Kazushi Yoshitomi
Remarks on cardinal inequalities in convergence spaces

REMARKS ON CARDINAL INEQUALITIES
IN CONVERGENCE SPACES

Kazushi Yoshitomi, Tokyo

Abstract. We extend the Noble and Ulmer theorem and the Juhász and Hajnal theorems in set-theoretic topology. We show that a statement analogous to that in the former theorem is valid for a family of almost topological convergences, whereas statements analogous to those in the latter theorems hold for a pretopologically Hausdorff convergence.

Keywords: convergence space; cardinal function; inequality; set-theoretic topology

MSC 2020: 54A20, 54A25

1. Introduction

For a set X we denote its cardinality by $\operatorname{card} X$. For a topological space X we designate the cellularity, the spread and the density of X by $c(X)$, $s(X)$ and $d(X)$, respectively. The starting point of our consideration is the following triplet of theorems in set-theoretic topology.

(I) (Noble and Ulmer) Let $\{X_i\}_{i\in I}$ be a family of topological spaces, α an infinite cardinal. Assume that $c\Big(\prod\limits_{i\in J} X_i\Big) \leqslant \alpha$ whenever J is a finite subset of I. Then we have $c\Big(\prod\limits_{i\in I} X_i\Big) \leqslant \alpha$.

(II) (Juhász and Hajnal) If X is a Hausdorff topological space, then

$$\operatorname{card} X \leqslant 2^{2^{s(X)}}.$$

(III) (Juhász and Hajnal) If X is a Hausdorff topological space, then

$$d(X) \leqslant 2^{s(X)}.$$

The proofs and the sources of these theorems are supplied in [9], Chapter II. The purpose of this paper is to extend these theorems in a setting of convergence theory. Our main theorems are Theorem 1, Theorem 6 and Theorem 8, which extend (I), (II) and (III) above, respectively.

Now we briefly review the history of convergence theory. To this end we recall the Kuratowski closure axioms:

(cl 1) $\overline{\overline{\emptyset}} = \emptyset$;

(cl 2) $A \subset \bar{A}$;

(cl 3) $\overline{A \cup B} = \bar{A} \cup \overline{B}$;

(cl 4) $\bar{\bar{A}} = \bar{A}$.

These axioms are equivalent to the standard definition of a topological space in the following sense: If X is a topological space, then its closure operator $2^X \ni A \mapsto \bar{A} \in 2^X$ satisfies (cl 1)–(cl 4). Conversely, given an operator cl: $2^X \ni A \mapsto \bar{A} \in 2^X$ satisfying (cl 1)–(cl 4), the family $\{O \in 2^X : \text{cl}(X \setminus O) = X \setminus O\}$ gives a topology on X, whose closure operator equals cl. Čech built the notion of closure spaces (cf. [2], Chapter 14): A set X endowed with a map $2^X \ni A \mapsto \bar{A} \in 2^X$ satisfying (cl 1)–(cl 3) is called a closure space or a pretopological space. We will see later the usefulness of this notion. Choquet (see [3]) put forward a further generalization of the notion of topological spaces. He built the notion of pseudotopologies by means of filters. He also investigated pretopologies. Pseudotopologies and pretopologies are important subclasses of convergence spaces. The theory of convergence spaces has been developed by many authors; see [5] and the references therein.

Let us describe the features of our work here. Theorem 1 asserts that the statement of Noble and Ulmer's theorem (I) remains true when $\{X_i\}_{i \in I}$ is replaced with a family of almost topological convergences. In its proof the following property of the pseudotopologizer, designated by S, plays a crucial role: we have

$$(1.1) \qquad \text{S}\left(\prod_{i \in \Lambda} \xi_i\right) = \prod_{i \in \Lambda} \text{S}\xi_i$$

for every family $\{\xi_i\}_{i \in \Lambda}$ of convergences (see [5], Theorem VIII.3.9). This equality is one of the most important results in convergence theory. A careful inspection of the proofs of Juhász and Hajnal's theorems (II) and (III) gives that the property (cl 4) is not actually used there. This suggests that it may be possible to relax the usual Hausdorffness condition when we extend these theorems in a Čech-Choquet pretopological setting. Theorems 6 and 8 show that this is the case; these theorems establish that the statement of (II) and that of (III) remain valid when X is replaced with a pretopologically Hausdorff convergence. We mention that a pretopologically Hausdorff convergence need not to be topologically Hausdorff; see [5], Example VII.1.9.

