

Sufficiency. Assume that \(\alpha \in \mathcal{R}(L)\) and that \(\perp \neq a \in L\). Then there exists an element \(\perp \neq x_a \in L\) such that \(x_a \leq a\) and \(\alpha|_{x_a} \in \mathcal{R}_c(\downarrow x_a)\). Hence, for any \(a \in L\), we have \((C_\alpha)^* \leq \bigwedge_{a \in L} x_a^*\). If \(t = \bigwedge_{a \in L} x_a^* \neq \perp\), then, there exists an element \(\perp \neq x_t \in L\) such that \(x_t \leq t = \bigwedge_{a \in L} x_a^*\) and \(\alpha|_{x_t} \in \mathcal{R}_c(\downarrow x_t)\). Therefore \(x_t \leq x_a^* \land x_t = \perp\), which is a contradiction. Hence \(\bigwedge_{a \in L} x_a^* = \perp\), which implies that \((C_\alpha)^* = \perp\) and we conclude \(\alpha \in \mathcal{R}_c(L)\).

Proposition 4.6. Consider the following conditions:

1. \(\mathcal{R}_c(L) = \mathcal{R}(L)\).
2. For every \(a \in \Sigma L\), there exists an element \(b \in L\) such that \(a \leq b\) and \(\mathcal{R}(\downarrow b) = \mathcal{R}(\downarrow b)\).

Then (1) implies (2), and if \(L\) is Lindelöf and \(\bigvee \Sigma L = \top\), then (2) implies (1).

Proof. (1) \(\Rightarrow\) (2) It is sufficient to take \(b = \top\) for every \(a \in \Sigma L\).

(2) \(\Rightarrow\) (1) Let \(L\) be Lindelöf and let \(\bigvee \Sigma L = \top\). For every \(a \in \Sigma L\), there exists an element \(x_a \in L\) such that \(a \leq x_a\) and \(\mathcal{R}(\downarrow x_a) = \mathcal{R}_c(\downarrow x_a)\). The assumptions imply that \(\bigvee_{a \in \Sigma L} x_a = \top\) and so there is a family \(\{a_n\}_{n \in \mathbb{N}} \subseteq \Sigma L\) such that \(\bigvee_{a \in \Sigma L} x_a = \top\), because \(L\) is Lindelöf. Now, let \(\alpha \in \mathcal{R}(L)\) be given. For every \(n \in \mathbb{N}\), since \(\alpha|_{x_{a_n}} \in \mathcal{R}(\downarrow x_{a_n}) = \mathcal{R}_c(\downarrow x_{a_n})\), we infer that there exists a countable subset \(S_n \subseteq \mathbb{R}\) such that \(\alpha|_{x_{a_n}} \downarrow S_n\). Put \(S := \bigcup_{n \in \mathbb{N}} S_n\). Suppose that \((p, q), v \in L(\mathbb{R})\) and that \(\tau(p, q) \cap S \subseteq \tau(v) \cap S\). Then for every \(n \in \mathbb{N}\), we have \(\tau(p, q) \cap S_n \subseteq \tau(v) \cap S_n\),
which follows that \( \alpha|_{x_{\alpha}(p, q)} \leq \alpha|_{x_{\alpha}(v)} \). Here
\[
\alpha(p, q) = \alpha(p, q) \land \bigvee_{n \in \mathbb{N}} x_{\alpha} = \bigvee_{n \in \mathbb{N}} \alpha|_{x_{\alpha}(p, q)} \leq \bigvee_{n \in \mathbb{N}} \alpha|_{x_{\alpha}(v)} = \alpha(v) \land \bigvee_{n \in \mathbb{N}} x_{\alpha} = \alpha(v).
\]
Therefore, \( \alpha \in \mathcal{R}_c(L) \).

**Proposition 4.7.** Let \( L \) be a frame such that for every \( a \in \Sigma L \), there exists an element \( b \in L \) such that \( a \leq b \) and \( \mathcal{R}(\downarrow b) = \mathcal{R}_c(\downarrow b) \) and moreover \( \bigvee \Sigma L = \top \). Then \( \mathcal{R}_c(L) = \mathcal{R}(L) \).

**Proof.** Let \( \alpha \in \mathcal{R}(L) \) and let \( p \in \Sigma L \). Then there exists an element \( a \in L \) such that \( p \leq a \) and \( \mathcal{R}(\downarrow a) = \mathcal{R}_c(\downarrow a) \). Since \( \alpha|_{\alpha} \in \mathcal{R}(\downarrow a) = \mathcal{R}_c(\downarrow a) \), then \( a \in \mathcal{C}_\alpha \). Therefore, \( \top = \bigvee_{p \in \Sigma L} p \leq \bigvee \mathcal{C}_\alpha \), and so \( \alpha \in \mathcal{R}_c(L) \). \( \square \)

We finish this section with some results on ring homomorphisms on \( \mathcal{R}_c(L) \).

