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DOUBLE WEIGHTED COMMUTATORS THEOREM FOR PSEUDO-DIFFERENTIAL OPERATORS WITH SMOOTH SYMBOLS

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Abstract. Let $-(n+1) < m \leqslant -(n+1)(1-\varrho)$ and let $T_a \in \mathcal{L}^m_{\varrho,\delta}$ be pseudo-differential operators with symbols $a(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, where $0 < \varrho \leqslant 1$, $0 \leqslant \delta < 1$ and $\delta \leqslant \varrho$. Let μ, λ be weights in Muckenhoupt classes A_p, $\nu = (\mu\lambda^{-1})^{1/p}$ for some $1 < p < \infty$. We establish a two-weight inequality for commutators generated by pseudo-differential operators T_a with weighted BMO functions $b \in \mathrm{BMO}_\nu$, namely, the commutator $[b, T_a]$ is bounded from $L^p(\mu)$ into $L^p(\lambda)$. Furthermore, the range of m can be extended to the whole $m \leqslant -(n+1)(1-\varrho)$.

Keywords: pseudo-differential operator; reverse Hölder inequality; A_p weight; commutator

MSC 2020: 47G30, 35S05, 42B25

1. Introduction

Let m be a real number. Following Stein in [26], a symbol in $S^m_{\varrho,\delta}$ is a smooth function $a(x,\xi)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leqslant C_{\alpha,\beta}(1+|\xi|)^{m-\varrho|\beta|+\delta|\alpha|}$$

holds for all multi-indices α and β, where $C_{\alpha,\beta}$ is independent of x and ξ. We now assume that the symbol $a(x,\xi)$ is smooth in both the spatial variable x and the frequency variable ξ.

Given $f \in C_0^\infty(\mathbb{R}^n)$, a pseudo-differential operator T_a, with symbol $a(x,\xi) \in S^m_{\varrho,\delta}$, is defined by

$$T_a f(x) = \int_{\mathbb{R}^n} a(x,\xi) \mathrm{e}^{2\pi \mathrm{i} x\cdot\xi} \hat{f}(\xi)\,\mathrm{d}\xi,$$

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where $\hat{f}$ denotes the Fourier transform of f. As usual, $\mathcal{L}^m_{\varrho,\delta}$ denotes the pseudo-differential operators T_a with symbols $a(x,\xi)$ in $S^m_{\varrho,\delta}$. Laptev in [17] proved that any pseudo-differential operator T_a in $\mathcal{L}^0_{1,0}$ is a standard Calderón-Zygmund operator and this result was extended to pseudo-differential operators in $\mathcal{L}^0_{1,\delta}$ with $0 < \delta < 1$.

The pseudo-differential operators play an important role in the theory of partial differential equations. The study of them was initiated by Kohn and Nirenberg, see [16] and Hörmander, see [13]. The L^p boundedness of these operators has been extensively studied, for example, the work of Calderón and Vaillancourt (see [5]) focused on the L^2 bounds of the operators T_a in $\mathcal{L}^0_{\varrho,\varrho}$ with $0 \leqslant \varrho < 1$. We also refer to [9], [13], [26] for more details about the L^p bounds of the operators T_a in $\mathcal{L}^m_{\varrho,\delta}$.

Weighted L^p boundedness of the pseudo-differential operators T_a in $\mathcal{L}^m_{\varrho,\delta}$ has also been studied. A pioneering investigation work of Miller (see [22]) showed the bounds of the operators T_a in $\mathcal{L}^0_{1,0}$ on weighted L^p spaces $L^p(\omega)$. Later on, Chanillo and Torchinsky in [7] proved that the operators T_a in $\mathcal{L}^{n(\varrho-1)/2}_{\varrho,\delta}$ are bounded on $L^p(\omega)$ when $2 \leqslant p < \infty$ and $\omega \in A_{p/2}$. Alvarez and Hounie in [1] presented the weighted L^p boundedness for $p > 1$ when T_a belongs to $\mathcal{L}^{n(\varrho-1)}_{\varrho,\delta}$ with $0 \leqslant \delta \leqslant \varrho \leqslant \frac{1}{2}$. Recently, Michalowski, Rule and Staubach in [21] improved this result to $0 \leqslant \delta < 1$, $0 < \varrho \leqslant 1$.

Let $b \in \mathrm{BMO}$ and let T be a Calderón-Zygmund operator. A classical result of Coifman-Rochberg-Weiss (see [8]) stated that the commutator operator $[b,T]$, defined by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x),$$

is bounded on L^p for $p > 1$.

Analogously to the above conclusion, when $b \in \mathrm{BMO}$, $T_a \in \mathcal{L}^m_{\varrho,\delta}$ and under certain conditions of m, ϱ, δ, there are numerous papers dealing with the L^p boundedness of the operators $[b,T_a]$ for $1 < p < \infty$. We refer to [2], [6], [15], [19], [29] and the references therein. The weighted L^p norm inequalities for commutators generated by pseudo-differential operators T_a with BMO functions b also attract a lot of interest and we refer to [4], [25], [27], [28] for more details. Furthermore, Michalowski, Rule and Staubach in [20] presented a weighted L^p norm inequality for the operators $[b,T_a]$ on the conditions of $0 \leqslant \delta < 1$, $0 < \varrho \leqslant 1$ and $m \leqslant -n(1-\varrho)$.

In 1985, Bloom in [3] presented a two-weighted result for the commutator of Hilbert transform H, i.e., $[b,H]$ is bounded from $L^p(\mu)$ into $L^p(\lambda)$, where $b \in \mathrm{BMO}_\nu$. Here, BMO_ν is the weighted BMO space of locally integrable functions b (see Definition 2.4 below). Very recently, Holmes, Lacey and Wick in [12] extended Bloom's result to $\mathbb{R}^n$.