Though we employ the basic frameworks of the proofs of Juhász and Hajnal's theorems in demonstrating our Theorems 6 and 8, we must deal with several technical matters proper to our situation; see (4.1) and Lemma 9. Furthermore, the notion of grills, which is due to Choquet, is effectively utilized in our proofs. Our work here builds bridges between set-theoretic topology and the theory of convergence spaces.

2. Preliminaries

From [5] we recall several notions and terminologies in the theory of convergence spaces for the reader's sake.

(a) We fix a set X. Let $\mathcal{A}$ and $\mathcal{B}$ be two families of subsets of X. $\mathcal{A}$ is said to be *coarser* than $\mathcal{B}$, in symbols, $\mathcal{A} \leqslant \mathcal{B}$, if for each $A \in \mathcal{A}$ there is a $B \in \mathcal{B}$ for which $B \subset A$. We designate by $\mathcal{A}^{\uparrow X}$ the *isotonization* of $\mathcal{A}$, that is,

$$\mathcal{A}^{\uparrow X} = \{B \in 2^X \colon \text{ there exists an } A \in \mathcal{A} \text{ such that } A \subset B\};$$

X is called an *ambient set*. When there is no risk of confusion we will write $\mathcal{A}^{\uparrow X}$ as $\mathcal{A}^\uparrow$. A family $\mathcal{F}$ of subsets of X is called a *filter base* on X if $\mathcal{F}^\uparrow$ is a filter.

(b) Fix a set X. We designate by $\mathbb{F}X$ the set of all filters on X. A relation ξ is said to be a *convergence* on X if it is contained in $\mathbb{F}X \times X$ and enjoys the following properties:

(isotone) $$\mathcal{F} \leqslant \mathcal{G} \Rightarrow \lim_\xi \mathcal{F} \subset \lim_\xi \mathcal{G};$$

(centered) $$x \in \lim_\xi \{x\}^\uparrow,$$

where we define $\lim_\xi \mathcal{F} = \{x \in X \colon (\mathcal{F}, x) \in \xi\}$ and use the abbreviation $\{x\}^\uparrow = \{\{x\}\}^\uparrow$; we say that X is the underlying set of ξ and write $|\xi| = X$. For a filter base $\mathcal{F}$ on X, we write $\lim_\xi \mathcal{F}$ for $\lim_\xi \mathcal{F}^\uparrow$.

(c) Let ξ and η be two convergences on X. We say that ξ is *coarser* than η, in symbols, $\xi \leqslant \eta$, if $\lim_\xi \mathcal{F} \supset \lim_\eta \mathcal{F}$ for every $\mathcal{F} \in \mathbb{F}X$.

(d) Let ξ be a convergence. A subset O of $|\xi|$ is said to be ξ-*open* if

$$O \cap \lim_\xi \mathcal{F} \neq \emptyset \Rightarrow O \in \mathcal{F}$$

for every filter $\mathcal{F}$. Let $x \in |\xi|$ and let N be a subset of $|\xi|$. We say that N is a ξ-*neighborhood* of x if there exists a ξ-open set O such that $x \in O \subset N$.

(e) Let ξ be a convergence, C a subset of $|\xi|$. We say that C is ξ-closed if $|\xi| \setminus C$ is ξ-open.

(f) A convergence ξ is said to be *Hausdorff* if $\mathrm{card}\big(\lim_\xi \mathcal{F}\big) \leqslant 1$ for every $\mathcal{F} \in \mathbb{F}|\xi|$.

(g) For a family $\{\mathcal{F}_i\}_{i \in K}$ of filters on a set X, we denote by $\bigwedge_{i \in K} \mathcal{F}_i$ the infimum of $\{\mathcal{F}_i\}_{i \in K}$ in the poset $(\mathbb{F}X, \leqslant)$; it holds that $\bigwedge_{i \in K} \mathcal{F}_i = \bigcap_{i \in K} \mathcal{F}_i$.

(h) Let ξ be a convergence and let $x \in |\xi|$. The *vicinity filter* of ξ at x is defined by

$$\mathcal{V}_\xi(x) = \bigwedge_{x \in \lim_\xi \mathcal{F}} \mathcal{F}.$$

A convergence ξ is said to be a *pretopology* if $x \in \lim_\xi \mathcal{V}_\xi(x)$ for each $x \in |\xi|$.

(i) Let $\mathcal{A}$ and $\mathcal{B}$ be two families of subsets of a set X. We say that $\mathcal{A}$ and $\mathcal{B}$ *mesh*, in symbols, $\mathcal{A} \# \mathcal{B}$, if $A \cap B \neq \emptyset$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$; otherwise we say that $\mathcal{A}$ and $\mathcal{B}$ are *dissociated*.