**Definition 4.8.** A frame \( L \) is said to be locally countably pseudocompact (briefly, \( lc \)-pseudocompact) if \( \mathcal{R}^*_c(L) = \mathcal{R}_c(L) \), where \( \mathcal{R}^*_c(L) = \mathcal{R}_c(L) \cap \mathcal{R}^*(L) \).

In what follows, by \([9]\), for every \( \alpha \in \mathcal{R}(L) \), we put
\[
\mathcal{R}_{\alpha} := \{ r \in \mathbb{R} : \text{coz}(\alpha - r) \neq \top \}.
\]

**Proposition 4.9.** \([9]\) If \( \alpha \in \mathcal{R}_c(L) \), then \( \mathcal{R}_\alpha \) is a countable subset of \( \mathbb{R} \).

Let us remind the reader that although apparently \( C_c(X) \) and \( C^F(X) \) are not defined algebraically, but they are in fact algebraic objects, in the sense that if \( C(X) \cong C(Y) \), then \( C_c(X) \cong C_c(Y) \) and \( C^F(X) \cong C^F(Y) \). For this, it is easy to see that whenever \( \varphi : C(X) \to C(Y) \) is a nonzero homomorphism, then \( \varphi(C_c(X)) \subseteq C_c(Y) \).

**Proposition 4.10.** If \( \varphi : \mathcal{R}(L) \longrightarrow \mathcal{R}(M) \) is a ring homomorphism such that \( \varphi(1) = 1 \) and \( \alpha \in \mathcal{R}_c(L) \), then \( \mathcal{R}_{\varphi(\alpha)} \) is countable.

**Proof.** Since \( \varphi \) preserves order and \( \varphi(1) = 1 \), we conclude that \( \varphi(r) = r \) for every \( r \in \mathbb{R} \). By Proposition 4.9, it is enough to show, \( \mathcal{R}_{\varphi(\alpha)} \subseteq \mathcal{R}_{\alpha} \). Let \( r \in \mathcal{R}_{\varphi(\alpha)} \setminus \mathcal{R}_{\alpha} \) be given. Therefore, \( \text{coz}(\alpha - r) = \top \), which follows that there exists an element \( \beta \in \mathcal{R}(L) \) such that \( (\alpha - r)\beta = 1 \). Thus \( \varphi(\alpha - r)\varphi(\beta) = \varphi(1) = 1 \) and hence \( \text{coz}(\varphi(\alpha - r)) = \top \), which is a contradiction. \( \square \)

**Remark 4.11.** Note that the converse of Proposition 4.9 is not true, in general. For example, we consider the isomorphism \( \varphi : \mathcal{O}(\mathbb{Q}) \longrightarrow \mathcal{O}(\mathbb{R}) \) given by \( \varphi(\tau(p, q) \cap \mathbb{Q}) = \tau(p, q) \). Then \( \psi : \mathcal{R}(\mathcal{O}(\mathbb{Q})) \longrightarrow \mathcal{R}(\mathcal{O}(\mathbb{R})) \) given by \( \psi(\alpha) = \varphi \circ \alpha \) is an isomorphism. We assume that \( \alpha : L \mathbb{R} \to \mathcal{O}(\mathbb{Q})) \) is given by \( \alpha(p, q) = \tau(p, q) \cap \mathbb{Q} \). Then
\[
\psi(\alpha)(p, q) = \varphi \circ \alpha(p, q) = \varphi(\tau(p, q) \cap \mathbb{Q}) = \tau(p, q)
\]
for every \( p, q \in \mathbb{Q} \). It is clear \( \alpha \in \mathcal{R}_c(\mathcal{O}(\mathbb{Q})) \). Indeed \( \psi(\alpha) \notin \mathcal{R}_c(\mathcal{O}(\mathbb{R})) \), because \( \psi(\alpha) \) is not an overlap of \( S \) for every \( S \subseteq \mathbb{R} \).

In \([5]\), Banaschewski showed that any \( 0 \leq \alpha \in \mathcal{R}(L) \) is a square. It is shown that this result holds for \( \mathcal{R}_c(L) \), that is if \( 0 \leq \alpha \in \mathcal{R}_c(L) \), then there exists an element \( \beta \in \mathcal{R}_c(L) \) such that \( \alpha = \beta^2 \). Here, we study this result for \( \mathcal{R}_c(L) \).
Proposition 4.12. If $0 \leq \alpha \in \mathcal{R}_{\ell c}(L)$ and $\alpha = \beta^2$, then $\beta \in \mathcal{R}_{\ell c}(L)$.