Let μ and λ be weights in A_p, and set $\nu = (\mu\lambda^{-1})^{1/p}$ for some $1 < p$. It is easy to have $\nu \in A_2$ (see Proposition 2.1 below). Thus, in view of Definitions 2.1 and 2.2 below, there are constants $C, \eta > 0$ such that for all balls B and all measurable subsets E of B,

$$\frac{\nu(E)}{\nu(B)} \leqslant C\left(\frac{|E|}{|B|}\right)^{\eta}. \tag{1.1}$$

The purpose of this paper focuses on the two-weighted norm inequality of the commutators $[b, T_a]$ when $b \in \mathrm{BMO}_\nu$ and $T_a \in \mathcal{L}^m_{\varrho,\delta}$, and we have the following main result.

Theorem 1.1. *Let us consider pseudo-differential operator $T_a \in \mathcal{L}^m_{\varrho,\delta}$ with $0 \leqslant \delta < \varrho \leqslant 1$, $\delta < 1$ and $\varrho > 0$, and*

$$-(n+1) < m \leqslant -(n+1)(1-\varrho).$$

For fixed $1 < p < \infty$ and given $\mu, \lambda \in A_p$, we define $\nu = (\mu\lambda^{-1})^{1/p}$. Let $c = \min\{1, (1+m+n)/\varrho\}$. Assume that $\eta > 1 - c/n$, where ν satisfies (1.1) for such η. Then for all $b \in \mathrm{BMO}_\nu$, the commutator operator $[b, T_a]$ is bounded from $L^p(\mu)$ into $L^p(\lambda)$ with

$$\int_{\mathbb{R}^n} |[b, T_a]f|^p \lambda \leqslant C \int_{\mathbb{R}^n} |f|^p \mu. \tag{1.2}$$

Furthermore, one can extend the range of m to the whole $m \leqslant -(n+1)(1-\varrho)$ and give a better range of admissible η as $\eta > 1 - 1/n$.

The remainder of this paper is organized as follows. In Section 2, we present some definitions, some notation and some well-known results we will need later. The aim of Section 3 is to prove Theorem 1.1. Our methods are similar to those of Bloom (see [3]) except that we deal with the estimation of the kernel in a different way. We first establish an estimate of the commutator sharp function (see Lemma 3.1 below). And then inspired by Hung and Ky (see [15]), we develop the method to handle the kernel estimate for a class of pseudo-differential operators (see Proposition 3.1 below). Finally, we prove Theorem 1.1 in a fashion similar to Bloom's, see [3]. It is enough to show that Theorem 1.1 is valid for $f \in C_0^\infty(\mathbb{R}^n)$. Once Theorem 1.1 holds for such f, it implies the weighted L^p boundedness for $1 < p < \infty$ and $\omega \in A_p$.

Throughout the whole paper, C denotes a constant that may change from line to line and we write $a \lesssim b$ as shorthand for $a \leqslant Cb$. If $a \lesssim b$ and $b \lesssim a$, we mean $a \sim b$. For a measurable set A, $|A|$ denotes the Lebesgue measure of A and χ_A the characteristic function. An exponent with a prime will denote the conjugate exponent, i.e., $1/p + 1/p' = 1$.

2. Auxiliary lemmas and well-known results

A weight is a locally integrable function on $\mathbb{R}^n$ which takes non-negative values almost everywhere. For a weight ω and a measurable set E, we write $\omega(E) = \int_E \omega$. Let ω be a weight. We denote by $L^p(\omega)$ the weighted L^p-space of all Lebesgue measurable functions f with norm

$$\|f\|_{L^p(\omega)} = \left(\int_{\mathbb{R}^n} |f|^p \omega\right)^{1/p}.$$

We will use the notation

$$f_B = \frac{1}{|B|}\int_B f$$

for the average of a locally integrable function f over the ball B so that the standard Hardy-Littlewood maximal function and the sharp maximal function are given by

$$M^* f(x) = \sup_{B \ni x} |f|_B \quad \text{and} \quad f^{\#}(x) = \sup_{B \ni x} |f - f_B|_B,$$

respectively, where the supremuma are taken over all balls B containing x. The pth maximal function $M_p^* f$ is defined by

$$M_p^* f(x) = \sup_{B \ni x}\left(\frac{1}{|B|}\int_B |f(y)|^p \,\mathrm{d}y\right)^{1/p},$$

where the supremum is taken over all balls B containing x.

Definition 2.1 (Muckenhoupt classes A_p)**.** A weight ω is said to be of *Muckenhoupt class* A_p for $1 < p < \infty$, if there exists $C > 1$ such that for all balls B we have

$$\left(\frac{1}{|B|}\int_B \omega\right)\left(\frac{1}{|B|}\int_B \omega^{-1/(p-1)}\right)^{p-1} \leqslant C. \tag{2.1}$$

The infimum of C satisfying the inequality (2.1) is denoted by $[\omega]_{A_p}$.

When $p = 1$, $\omega \in A_1$ if there exists $C > 1$ such that for almost every x we have

$$M^*\omega(x) \leqslant C\omega(x). \tag{2.2}$$

The infimum of C satisfying the inequality (2.2) is denoted by $[\omega]_{A_1}$.

Definition 2.2 (A_∞ condition). A *weight* ω is said to be of class A_∞ if there exist constants $0 < C, \eta < \infty$, depending only on the dimension n, and $[\omega]_{A_\infty}$ such that for all balls B and all measurable subsets E of B,

$$\frac{\omega(E)}{\omega(B)} \leqslant C\Big(\frac{|E|}{|B|}\Big)^{\eta},$$

where we define $[\omega]_{A_\infty} = \inf_{1\leqslant p<\infty} [\omega]_{A_p}$.

Muckenhoupt in [23] and [24] showed that the A_∞ condition is equivalent to the reverse Hölder condition.

Definition 2.3 (Reverse Hölder condition). A weight ω is said to satisfy the *reverse Hölder condition* if there exist constants $0 < C, \eta < \infty$, depending only on the dimension n, and $[\omega]_{A_\infty}$ such that for all balls B,

$$\Big(\frac{1}{|B|}\int_B \omega^{1+\varepsilon}\Big)^{1/(1+\varepsilon)} \leqslant \frac{C}{|B|}\int_B \omega.$$

For more details of A_p weights, we refer to Grafakos (see [11]) or Stein (see [26]), and we also have the following proposition.