(j) Let $\mathcal{A}$ and $\mathcal{B}$ be two filters on a set X. The set $\{\mathcal{A}, \mathcal{B}\}$ admits the least upper bound in the poset $(\mathbb{F}X, \leqslant)$ if and only if $\mathcal{A}$ and $\mathcal{B}$ mesh.

(k) For a convergence ξ and a subset A of $|\xi|$, we designate by $\mathrm{adh}_\xi A$ the (*principal*) *adherence* of A, that is,

$$\mathrm{adh}_\xi A = \bigcup_{\mathcal{F} \# A} \lim_\xi \mathcal{F},$$

where we write $\mathcal{F} \# A$ for $\mathcal{F} \# \{A\}$. For a family $\mathcal{G}$ of subsets of $|\xi|$, we define $\mathrm{adh}_\xi^\natural \mathcal{G} = \{\mathrm{adh}_\xi A \colon A \in \mathcal{G}\}$. Note that if $\mathcal{G}$ is a filter, then $\mathrm{adh}_\xi^\natural \mathcal{G}$ is a filter base.

(l) A pretopology ξ is called a *topology* if the operator adh_ξ is idempotent, that is,

$$\mathrm{adh}_\xi\big(\mathrm{adh}_\xi A\big) = \mathrm{adh}_\xi A$$

for every $A \in 2^{|\xi|}$.

(m) A convergence ξ is said to be *regular* at a point x if, for any filter $\mathcal{F}$,

$$x \in \lim_\xi \mathcal{F} \Rightarrow x \in \lim_\xi \mathrm{adh}_\xi^\natural \mathcal{F}.$$

A convergence ξ is called *regular* if ξ is regular at each $x \in |\xi|$.

(n) Given a convergence ξ and a family $\mathcal{A}$ of subsets of $|\xi|$, let $\mathrm{adh}_\xi \mathcal{A}$ stand for the *adherence* of $\mathcal{A}$:

$$\mathrm{adh}_\xi \mathcal{A} = \bigcup_{\mathbb{F}|\xi| \ni \mathcal{H} \# \mathcal{A}} \lim_\xi \mathcal{H}.$$

We say that a convergence ξ is *compact* if $\mathrm{adh}_\xi \mathcal{F} \neq \emptyset$ for every filter $\mathcal{F}$.

(o) Let X be a set. A maximal element of the poset $(\mathbb{F}X, \leqslant)$ is called an *ultrafilter* on X. The set of all ultrafilters on X is designated by $\mathbb{U}X$. For a filter $\mathcal{F}$ on X, we set $\beta(\mathcal{F}) = \{\mathcal{U} \in \mathbb{U}X \colon \mathcal{F} \leqslant \mathcal{U}\}$.

(p) A convergence ξ is called a *pseudotopology* if

$$\lim_{\xi} \mathcal{F} = \bigcap_{\mathcal{U} \in \beta(\mathcal{F})} \lim_{\xi} \mathcal{U}$$

for each filter $\mathcal{F}$.

(q) For a set X, we designate by $o = o_X$ the coarsest convergence on X, that is, $\lim_{o_X} \mathcal{F} = X$ for every $\mathcal{F} \in \mathbb{F}X$; we call o_X the *chaotic convergence* on X.

(r) A convergence ξ is said to be T_1 if $\lim_{\xi}\{x\}^{\uparrow} = \{x\}$ for each $x \in |\xi|$.

For further background materials, one can consult [5].

3. Extension of Noble and Ulmer's theorem (I)

By S and T we designate the pseudotopologizer and the topologizer, respectively; for a convergence ξ, $S\xi$ is the finest among pseudotopologies coarser than ξ, and $T\xi$ is the finest among topologies coarser than ξ. A convergence ξ is said to be *almost topological* if $S\xi = T\xi$.

For a convergence ξ, we define its *cellularity* $c(\xi)$ by

$$c(\xi) = \sup\{\operatorname{card}\mathcal{G}\colon \mathcal{G} \text{ is a family of disjoint nonempty } \xi\text{-open sets}\}.$$

The following result extends Noble and Ulmer's theorem (I).

Theorem 1. *Let $\{\xi_i\}_{i \in I}$ be a family of almost topological convergences, α an infinite cardinal. Suppose that $c\left(\prod_{i \in J} \xi_i\right) \leqslant \alpha$ whenever J is a finite subset of I. It then holds that $c\left(\prod_{i \in I} \xi_i\right) \leqslant \alpha$.*

For a convergence ξ and an element x of $|\xi|$, we denote by $\mathcal{N}_{\xi}(x)$ the set of all the ξ-neighborhoods of x. Since $\mathcal{N}_{\xi}(x) = \mathcal{N}_{T\xi}(x)$ (see [5], page 141), the following assertion holds.