Proof. Since $\alpha|_a = \beta^2|_a = (\beta|_a)^2$ and $\alpha|_a \in \mathcal{R}_c(\downarrow a)$ for every $a \in \mathfrak{C}_\alpha$, then $\beta|_a \in \mathcal{R}_c(\downarrow a)$. Therefore

$$C_\alpha = \bigvee \{a \in L : \alpha|_a \in \mathcal{R}_c(\downarrow a)\} \leq \bigvee \{a \in L : \beta|_a \in \mathcal{R}_c(\downarrow a)\} = C_\beta,$$

which implies that $(C_\beta)^* = \perp$, then $\beta \in \mathcal{R}_{\ell c}(L)$. □

Now, an interesting function is introduced as below; see [2]. For a complemented element $a$ of $L$, define the frame map $e_a : \mathcal{L}(\mathbb{R}) \rightarrow L$ given by

$$e_a(p, q) = \begin{cases} \top & \text{if } p < 0 < 1 < q, \\ a' & \text{if } p < 0 < q \leq 1, \\ a & \text{if } 0 \leq p < 1 < q, \\ \perp & \text{otherwise,} \end{cases}$$

for each $p, q \in \mathbb{Q}$.

Proposition 4.13. Every homomorphism $\varphi : \mathcal{R}_{\ell c}(L) \rightarrow \mathcal{R}_{\ell c}(M)$, takes $\mathcal{R}^*_\ell_{\ell c}(L)$ into $\mathcal{R}^*_\ell_{\ell c}(M)$.

Proof. If $\varphi = 0$, then it is trivial. Let $\varphi \neq 0$; then $\varphi(1) \neq 0$. Since $\varphi(1)$ is an idempotent element in $\mathcal{R}(M)$, then $\text{coz}(\varphi(1))$ is complemented and

$$\varphi(1)(p, q) = \begin{cases} \top & 0, 1 \in \tau(p, q), \\ \text{coz}(\varphi(1)) & 0 \not\in \tau(p, q), 1 \in \tau(p, q), \\ \text{coz}(\varphi(1))' & 0 \in \tau(p, q), 1 \not\in \tau(p, q), \\ \perp & 0, 1 \not\in \tau(p, q). \end{cases}$$

Therefore $\varphi(1) \leq 1$, which implies that $\varphi(\mathfrak{n}) \leq \mathfrak{n}$. Let $0 \leq \alpha \in \mathcal{R}_{\ell c}(L)$ be given. Then, by Proposition 4.12, there exists an element $\beta \in \mathcal{R}_{\ell c}(L)$ such that $\alpha = \beta^2$. Therefore, $\varphi(\alpha) = \varphi(\beta^2) \geq 0$. Now, if $\alpha \in \mathcal{R}^*_\ell_{\ell c}(L)$, then $|\alpha| \leq n$ for some $n \in \mathbb{N}$, which implies that $\varphi(|\alpha|) \leq \varphi(n)$, and so $|\varphi(\alpha)| \leq \mathfrak{n}$. Therefore, $\varphi(\alpha) \in \mathcal{R}^*_\ell_{\ell c}(M)$. □

Corollary 4.14. If $M$ is not an lc-pseudocompact frame, then $\mathcal{R}_{\ell c}(M)$ cannot be a homomorphic image of $\mathcal{R}^*_\ell_{\ell c}(L)$ for any frame $L$.

Proof. Suppose that there is a frame map $\varphi : \mathcal{R}_{\ell c}(L) \rightarrow \mathcal{R}_{\ell c}(M)$ such that $\varphi(\mathcal{R}^*_\ell_{\ell c}(L)) = \mathcal{R}_{\ell c}(M)$. By Proposition 4.13, we have

$$\varphi(\mathcal{R}^*_\ell_{\ell c}(L)) \subseteq \mathcal{R}^*_\ell_{\ell c}(M) \subseteq \mathcal{R}_{\ell c}(M) = \varphi(\mathcal{R}^*_\ell_{\ell c}(L)),$$

which shows that $\mathcal{R}^*_\ell_{\ell c}(M) = \mathcal{R}_{\ell c}(M)$. That is a contradiction. □

Corollary 4.15. If $\varphi$ is a homomorphism from $\mathcal{R}_{\ell c}(L)$ into $\mathcal{R}_{\ell c}(M)$ whose image contains $\mathcal{R}^*_\ell_{\ell c}(M)$, then $\varphi(\mathcal{R}^*_\ell_{\ell c}(L)) = \mathcal{R}^*_\ell_{\ell c}(M)$. 
5. Constant functions and sublocales