Proposition 2.1. *Let $\lambda, \mu \in A_p$ and $\nu = (\mu\lambda^{-1})^{1/p}$ for some $p > 1$, then $\nu \in A_2$.*

P r o o f. Indeed, to get $\nu \in A_2$, we just need to show

$$\Big(\frac{1}{|B|}\int_B \nu\Big)\Big(\frac{1}{|B|}\int_B \nu^{-1}\Big) \leqslant C$$

holds for all balls B. By Hölder's inequality, we have

$$\frac{1}{|B|}\int_B \nu = \frac{1}{|B|}\int_B \mu^{1/p}\lambda^{-1/p} \leqslant \Big(\frac{1}{|B|}\int_B \mu\Big)^{1/p}\Big(\frac{1}{|B|}\int_B \lambda^{-p'/p}\Big)^{1/p'},$$

and

$$\frac{1}{|B|}\int_B \nu^{-1} = \frac{1}{|B|}\int_B \mu^{-1/p}\lambda^{1/p} \leqslant \Big(\frac{1}{|B|}\int_B \mu^{-p'/p}\Big)^{1/p'}\Big(\frac{1}{|B|}\int_B \lambda\Big)^{1/p}.$$

Thus, the A_p weight condition of $\lambda, \mu \in A_p$ yields $\nu \in A_2$. □

It is worth pointing out that $A_\infty = \bigcup_{1\leqslant p<\infty} A_p$, due to the following results.

Lemma 2.1. *Suppose $p > 1$ and $\omega \in A_p$. There is an exponent $q < p$ which depends only on p and $[\omega]_{A_p}$, such that $\omega \in A_q$.*

Lemma 2.2. *For $1 < q < \infty$, the Hardy-Littlewood maximal operator M^* is bounded on $L^q(\omega)$ if and only if $\omega \in A_q$. Consequently, for $1 \leqslant p < q < \infty$, M_p^* is bounded on $L^p(\omega)$ if and only if $\omega \in A_{q/p}$.*

Lemmas 2.1 and 2.2 are classical results in the theory of A_p weights and we refer to Stein, see [26]. The following lemma is known as Fefferman-Stein sharp function theorem.

Lemma 2.3 (Sharp function theorem)**.** *Let $f \in L^1(\omega)$ and $f^\# \in L^p(\omega)$ for some $1 < p < \infty$. If $\omega \in A_\infty$, then we have*

$$\|M^* f\|_{L^p(\omega)} \leqslant C_{n,p,\omega} \|f^\#\|_{L^p(\omega)}.$$

An unweighted version of Lemma 2.3 was given by Fefferman and Stein (see [10], Theorem 5) and the weighted version can be found in Lerner, see [18].

Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz class of test functions and let $\mathcal{S}'(\mathbb{R}^n)$ be the dual of $\mathcal{S}(\mathbb{R}^n)$. The space of C^∞-functions with compact support is denoted by $C_0^\infty(\mathbb{R}^n)$. Consider the pseudo-differential operators $T_a \in \mathcal{L}^m_{\varrho,\delta}$ with $0 < \varrho \leqslant 1$, $0 \leqslant \delta < 1$. It is well known that T_a is bounded from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ and as such possesses the distribution kernel $K(x, y) \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ which is given by

$$K(x, y) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \mathrm{e}^{2\pi \mathrm{i}(x-y)\xi} a(x, \xi) \psi(\varepsilon \xi) \, \mathrm{d}\xi,$$

where $\psi \in C_0^\infty(\mathbb{R}^n)$ satisfies $\psi(\xi) = 1$ for $|\xi| \leqslant 1$ and the limit is taken in $\mathcal{S}'(\mathbb{R}^n)$ and independent of the choice of ψ (see Hounie and Kapp [14], Proposition 3.1). The following kernel estimates of the pseudo-differential operator T_a due to Alvarez and Hounie (see [1]) are useful.

Lemma 2.4. *Let $0 < \varrho \leqslant 1$, $0 \leqslant \delta < 1$ and $T_a \in \mathcal{L}^m_{\varrho,\delta}$. Then, the distribution kernel $K(x, y)$ of T_a is smooth away from the diagonal $\{(x, x) \colon x \in \mathbb{R}^n\}$. Moreover:*

(i) *For any multi-index α, β, $N > 0$,*

$$\sup_{|x-y| \geqslant 1} |x - y|^N |D_x^\alpha D_y^\beta K(x, y)| \leqslant C_{\alpha,\beta,N}.$$

(ii) *Suppose $M + m + n > 0$ for some $M \in \mathbb{Z}_+$. Then there exists a constant $C > 0$ such that*

$$\sup_{|\alpha+\beta|=M} |D_x^\alpha D_y^\beta K(x, y)| \leqslant C_M \frac{1}{|x - y|^{(M+m+n)/\varrho}}, \quad x \neq y.$$

Given $f \in C_0^\infty(\mathbb{R}^n)$, in order to prove Theorem 1.1 we also need the L^p and weighted L^p estimates of pseudo-differential operators T_a in $\mathcal{L}^m_{\varrho,\delta}$.

Lemma 2.5. *Consider a pseudo-differential operator $T_a \in \mathcal{L}^m_{\varrho,\delta}$ with $0 < \varrho \leqslant 1$, $\delta \leqslant \varrho$ and $\delta < 1$. If $m \leqslant -n(1-\varrho)|1/p - 1/2|$, then T_a is bounded on L^p for each $1 < p < \infty$, i.e., there exists a constant $C > 0$ such that*

$$\|T_a f\|_{L^p} \leqslant C\|f\|_{L^p}.$$

Lemma 2.5 involves the work of Fefferman and Stein. The full version can be found in Stein, see [26], page 322. The result of Lemma 2.5 is sharp, it is well known as Hardy-Littlewood-Hirschman-Wainger's lemma.