Proposition 2. *For a convergence ξ, we have $c(\xi) = c(T\xi)$.*

Let us prove the following implication.

Lemma 3. *Let $\{\xi_i\}_{i \in K}$ be a family of almost topological convergences. Then*

$$T\left(\prod_{i \in K} \xi_i\right) = \prod_{i \in K} T\xi_i.$$

P r o o f.   We have $\mathrm{S}\xi_i = \mathrm{T}\xi_i$ for each $i \in K$ by assumption, from which and (1.1) we obtain

$$(3.1) \qquad\qquad \mathrm{S}\left(\prod_{i \in K} \xi_i\right) = \prod_{i \in K} \mathrm{T}\xi_i.$$

Since $\prod_{i \in K} \mathrm{T}\xi_i$ is a topology by [5], Corollary V.4.33, so is $\mathrm{S}\left(\prod_{i \in K} \xi_i\right)$, and thus $\mathrm{S}\left(\prod_{i \in K} \xi_i\right) \leqslant \mathrm{T}\left(\prod_{i \in K} \xi_i\right)$. But $\mathrm{S}\eta \geqslant \mathrm{T}\eta$ for every convergence η; see [5], equation (VIII.3.2). So $\mathrm{S}\left(\prod_{i \in K} \xi_i\right) = \mathrm{T}\left(\prod_{i \in K} \xi_i\right)$. This, together with (3.1), yields the implication. $\qquad\square$

We are now in a position to complete the proof of Theorem 1.

P r o o f of Theorem 1.   Combining the assumption with Proposition 2 and Lemma 3, we have $c\left(\prod_{i \in J} \mathrm{T}\xi_i\right) \leqslant \alpha$ for any finite subset J of I. This, together with Noble and Ulmer's theorem (I), yields that $c\left(\prod_{i \in I} \mathrm{T}\xi_i\right) \leqslant \alpha$. Combining this with Proposition 2 and Lemma 3 again, we arrive at the conclusion. $\qquad\square$

An almost topological convergence is not necessarily topological. See [5], Exercise VIII.3.14 for a simple example. Here is an even simpler one:

E x a m p l e 4 (An almost topological convergence on $\{0, 1\}$ which is not a pseudotopology).   Let $X = \{0, 1\}$. The filters on X are $\{0\}^\uparrow$, $\{1\}^\uparrow$ and $\{0, 1\}^\uparrow$, and the ultrafilters on X are $\{0\}^\uparrow$ and $\{1\}^\uparrow$. We define a relation ξ on $\mathbb{F}X \times X$ by

$$\lim_\xi \{0\}^\uparrow = \{0, 1\}, \quad \lim_\xi \{1\}^\uparrow = \{0, 1\} \quad \text{and} \quad \lim_\xi \{0, 1\}^\uparrow = \{0\}.$$

One can easily check that ξ is a convergence on X. Because

$$\bigcap_{\mathcal{U} \in \beta(\{0,1\}^\uparrow)} \lim_\xi \mathcal{U} = \lim_\xi \{0\}^\uparrow \cap \lim_\xi \{1\}^\uparrow = \{0, 1\} \neq \lim_\xi \{0, 1\}^\uparrow,$$

the relation ξ is not a pseudotopology. Next, we will verify that ξ is almost topological. To this end, let η be a convergence satisfying $\eta \leqslant \xi$. We then have $\eta = o$ or $\eta = \xi$. Indeed, it holds that $\lim_o \mathcal{F} \supset \lim_\eta \mathcal{F} \supset \lim_\xi \mathcal{F}$ for every filter $\mathcal{F}$ on X since $o \leqslant \eta \leqslant \xi$, from which we have $\lim_\eta \{0\}^\uparrow = \lim_\eta \{1\}^\uparrow = \{0, 1\}$ and either $\lim_\eta \{0, 1\}^\uparrow = \{0\}$ or $\lim_\eta \{0, 1\}^\uparrow = \{0, 1\}$. In the former case, $\eta = \xi$, while in the latter case, $\eta = o$. Hence, $\mathrm{S}\xi = \mathrm{T}\xi = o$, that is, ξ is almost topological.

It is useful to recall the following result, which gives a sufficient condition for a convergence to be almost topological.