First, we recall some concepts of sublocales. For more information on locales, see [19]. If $M$ is a sublocale of $L$, then the associated frame surjection is the surjective frame homomorphism $\nu_M : L \rightarrow M$ given by

$$\nu_M(a) = \bigwedge \{ m \in M : a \leq m \} = \bigwedge (M \cap c_L(a)),$$

Let $M$ be a sublocale of $L$ and let $\alpha \in \mathcal{R}(L)$. We define the frame map $\alpha^M : \mathcal{L}(\mathbb{R}) \rightarrow M$ given by

$$\alpha^M(p, q) = \nu_M(\alpha(p, q)) = \bigwedge \{ m \in M : \alpha(p, q) \leq m \},$$

and we denote $\mathcal{R}_{(M,\text{constant})}(L)$ to be the set of all $\alpha \in \mathcal{R}(L)$ such that $\alpha^M \in \mathcal{R}^1(M)$.

**Remark 5.1.** Let $\nu_M : L \rightarrow M$ be the associated frame surjection to $M$ and let $\alpha, \beta \in \mathcal{R}(L)$. Then $\nu_M \circ (\alpha \triangleleft \beta) = (\nu_M \circ \alpha) \triangleleft (\nu_M \circ \beta)$ for $\triangleleft \in \{ +, \cdot, \wedge, \vee \}$. Therefore, $\mathcal{R}_{(M,\text{constant})}(L)$ is a sub-$f$-ring and an $\mathbb{R}$-subalgebra of $\mathcal{R}(L)$. Moreover, note that for any frame $L$, it is clear that $\mathcal{R}_{(L,\text{constant})}(L) = \mathcal{R}(L)$ if and only if every function in $\mathcal{R}(L)$ is constant.

**Proposition 5.2.** Let $L$ be a completely regular frame. Then $\mathcal{R}_{(L,\text{constant})}(L) = \mathcal{R}(L)$ if and only if $L = 2$, where 2 $= \{ \bot, \top \}$.

**Proof.** Suppose that there exists an element $a \in L$ such that $\top \neq a \neq \bot$. Since $L$ is a completely regular frame, there exists a subset $\{ \alpha_\gamma \}_{\gamma \in \Lambda} \subseteq \mathcal{R}(L)$ such that $a = \bigvee_{\gamma \in \Lambda} \text{coz}(\alpha_\gamma)$. Hence for every $\gamma \in \lambda$, we have $\bot \leq \text{coz}(\alpha_\gamma) \leq a < \top$, which follows that $a = \bot$, a contradiction. The converse is evident. \qed

**Proposition 5.3** ([9]). If $L$ is a connected frame, then $\mathcal{R}_c(L) \cong \mathbb{R}$. In fact, $|R_\alpha| = 1$ and $\alpha \triangleleft R_\alpha$ for every $\alpha \in \mathcal{R}_c(L)$.

**Proposition 5.4** ([16, Proposition 3.19]). Let $\alpha : \mathcal{L}(\mathbb{R}) \rightarrow L$ and $\beta : L \rightarrow M$ be frame maps.

1. If $\alpha \triangleleft S$, then $\beta \circ \alpha \triangleleft S$.
2. If $\beta$ is monomorphism and $\beta \circ \alpha \triangleleft S$, then $\alpha \triangleleft S$.

By these propositions, we have the next result. We conclude this section with the following fact.

**Proposition 5.5.** Let $M$ be a connected sublocale of $L$. Then $\mathcal{R}_c(L) \subseteq \mathcal{R}_{(M,\text{constant})}(L)$. In particular, if $M$ is a connected sublocale of $L$ and $\nu_M : L \rightarrow M$ is a monomorphism, then $\mathcal{R}_c(L) = \mathcal{R}_{(M,\text{constant})}(L)$.

**Proof.** Since $M$ is connected, by Proposition 5.3, we have $\mathcal{R}_c(M) \cong \mathbb{R}$. Suppose that $\alpha \in \mathcal{R}_c(L)$. Then, there exists a countable subset $S \subseteq \mathbb{R}$ such that $\alpha \triangleleft S$. Therefore, Proposition 5.4 implies $\alpha^M = \nu_M \circ \alpha \triangleleft S$, which follows $\alpha^M \in \mathcal{R}_c(M) \cong \mathbb{R}$. Then $\alpha \in \mathcal{R}_{(M,\text{constant})}(L)$, as desired. \qed

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