Lemma 2.6 ([21], Theorem 3.4)**.** *Consider a pseudo-differential operator $T_a \in \mathcal{L}^m_{\varrho,\delta}$ with $0 < \varrho \leqslant 1$, $0 \leqslant \delta < 1$ and $m \leqslant -n(1-\varrho)$. Then for each $1 < p < \infty$ and $\omega \in A_p$ there exists a constant $C > 0$ such that*

$$\|T_a f\|_{L^p(\omega)} \leqslant C\|f\|_{L^p(\omega)}.$$

Given locally integrable functions f and b, the following notation is useful. Let $q > 1$ be a number near p but less than p, $r \geqslant 1$ and ω a weight. Define

$$S_r(b;\omega,B) = \left(\frac{1}{|B|}\int_B |b - b_B|^r\omega^r\right)^{1/r}, \quad \Lambda_r(f;\omega,B) = \left(\frac{1}{|B|}\int_B |f\omega|^r\right)^{1/r},$$
$$K_r^*(b,f,\omega)(x) = \sup_{x\in B} S_{rq'}(b;\omega,B)\Lambda_{rq}(f;\omega^{-1},B)$$

and write $K^* = K_1^*$, and let M_ω^* be the weighted maximal function

$$M_\omega^* f(x) = \sup_{B\ni x}\frac{1}{\omega(B)}\int_B |f(y)|\omega(y)\,\mathrm{d}y,$$

where the supremum is taken over all balls B containing x.

The following result is due to Bloom (see [3], Lemma 4.4 for more details; extending his proof to $\mathbb{R}^n$ is straightforward).

Lemma 2.7. *Let λ, μ be weights in A_p. Then for an appropriate choice of $1 < q < p$ and for r with $1 \leqslant r < p/q$ there exists a weight ω depending on r such that $\omega^{rq'} \in A_{q'}$ and*

$$\int [K_r^*(b,f,\omega)(x)]^p\lambda(x)\,\mathrm{d}x \leqslant C\int |f|^p\mu(x)\,\mathrm{d}x.$$

We will end this section by defining the weighted BMO class BMO_ω.

Definition 2.4. Let $\omega \in A_\infty$. We define the *class* BMO_ω as the space of classes of locally integrable functions b such that

$$\|b\|_{\mathrm{BMO}_\omega} = \sup_B \frac{1}{\omega(B)} \int_B |b - b_B| < \infty,$$

where $b_B = 1/|B| \int_B b$ and the supremum is taken over all balls B in $\mathbb{R}^n$.

3. Proofs of Theorem 1.1

Recall that p' is the conjugate exponent of p, $\lambda, \mu \in A_p$ and $\nu = (\mu\lambda^{-1})^{1/p}$. The proof of Theorem 1.1 is mainly about an estimate of the sharp function $([b, T_a]f)^{\#}$, which we set out in the lemma below:

Lemma 3.1. *Let us consider a pseudo-differential operator $T_a \in \mathcal{L}^m_{\varrho,\delta}$ with $0 \leqslant \delta < \varrho \leqslant 1$, $\delta < 1$ and $\varrho > 0$. Let ω and $\widetilde{\omega}$ be weights with $\omega^{q'}, \widetilde{\omega}^{rq'} \in A_{q'}$ and $c = \min\{1, (1 + m + n)/\varrho\}$. Assume that $\eta > 1 - c/n$, where ν satisfies* (1.1) *for such η. For an appropriate choice of $1 < q < p < \infty$ and for some r with $1 < r < p/q$, if $-(n + 1) < m \leqslant -(n + 1)(1 - \varrho)$ and $b \in \mathrm{BMO}_\nu$, then*

$$\begin{aligned}([b, T_a]f)^{\#}(x)&\\ &\leqslant C[K^*(b, f, \omega)(x) + K^*(b, T_a f, \omega)(x) + K^*_r(b, f, \widetilde{\omega})(x) + (M^*_\lambda(|f\nu|^q)(x))^{1/q}],\end{aligned}$$

where $f \in C_0^\infty(\mathbb{R}^n)$.

Proof. Let $g = [b, T_a]f$, we shall estimate $g^{\#}$. Fix x and a ball B containing x. Let x_0 be the center of $B = B(x_0, l)$ with the radius l. Decompose $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$. Noting that for any constant $\tilde{c}$,

$$\frac{1}{|B|} \int_B |g - g_B| \lesssim \frac{1}{|B|} \int_B |g - \tilde{c}|,$$

we can take $\tilde{c} = T_a((b - b_B)f_2)(x_0)$ without losing more than a factor of constant,

$$\frac{1}{|B|} \int_B |g - g_B| \lesssim \frac{1}{|B|} \int_B |g - T_a((b - b_B)f_2)(x_0)|.$$

Now $g = [b - b_B, T_a]f = T_a((b - b_B)f_1) + T_a((b - b_B)f_2) - (b - b_B)T_a f$, so we have

$$\begin{aligned}\frac{1}{|B|} \int_B |g - g_B| &\lesssim \frac{1}{|B|} \int_B |b - b_B|\,|T_a f| + \frac{1}{|B|} \int_B |T_a((b - b_B)f_1)|\\ &\quad + \frac{1}{|B|} \int_B |T_a((b - b_B)f_2)(t) - T_a((b - b_B)f_2)(x_0)|\,\mathrm{d}t\\ &= K_1 + K_2 + K_3.\end{aligned}$$

For the first term K_1,

$$\begin{aligned}K_1 &= \frac{1}{|B|}\int_B |b-b_B|\omega|T_a f|\omega^{-1}\\ &\leqslant \Big(\frac{1}{|B|}\int_B |b-b_B|^{q'}\omega^{q'}\Big)^{1/q'}\Big(\frac{1}{|B|}\int_B |T_a f|^q\omega^{-q}\Big)^{1/q} \quad \text{(by Hölder)}\\ &= S_{q'}(b;\omega,B)\Lambda_q(T_a f;\omega^{-1},B) \leqslant K^*(b,T_a f,\omega)(x).\end{aligned}$$

For the second piece K_2, by Hölder's inequality, we have

$$K_2 \leqslant \Big(\frac{1}{|B|}\int_B |T_a((b-b_B)f_1)|^r\Big)^{1/r} \lesssim |B|^{-1/r}\Big(\int |b-b_B|^r|f_1|^r\Big)^{1/r},$$

where Lemma 2.5 yields the second inequality with $m \leqslant -n(1-\varrho)|1/r-1/2|$. Here, we have to point out that if $m \leqslant -(n+1)(1-\varrho)$, we indeed have that m satisfies Lemma 2.5. This is the case because $1<r<\infty$ and

$$-(n+1)(1-\varrho) \leqslant -\frac{n}{2}(1-\varrho) \leqslant -n(1-\varrho)\Big|\frac{1}{r}-\frac{1}{2}\Big|.$$