Theorem 5 ([5], Corollary IX.2.4). *A compact Hausdorff regular convergence is almost topological.*

We will give an extension of this theorem. To this end we recall a terminology from [4]. For a convergence ξ, let $r\xi$ stand for the partial regularization of ξ, that is, $x \in \lim_{r\xi} \mathcal{F}$ if there is a filter $\mathcal{G}$ such that $x \in \lim_\xi \mathcal{G}$ and $\mathcal{F} \geqslant \mathrm{adh}^\natural_\xi \mathcal{G}$. We have $\xi \geqslant r\xi$. We also recall that a Hausdorff topological space X is called H-closed if X is closed in any Hausdorff topological space in which it is embedded. In [8] a reasonable extension of the notion of an H-closed topological space is presented in a pretopological setting. In the same spirit we define the H-closedness of a convergence as follows: A convergence ξ is said to be H-closed if it is Hausdorff and $r\xi$ is compact. Now, the following assertion holds true: *A regular H-closed convergence is almost topological.* This follows at once from Theorem 5 and the fact that $\xi \leqslant r\xi$ if ξ is regular. We mention that the theory of H-closed topological spaces was initiated by Alexandroff and Urysohn (see [1]) and has been extensively developed (cf. [7], [6]).

4. Extension of the Juhász-Hajnal theorem (II)

We recall several notions and terminologies from [5], which are needed to state our second main theorem. For a convergence ξ, we denote by $\mathrm{S}_0\xi$ the *pretopological modification* of ξ, that is, $\mathrm{S}_0\xi$ is the finest one among pretopologies coarser than ξ; we say that ξ is *pretopologically Hausdorff* if $\mathrm{S}_0\xi$ is Hausdorff. Since $\mathcal{V}_\xi(x) = \mathcal{V}_{\mathrm{S}_0\xi}(x)$ for each $x \in |\xi|$ (see [5], equation (V.1.9)), we infer that ξ is pretopologically Hausdorff if and only if it holds that whenever $x_0, x_1 \in |\xi|$ satisfy $x_0 \neq x_1$, $\mathcal{V}_\xi(x_0)$ and $\mathcal{V}_\xi(x_1)$ are dissociated.

For a map $f\colon X \to Y$ and a filter $\mathcal{F}$ on X, we define $f[\mathcal{F}] = \{f(F)\colon F \in \mathcal{F}\}$, which is a filter base on Y.

Let (X, ξ) be a convergence space, A a subset of X. By $i^A_X\colon A \to X$ we denote the *inclusion map* of A in X, i.e. $i^A_X(x) = x$ for each $x \in A$. The convergence $\xi_{|A}$ induced on A by ξ is defined by

$$x \in \lim_{\xi_{|A}} \mathcal{F} \Leftrightarrow x \in \lim_\xi i^A_X[\mathcal{F}].$$

For a set X, we designate by $\iota = \iota_X$ the finest convergence on X, that is, for any $\mathcal{F} \in \mathbb{F}X$ and $x \in X$,

$$x \in \lim_\iota \mathcal{F} \Rightarrow \mathcal{F} = \{x\}^\uparrow.$$

We call ι_X the *discrete convergence* on X. For a convergence ξ, we define its *spread* by

$$s(\xi) = \sup\{\mathrm{card}\, Y\colon Y \text{ is a subset of } |\xi| \text{ for which } \xi_{|Y} \text{ is discrete}\}.$$

We are now ready to state our second main result, which extends the Juhász-Hajnal theorem (II).

Theorem 6. *Let ξ be a pretopologically Hausdorff convergence. It then holds that*

$$\operatorname{card} |\xi| \leqslant 2^{2^{s(\xi)}}.$$

We introduce some notation which is needed in the proof of this theorem. For a cardinal α, we denote its cardinal successor by α^+. For a set S and a positive integer r, we put $[S]^r = \{X \in 2^S : \operatorname{card} X = r\}$. For a set X and a family $\mathcal{A}$ of subsets of X, we designate by $\mathcal{A}^\#$ the *grill* of $\mathcal{A}$, that is,

$$\mathcal{A}^\# = \{H \in 2^X : A \cap H \neq \emptyset \text{ whenever } A \in \mathcal{A}\}.$$

The following claim is also needed, which is an immediate consequence of [5], Proposition III.1.11.