Thus,

$$\begin{aligned}K_2 &\leqslant C\Big(\frac{1}{|2B|}\int_{2B} |b-b_B|^r|f|^r\Big)^{1/r}\\ &\lesssim \Big(\frac{1}{|2B|}\int_{2B} |b-b_{2B}|^r|f|^r\Big)^{1/r} + |b_B-b_{2B}|\Big(\frac{1}{|2B|}\int_{2B} |f|^r\Big)^{1/r} = K_{21}+K_{22}.\end{aligned}$$

Here, by Hölder's inequality, we have

$$\begin{aligned}K_{21} &= \Big(\frac{1}{|2B|}\int_{2B} |b-b_{2B}|^r\widetilde{\omega}^r|f|^r\widetilde{\omega}^{-r}\Big)^{1/r}\\ &\leqslant \Big(\frac{1}{|2B|}\int_{2B} |b-b_{2B}|^{rq'}\widetilde{\omega}^{rq'}\Big)^{1/rq'}\Big(\frac{1}{|2B|}\int_{2B} |f|^{rq}\widetilde{\omega}^{-rq}\Big)^{1/rq}\\ &= S_{rq'}(b;\widetilde{\omega},2B)\Lambda_{rq}(f;\widetilde{\omega}^{-1},2B) \leqslant K_r^*(b,f,\widetilde{\omega})(x).\end{aligned}$$

To estimate K_{22}, we note that

$$\begin{aligned}\Big(\frac{1}{|2B|}\int_{2B} |f|^r\Big)^{1/r} &\leqslant \Big(\frac{1}{|2B|}\int_{2B} |f|^{rq}\widetilde{\omega}^{-rq}\Big)^{1/rq}\Big(\frac{1}{|2B|}\int_{2B} \widetilde{\omega}^{rq'}\Big)^{1/rq'} \quad \text{(by Hölder)}\\ &= \Lambda_{rq}(f;\widetilde{\omega}^{-1},2B)\Big(\frac{1}{|2B|}\int_{2B} \widetilde{\omega}^{rq'}\Big)^{1/rq'}.\end{aligned}$$

and

$$\begin{aligned}|b_B - b_{2B}| &\leqslant \frac{1}{|B|}\int_B |b - b_{2B}| \lesssim \frac{1}{|2B|}\int_{2B} |b - b_{2B}| \\ &\leqslant \biggl(\frac{1}{|2B|}\int_{2B} |b - b_{2B}|^r\biggr)^{1/r} \quad \text{(by Hölder)} \\ &\leqslant \biggl(\frac{1}{|2B|}\int_{2B} |b - b_{2B}|^{rq'}\widetilde{\omega}^{rq'}\biggr)^{1/rq'} \biggl(\frac{1}{|2B|}\int_{2B} \widetilde{\omega}^{-rq}\biggr)^{1/rq} \quad \text{(by Hölder)} \\ &= S_{rq'}(b;\widetilde{\omega}, 2B)\biggl(\frac{1}{|2B|}\int_{2B} \widetilde{\omega}^{-rq}\biggr)^{1/rq}.\end{aligned}$$

We then have $K_{22} \lesssim K_r^*(b, f, \widetilde{\omega})(x)$ since $\widetilde{\omega}^{rq'} \in A_{q'}$. Hence $K_2 \lesssim K_r^*(b, f, \widetilde{\omega})(x)$. Finally, to estimate K_3, we express T_a by a smooth distribution kernel $K(x, y)$ as

$$T_a f(x) = \int K(x, y) f(y)\,\mathrm{d}y.$$

Let $t \in B(x_0, l)$. Then

$$\begin{aligned}K_3 &= \frac{1}{|B|}\int_B |T_a((b - b_B)f_2)(t) - T_a((b - b_B)f_2)(x_0)|\,\mathrm{d}t \\ &\leqslant \frac{1}{|B|}\int_B \int |K(t, y) - K(x_0, y)||b(y) - b_B||f_2(y)|\,\mathrm{d}y\,\mathrm{d}t \\ &\leqslant \frac{1}{|B|}\int_B \sum_{k=1}^{\infty} \int_{2^k l \leqslant |x_0 - y| < 2^{k+1} l} |K(t, y) - K(x_0, y)||b(y) - b_B||f(y)|\,\mathrm{d}y\,\mathrm{d}t.\end{aligned}$$

The following proposition allows us to control the inner integral.

Proposition 3.1. *Let m, ϱ, δ be as in Lemma 3.1 and $T_a \in \mathcal{L}^m_{\varrho,\delta}$. Then for any $t \in B(x_0, l)$ and $2^k l \leqslant |x_0 - y| < 2^{k+1} l$, or $y \in 2^{k+1}B \setminus 2^k B$, we have*

(i) $|t - y| \sim |x_0 - y| \sim 2^k l$

(ii)

$$|K(t, y) - K(x_0, y)| \lesssim \frac{2^{-ck}}{(2^k l)^n}, \tag{3.1}$$

where $c = \min\{1, (1 + m + n)/\varrho\}$.

P r o o f. (i) Obviously, $|x_0 - y| \sim 2^k l$. To see $|t - y| \sim |x_0 - y|$, we note that $|x_0 - y| \geqslant 2^k l \geqslant 2l$ since $k \geqslant 1$, and on the one hand,

$$|t - y| \leqslant |t - x_0| + |x_0 - y| \leqslant \frac{3}{2}|x_0 - y|,$$

and on the other hand,

$$|t-y| \geqslant |x_0-y| - |t-x_0| \geqslant \frac{1}{2}|x_0-y|.$$

So, $|t-y| \sim |x_0-y| \sim 2^k l$ holds.

(ii) To get the kernel estimate (3.1), we consider two cases about l.