Proposition 7. *If ξ is a T_1 convergence and Y is a finite subset of $|\xi|$, then $\xi_{|Y}$ is discrete.*

P r o o f of Theorem 6. We proceed by an argument similar to that in [9], Chapter II, (6). Set $X = |\xi|$. By the Zermelo theorem there exists a relation "$<$" which well-orders X. Since ξ is pretopologically Hausdorff, we infer that, for each $(x, y) \in X^2$ with $x \neq y$, there exist $U_{xy} \in \mathcal{V}_\xi(x)$ and $V_{xy} \in \mathcal{V}_\xi(y)$ such that $U_{xy} \cap V_{xy} = \emptyset$. We consider the four subsets F_1, F_2, F_3 and F_4 of $[X]^3$ defined as follows: for $x, y, z \in X$ with $x < y < z$,

$$\{x, y, z\} \in F_1 \Leftrightarrow z \in U_{xy} \text{ and } x \in V_{yz};$$
$$\{x, y, z\} \in F_2 \Leftrightarrow z \notin U_{xy} \text{ and } x \in V_{yz};$$
$$\{x, y, z\} \in F_3 \Leftrightarrow z \notin U_{xy} \text{ and } x \notin V_{yz};$$
$$\{x, y, z\} \in F_4 \Leftrightarrow z \in U_{xy} \text{ and } x \notin V_{yz}.$$

The sets F_1, F_2, F_3 and F_4 form a partition of $[X]^3$.

First, we consider the case where $\alpha \equiv s(\xi)$ is infinite. Seeking a contradiction, we assume that $\operatorname{card} X > 2^{2^\alpha}$. It follows from the Erdős-Rado theorem (see [9], Chapter II, (4)) that there is a subset Y of X with $\operatorname{card} Y = \alpha^+$ such that $[Y]^3$ is a subset of one of F_1, F_2, F_3 and F_4. We define

$$Z = \{t \in Y : t \text{ has an immediate predecessor in the well-ordered set } (Y, <)\}.$$

We prove that $\xi_{|Z}$ is discrete. First, consider the case $[Y]^3 \subset F_2$. Pick a $t \in Z$ arbitrarily. We designate by z the immediate successor of t in $(Y, <)$. Let us show that $t \notin \mathrm{adh}_{\xi_{|Z}}\{s \in Z\colon s < t\}$. For any $s \in Z$ with $s < t$, we have $s \in V_{tz}$, so that $s \in U_{tz}^c$, U_{tz}^c being the complement of U_{tz} in X. Thus, $\{s \in Z\colon s < t\} \subset U_{tz}^c \cap Z$, from which

$$\mathrm{adh}_{\xi_{|Z}}\{s \in Z\colon s < t\} \subset \mathrm{adh}_{\xi_{|Z}}(U_{tz}^c \cap Z).$$

So, it suffices to show that

$$(4.1) \qquad\qquad t \notin \mathrm{adh}_{\xi_{|Z}}(U_{tz}^c \cap Z).$$

By definition we have

$$\mathcal{V}_{\xi_{|Z}}(t) = \bigwedge_{\substack{\mathcal{F} \in \mathbb{F}Z \\ t \in \lim_\xi i_X^Z[\mathcal{F}]}} \mathcal{F}.$$

We will verify that

$$(4.2) \qquad\qquad U_{tz} \cap Z \in \mathcal{V}_{\xi_{|Z}}(t).$$

Pick an $\mathcal{F} \in \mathbb{F}Z$ with $t \in \lim_\xi i_X^Z[\mathcal{F}]$ arbitrarily. Since $U_{tz} \in \mathcal{V}_\xi(t)$, we have $U_{tz} \in i_X^Z[\mathcal{F}]^\uparrow$, that is, there is an $F \in \mathcal{F}$ satisfying $F \subset U_{tz}$. Thus $U_{tz} \cap Z \in \mathcal{F}$, so that (4.2) holds. We have from (4.2) that $U_{tz}^c \cap Z \notin \mathcal{V}_{\xi_{|Z}}(t)^\#$. This, together with [5], Proposition V.2.12, yields (4.1), and therefore $t \notin \mathrm{adh}_{\xi_{|Z}}\{s \in Z\colon s < t\}$. Next, we will show that $t \notin \mathrm{adh}_{\xi_{|Z}}\{s \in Z\colon t < s\}$. For $s \in Z$ with $z < s$, we have $s \in U_{tz}^c$. Since $z \in V_{tz}$, we have also $z \in U_{tz}^c$. Thus $\{s \in Z\colon t < s\} \subset U_{tz}^c \cap Z$, whereupon

$$\mathrm{adh}_{\xi_{|Z}}\{s \in Z\colon t < s\} \subset \mathrm{adh}_{\xi_{|Z}}(U_{tz}^c \cap Z).$$

This, together with (4.1), yields $t \notin \mathrm{adh}_{\xi_{|Z}}\{s \in Z\colon t < s\}$. Hence,

$$t \notin \mathrm{adh}_{\xi_{|Z}}\{s \in Z\colon s < t\} \cup \mathrm{adh}_{\xi_{|Z}}\{s \in Z\colon t < s\} = \mathrm{adh}_{\xi_{|Z}}\{s \in Z\colon s \neq t\},$$

that is, $\mathrm{adh}_{\xi_{|Z}}\{s \in Z\colon s \neq t\} \subset \{s \in Z\colon s \neq t\}$. So, $Z \setminus \{t\}$ is $\xi_{|Z}$-closed.

Let us consider the case where $[Y]^3 \subset F_1$. Pick a $t \in Z$ arbitrarily. We designate by z the immediate successor of t in $(Y, <)$. For $s \in Z$ with $s < t$, we have $s \in V_{tz} \subset U_{tz}^c$, and therefore $\{s \in Z\colon s < t\} \subset U_{tz}^c \cap Z$. So, we obtain $t \notin \mathrm{adh}_{\xi_{|Z}}\{s \in Z\colon s < t\}$ as in the discussion above. By w we denote the immediate predecessor of t in $(Y, <)$. For $s \in Z$ with $t < s$, we have $s \in U_{wt} \subset V_{wt}^c$. So,

$$\mathrm{adh}_{\xi_{|Z}}\{s \in Z\colon t < s\} \subset \mathrm{adh}_{\xi_{|Z}}(V_{wt}^c \cap Z).$$

Now, we have

$$t \notin \mathrm{adh}_{\xi_{|Z}}(V_{wt}^c \cap Z)$$

as in the derivation of (4.1). Thus $t \notin \mathrm{adh}_{\xi_{|Z}}\{s \in Z \colon t < s\}$, and hence $Z \setminus \{t\}$ is $\xi_{|Z}$-closed.

In the remaining two cases one can similarly verify that, for each $t \in Z$, $Z \setminus \{t\}$ is $\xi_{|Z}$-closed. In any case, $\xi_{|Z}$ is a discrete convergence and $\mathrm{card}\, Z = \alpha^+$, which is absurd. Therefore, we obtain

$$(4.3) \qquad\qquad \mathrm{card}\, X \leqslant 2^{2^\alpha}.$$

Finally, we consider the case where $\alpha = s(\xi)$ is finite. It follows from Proposition 7 that X is a finite set. Using Proposition 7 again, we have $s(\xi) = \mathrm{card}\, |\xi|$, so that (4.3) holds. $\qquad\square$

5. Extension of the Juhász-Hajnal theorem (III)

For a convergence ξ, we define its *density* by

$$d(\xi) = \min\{\mathrm{card}\, S \colon S \text{ is a subset of } |\xi| \text{ such that } \mathrm{adh}_\xi S = |\xi|\}.$$

This definition of the density is equivalent to the one given in [5], Definition IV.9.28 and extends the notion of the density of a topological space.

We state our third main theorem, which extends the Juhász-Hajnal theorem (III).

Theorem 8. *Let ξ be a pretopologically Hausdorff convergence. It then holds that*

$$d(\xi) \leqslant 2^{s(\xi)}.$$

The following simple lemma is used in the proof of this theorem.

Lemma 9. *Let ξ be a convergence and let Z and A be two sets with $Z \subset A \subset |\xi|$. Then we have*

$$\mathrm{adh}_{\xi_{|A}} Z \subset \big(\mathrm{adh}_\xi Z\big) \cap A.$$

P r o o f.   Put $X = |\xi|$. We obtain

$$\mathrm{adh}_{\xi_{|A}} Z = \bigcup_{\substack{\mathcal{F} \# Z \\ \mathcal{F} \in \mathbb{F}A}} \lim_{\xi_{|A}} \mathcal{F} = \bigcup_{\substack{\mathcal{F} \# Z \\ \mathcal{F} \in \mathbb{F}A}} \lim_\xi i_X^A[\mathcal{F}]^\uparrow \subset \bigcup_{\substack{\mathcal{G} \# Z \\ \mathcal{G} \in \mathbb{F}X}} \lim_\xi \mathcal{G} = \mathrm{adh}_\xi Z.$$

$\square$

We introduce a new term. We designate by Ord the class of all ordinals. Fix $\alpha \in \mathrm{Ord}$. A function is said to be an α-*sequence* if its domain is $\{\beta \in \mathrm{Ord}\colon \beta < \alpha\}$. We will identify the latter set with α.