Case 1: Let $l \geqslant 1$. Recall (i) of Lemma 2.4. The kernel $K(x,y)$ satisfies

$$\sup_{|x-y|\geqslant 1} |x-y|^N |D_x^\alpha D_y^\beta K(x,y)| \leqslant C_{\alpha,\beta,N} \tag{3.2}$$

for any multi-index α, β and for $N>0$.

Since $|x_0-y| \geqslant 2l > 1$ and $|t-y| \geqslant \frac{1}{2}|x_0-y| \geqslant l \geqslant 1$, statement (i) of Proposition 3.1 and (3.2) yield

$$\begin{aligned}|K(t,y)-K(x_0,y)| &\leqslant |K(t,y)|+|K(x_0,y)| \leqslant \frac{C}{|x_0-y|^N} \\ &\lesssim \frac{1}{|x_0-y|^{n+1}} \lesssim \frac{1}{(2^k l)^{n+1}} \leqslant \frac{2^{-k}}{(2^k l)^n} \leqslant \frac{2^{-ck}}{(2^k l)^n},\end{aligned}$$

where we take $N=n+1$.

Case 2: Let $0<l<1$. Lemma 2.4 says the distribution kernel $K(x,y)$ of T_a is smooth outside the diagonal $\{(x,x)\colon\ x\in\mathbb{R}^n\}$ and satisfies

$$\sup_{|\alpha+\beta|=M} |D_x^\alpha D_y^\beta K(x,y)| \leqslant C_M \frac{1}{|x-y|^{(M+m+n)/\varrho}}, \quad x\neq y, \tag{3.3}$$

if $M+m+n>0$.

Thus for every $t\in B(x_0,l)$ and $y\in 2^{k+1}B\setminus 2^kB$, from the mean value theorem and $1+n+m>0$, statement (i) of Proposition 3.1 and (3.3) imply

$$|K(t,y)-K(x_0,y)| = |D_xK(\xi,y)||t-x_0| \leqslant C\frac{|t-x_0|}{|x_0-y|^{(1+m+n)/\varrho}}, \tag{3.4}$$

where we choose $M=1$ and use the fact that $|\xi-y|\sim|x_0-y|$ if $\xi\in B(x_0,l)$. Let us now consider two subcases.

Subcase 2.1: If $(2^k-1)l\geqslant 1$, then, for any $t\in B(x_0,l)$ and $y\in 2^{k+1}B\setminus 2^kB$,

$$|t-y| \geqslant |y-x_0|-|t-x_0| \geqslant (2^k-1)l \geqslant 1.$$

It is similar to the case $l\geqslant 1$. Noting that $t\neq x_0$ and $l>0$, we employ the mean value theorem, statement (i) of Proposition 3.1 and (3.2) to obtain

$$|K(t,y)-K(x_0,y)| \leqslant \frac{C|t-x_0|}{|x_0-y|^N} \lesssim \frac{l}{|x_0-y|^{n+(1+m+n)/\varrho}} \lesssim \frac{l}{(2^k l)^{n+(1+m+n)/\varrho}},$$

where we choose $N = n + (1+m+n)/\varrho$. Meanwhile,

$$\frac{l}{(2^k l)^{(1+m+n)/\varrho}} \leqslant \begin{cases} 2^{-k(1+m+n)/\varrho}, & \dfrac{1+m+n}{\varrho} \leqslant 1, \\ 2^{-k}, & \dfrac{1+m+n}{\varrho} \geqslant 1, \end{cases}$$

since $l < 1$ and $2^k l > 1$. Therefore, (3.1) holds if we let $c = \min\{1, (1+m+n)/\varrho\}$.

Subcase 2.2: If $(2^k - 1)l < 1$, then

$$\frac{1}{2} \cdot 2^k l \leqslant (2^k - 1)l < 1.$$

Hence, statement (i) of Proposition 3.1 and (3.4) yield

$$|K(t,y) - K(x_0,y)| \leqslant C \frac{l}{(2^k l)^{(1+m+n)/\varrho}} \leqslant C \frac{l}{(2^k l)^{n+1}} \lesssim \frac{2^{-k}}{(2^k l)^n},$$

provided $m \leqslant -(n+1)(1-\varrho)$. Combining all the conditions on m, we require

$$-(n+1) < m \leqslant -(n+1)(1-\varrho),$$

which ends the proof of (3.1) and Proposition 3.1. □

Let us continue to estimate K_3. Equation (3.1) of Proposition 3.1 yields

$$\begin{aligned} K_3 &\leqslant \frac{C}{|B|} \int_B \sum_{k=1}^{\infty} \int_{2^k l \leqslant |x_0 - y| < 2^{k+1} l} \frac{2^{-ck}}{(2^k l)^n} |b(y) - b_B||f(y)| \,\mathrm{d}y \,\mathrm{d}t \\ &\leqslant C \sum_{k=1}^{\infty} \frac{2^{-ck}}{(2^k l)^n} \int_{2^{k+1} B} |b(y) - b_B||f(y)| \,\mathrm{d}y \\ &\leqslant C \sum_{k=1}^{\infty} 2^{-ck} \frac{1}{|2^{k+1} B|} \int_{2^{k+1} B} |b - b_B||f|. \end{aligned}$$

Let $B_k = 2^k B$. Then

$$\begin{aligned} K_3 &\leqslant C \sum_{k=1}^{\infty} 2^{-ck} \frac{1}{|B_{k+1}|} \int_{B_{k+1}} |b - b_B||f| \\ &\leqslant C \sum_k 2^{-ck} \left(\frac{1}{|B_{k+1}|} \int_{B_{k+1}} |b - b_{B_{k+1}}||f| + \frac{1}{|B_{k+1}|} \int_{B_{k+1}} |b_B - b_{B_{k+1}}||f| \right) \\ &= C \sum_k 2^{-ck} \left(\frac{1}{|B_{k+1}|} \int_{B_{k+1}} |b - b_{B_{k+1}}||f| + |b_B - b_{B_{k+1}}| \frac{1}{|B_{k+1}|} \int_{B_{k+1}} |f| \right) \\ &= C \sum_k 2^{-ck} (L_{k+1} + M_{k+1}). \end{aligned}$$

However,

$$L_{k+1} \leqslant S_{q'}(b;\omega,B_{k+1})\Lambda_q(f;\omega^{-1},B_{k+1}) \leqslant K^*(b,f,\omega)(x), \quad \text{(by Hölder)}$$

so that

$$K_3 \leqslant C\biggl(K^*(b,f,\omega)(x) + \sum_k 2^{-ck} M_{k+1}\biggr).$$

Now, we will show

$$\sum_k 2^{-ck} M_{k+1} \leqslant C[M_\lambda^*(|f\nu|^q)(x)]^{1/q}. \tag{3.5}$$

To prove (3.5), we will use the fact that

$$\int_B |b - b_B| \leqslant C\nu(B) \tag{3.6}$$

for each ball B, since $b \in \mathrm{BMO}_\nu$. Meanwhile, we recall (1.1). Since $\nu \in A_\infty$, there exists a $\eta > 0$ such that

$$\frac{\nu(E)}{\nu(B)} \leqslant C\biggl(\frac{|E|}{|B|}\biggr)^\eta \tag{3.7}$$

holds for all measurable sets $E \subset B$. Thus,

$$\begin{aligned}
|b_B - b_{B_{k+1}}| \leqslant \sum_{j=0}^k |b_{B_j} - b_{B_{j+1}}| \leqslant \sum_{j=0}^k \frac{1}{|B_j|}\int_{B_j} |b - b_{B_{j+1}}| & \\
\leqslant & C\sum_{j=0}^k \frac{1}{|B_{j+1}|}\int_{B_{j+1}} |b - b_{B_{j+1}}| \\
\lesssim & \sum_{j=0}^k \frac{\nu(B_{j+1})}{|B_{j+1}|} && \text{(by (3.6))} \\
= & \nu_{B_{k+1}} \sum_{j=0}^k \frac{\nu(B_{j+1})}{\nu(B_{k+1})}\frac{|B_{k+1}|}{|B_{j+1}|} \\
\leqslant & C\nu_{B_{k+1}} \sum_{j=0}^k \biggl(\frac{|B_{k+1}|}{|B_{j+1}|}\biggr)^{1-\eta} && \text{(by (3.7))} \\
= & C\nu_{B_{k+1}} \sum_{j=0}^k 2^{(k-j)n(1-\eta)} = C\nu_{B_{k+1}} \sum_{j=0}^k 2^{jn(1-\eta)} \\
:= & C\nu_{B_{k+1}} h(k).
\end{aligned}$$

Hence, for $k \geqslant 0$ we have

$$\begin{aligned}\sum_{k\geqslant 0} 2^{-ck} M_k &\leqslant C \sum_{k\geqslant 0} 2^{-ck} h(k)\nu_{B_k} \frac{1}{|B_k|}\int_{B_k} |f| \\ &= C \sum_{k\geqslant 0} 2^{-ck} h(k)\nu_{B_k} \frac{1}{|B_k|}\int_{B_k} |f|\nu\lambda^{1/q}\nu^{-1}\lambda^{-1/q} \\ &\lesssim \sum_{k\geqslant 0} 2^{-ck} h(k)\nu_{B_k} \left(\frac{1}{|B_k|}\int_{B_k} |f\nu|^q\lambda\right)^{1/q} \left(\frac{1}{|B_k|}\int_{B_k} \nu^{-q'}\lambda^{-q'/q}\right)^{1/q'} \\ &\leqslant [M_\lambda^*(|f\nu|^q)(x)]^{1/q} \sum_{k\geqslant 0} 2^{-ck} h(k)\nu_{B_k}(\lambda_{B_k})^{1/q} \left(\frac{1}{|B_k|}\int_{B_k} \nu^{-q'}\lambda^{-q'/q}\right)^{1/q'},\end{aligned}$$

since $\lambda_{B_k} = 1/|B_k| \int_{B_k} \lambda$. Now, we consider the series $\sum_{k\geqslant 0} 2^{-ck} h(k)$. Changing the order of summation, we have

$$\sum_{k\geqslant 0} 2^{-ck} h(k) = \sum_{j\geqslant 0} 2^{jn(1-\eta)} \sum_{k\geqslant j} 2^{-ck} \lesssim \sum_{j\geqslant 0} 2^{-j(c-n(1-\eta))},$$

which is convergent only with the assumption that $c - n(1-\eta) > 0$. Therefore, (3.5) will be proved if

$$I = \nu_B(\lambda_B)^{1/q}\left(\frac{1}{|B|}\int_B \nu^{-q'}\lambda^{-q'/q}\right)^{1/q'} \leqslant C \tag{3.8}$$

holds for all balls B provided

$$\eta > 1 - \frac{c}{n},$$

where $c = \min\{1, (1+m+n)/\varrho\}$. To show (3.8), we first note that

$$\nu^{-q'}\lambda^{-q'/q} = \mu^{-q'/p}\lambda^{-q'(1/q-1/p)}.$$

Then choose s large enough such that $sq'(1/q - 1/p) = p'/p$. Applying the reverse Hölder inequality to $\mu^{-q'/p}$ with the exponent $q's'/p'$ for q near p, we get

$$\begin{aligned}I &= (\mu_B\lambda_B^{-1})^{1/p}(\lambda_B)^{1/q}\left(\frac{1}{|B|}\int_B \mu^{-q'/p}\lambda^{-q'(1/q-1/p)}\right)^{1/q'} \\ &\leqslant (\lambda_B)^{1/q-1/p}(\mu_B)^{1/p}\left(\frac{1}{|B|}\int_B \mu^{-q's'/p}\right)^{1/s'q'}\left(\frac{1}{|B|}\int_B \lambda^{-p'/p}\right)^{1/sq'} \quad \text{(by Hölder)} \\ &\leqslant (\mu_B)^{1/p}(\mu_B^{-p'/p})^{1/p'}(\lambda_B^{p'/p})^{1/sq'}(\lambda_B^{-p'/p})^{1/sq'} \quad \text{(by reverse Hölder)}\end{aligned}$$

which is bounded, since μ in A_p. This completes the proof of Lemma 3.1. □

P r o o f. We now will prove Theorem 1.1. By Lemma 3.1,

$$\int ([b,T_a]f)^{\#p}\lambda \leqslant \int K^*(b,f,\omega)^p\lambda + \int K^*(b,T_af,\omega)^p\lambda + \int K_r^*(b,f,\widetilde{\omega})^p\lambda + \int (M_\lambda^*(|f\nu|^q))^{p/q}\lambda$$

for ω and $\widetilde{\omega}$ satisfying $\omega^{q'}, \widetilde{\omega}^{rq'} \in A_{q'}$. By Lemma 2.7, we can choose an appropriate $r > 1$ and such weights ω and $\widetilde{\omega}$ that

$$\int K^*(b,f,\omega)^p\lambda \leqslant C\int |f|^p\mu \quad \text{and} \quad \int K_r^*(b,f,\widetilde{\omega})^p\lambda \leqslant C\int |f|^p\mu.$$

Therefore,

$$\int ([b,T_a]f)^{\#p}\lambda \lesssim \int |f|^p\mu + \int |T_af|^p\mu + \int (M_\lambda^*(|f\nu|^q))^{p/q}\lambda.$$

Noting our $m \leqslant -(n+1)(1-\varrho)$ also satisfies the condition of Lemma 2.6 by using Lemma 2.6 we have

$$\int |T_af|^p\mu \leqslant C\int |f|^p\mu.$$

Since $\lambda \in A_p$, by Lemma 2.1 there is some $q < p$ such that $\omega \in A_{p/q}$. Then by Lemma 2.2 we obtain

$$\int (M_\lambda^*(|f\nu|^q))^{p/q}\lambda \leqslant C\int |f\nu|^p\lambda = C\int |f|^p\mu.$$

Hence, we have the two-weighted estimate for the sharp function

$$\int ([b,T_a]f)^{\#p}\lambda \lesssim \int |f|^p\mu. \tag{3.9}$$

Now, for any fixed ball B, let

$$k = \frac{1}{|B|}\int_B [b,T_a](f\chi_B)$$

be the average of $[b,T_a](f\chi_B)$ over B and let us estimate

$$\int_B |[b,T_a]f|^p\lambda \lesssim \int_B |[b,T_a]f-k|^p\lambda + \lambda(B)|k|^p.$$

The first term can be bounded by bounding the inner term by the p-power of the Hardy-Littlewood maximal function, then using the Fefferman-Stein result (see Lemma 2.3), and the two-weighted estimate for the sharp function (3.9),

$$\begin{aligned}\int_B |[b,T_a]f-k|^p\lambda \leqslant C\int_B (M^*([b,T_a]f))^p\lambda \lesssim \int (M^*([b,T_a]f))^p \\ \lesssim \int ([b,T_a]f)^{\#p}\lambda \lesssim \int |f|^p\mu\end{aligned}$$

with constants uniform on B. On the other hand, arguing as in the proof of Lemma 3.1, we deduce

$$\begin{aligned}|k| &\leqslant \frac{1}{|B|}\int_B |b-b_B||T_a(f\chi_B)| + \frac{1}{|B|}\int_B |T_a((b-b_B)f\chi_B)| \\ &\leqslant K^*(b,T_a(f\chi_B),\omega)(x) + K_r^*(b,f,\widetilde{\omega})(x).\end{aligned}$$

Combining this estimate with the monotonicity of the integral, and the double-weighted L^p boundedness of the operators K^* and K_r^* for an appropriate choice of ω, $\widetilde{\omega}$ and $r>1$ (see Lemma 2.7), we have

$$\begin{aligned}\lambda(B)|k|^p = \int_B |k|^p\lambda &\leqslant \int_B (K^*(b,T_a(f\chi_B),\omega)(x) + K_r^*(b,f,\widetilde{\omega})(x))^p\lambda \\ &\leqslant \|K^*(b,T_a(f\chi_B),\omega)\|^p_{L^p(\lambda)} + \|K_r^*(b,f,\widetilde{\omega})\|^p_{L^p(\lambda)} \\ &\leqslant C\|T_a(f\chi_B)\|^p_{L^p(\mu)} + \|f\|_{L^p(\mu)} \lesssim \|f\|^p_{L^p(\mu)},\end{aligned}$$

where all constants are independent of B. Here we use the double-weighted L^p boundedness of the operators K_r^* (see Lemma 2.7), $L^p(\omega)$ boundedness of T_a (see Lemma 2.6) and the monotonicity of the integral to get

$$\int K^*(b,T_a(f\chi_B),\omega)(x)^p\lambda \leqslant C\int |T_a(f\chi_B)|^p\mu \leqslant C\int_B |f|^p\mu \lesssim \int |f|^p\mu$$

with constants independent of B. Putting these estimates together, we conclude that for all balls B

$$\int_B |[b,T_a]f|^p\lambda \lesssim \|f\|^p_{L^p(\mu)}$$

with constants independent of B. And so, by the monotone convergence theorem, as the constants are independent of B, this yields

$$\|[b,T_a]f\|_{L^p(\lambda)} \lesssim \|f\|_{L^p(\mu)}.$$

Since the Hörmander classes $S^m_{\varrho,\delta}$ satisfy that if $m_1 \leqslant m_2 \leqslant 0$, then

$$S^{m_1}_{\varrho,\delta} \subset S^{m_2}_{\varrho,\delta},$$

we take directly the critical index of m as $m_c = -(n+1)(1-\varrho)$. So, for every $a(x,\xi) \in S^m_{\varrho,\delta}$ with $m \leqslant -(n+1)(1-\varrho)$, it follows that for $1 \geqslant \varrho > 0$,

$$-(n+1) < m_c = -(n+1)(1-\varrho),$$

and also

$$a(x,\xi) \in S^m_{\varrho,\delta} \subset S^{m_c}_{\varrho,\delta}.$$

Hence, applying the argument of the proof to m_c and taking also

$$c = \min\Big\{1, \frac{m_c+n+1}{\varrho}\Big\} = \min\{1, n+1\} = 1,$$

we get a better range of admissible η as $\eta > 1 - 1/n$. Thus, we complete the proof of Theorem 1.1. □

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