P r o o f of Theorem 8.   We employ a method parallel to that in [9], Chapter II, (5). Put $\alpha = s(\xi)$ and $X = |\xi|$. First, we consider the case where α is infinite. Seeking a contradiction, we suppose that $d(\xi) > 2^\alpha$. For each ordinal β with $\beta < (2^\alpha)^+$ and for each X-valued β-sequence $\{r_\gamma\}_{\gamma<\beta}$, we have $X \setminus \mathrm{adh}_\xi \{r_\gamma\}_{\gamma<\beta} \neq \emptyset$, because $\beta < d(\xi)$. By transfinite induction one can construct a sequence $P = \{p_\beta\}_{\beta<(2^\alpha)^+}$ such that $p_\beta \in X \setminus \mathrm{adh}_\xi \{p_\gamma\}_{\gamma<\beta}$ for each $\beta < (2^\alpha)^+$.

We will construct a sequence $\{q_\beta\}_{\beta<\alpha^+}$ such that $q_\beta \notin \mathrm{adh}_\xi \{q_\gamma\}_{\gamma>\beta}$ for each $\beta < \alpha^+$. We proceed by a ramification argument. For $\gamma < \alpha^+$, let

$$F_\gamma = \{f\colon f \text{ is a } \{0,1\}\text{-valued } \gamma\text{-sequence}\}.$$

Set $F = \bigcup_{\gamma<\alpha^+} F_\gamma$. By transfinite induction we will construct a family $\{P_f\}_{f\in F}$ of subsets of P as follows. We fix $\gamma < \alpha^+$ and suppose that we have defined P_g for every $g \in \bigcup_{\delta<\gamma} F_\delta$.

First, we consider the case where $\gamma = 0$. We have $F_\gamma = \{\emptyset\}$. Put $P_\emptyset = P$. Next, we consider the case where γ is an ordinal successor; let δ be such that $\gamma = \delta + 1$. Consider two elements f and g of F_γ such that $f|_\delta = g|_\delta$ and $f(\delta) \neq g(\delta)$. We consider the following two cases:

(a) $\mathrm{card}\, P_{f|_\delta} \leqslant 1$. We put $P_f = P_g = \emptyset$.
(b) $\mathrm{card}\, P_{f|_\delta} > 1$. Choose $a, b \in P_{f|_\delta}$ with $a \neq b$. Since ξ is pretopologically Hausdorff, there exist $U_a \in \mathcal{V}_\xi(a)$ and $U_b \in \mathcal{V}_\xi(b)$ for which $U_a \cap U_b = \emptyset$. Put $P_f = P_{f|_\delta} \setminus U_a$ and $P_g = P_{f|_\delta} \setminus U_b$. We have $P_f \cup P_g = P_{f|_\delta}$.

Finally, we treat the case where γ is a limit ordinal which is not 0. For each $f \in F_\gamma$, define $P_f = \bigcap_{\delta<\gamma} P_{f|_\delta}$.

We will prove that there is a $p \in P$ such that $\{p\} \neq P_f$ for each $f \in F$. Since $\mathrm{card}\, F_\gamma = 2^{\mathrm{card}\,\gamma} \leqslant 2^\alpha$ for each $\gamma < \alpha^+$, we have $\mathrm{card}\, F \leqslant 2^\alpha$. We define $F' = \{f \in F\colon P_f \neq \emptyset\}$. For each $f \in F'$, we choose an element l_f of P_f. Because $\mathrm{card}\, P = (2^\alpha)^+ > \mathrm{card}\, F'$, there is a $p \in P$ such that $l_f \neq p$ for each $f \in F'$. Hence, $\{p\} \neq P_f$ for every $f \in F$.

Let us prove that there is a sequence of functions $f^\gamma \in F_\gamma$, $\gamma < \alpha^+$, such that $p \in P_{f^\gamma}$ for each $\gamma < \alpha^+$ and $f^\gamma|_\delta = f^\delta$ whenever $\delta < \gamma < \alpha^+$. We proceed by transfinite induction. We fix $\gamma_0 < \alpha^+$ and suppose that there is a sequence of functions $f^\gamma \in F_\gamma, \gamma < \gamma_0$, such that $p \in P_{f^\gamma}$ for each $\gamma < \gamma_0$ and $f^\gamma|_\delta = f^\delta$ whenever $\delta < \gamma < \gamma_0$. We define f^{γ_0} as follows. First, consider the case where γ_0 is an ordinal

successor. Let γ_0' be such that $\gamma_0 = \gamma_0' + 1$. Since $p \in P_{f^{\gamma_0'}}$ and $\{p\} \neq P_{f^{\gamma_0'}}$, we have card $P_{f^{\gamma_0'}} > 1$. Pick $f, g \in F_{\gamma_0}$ for which $f^{\gamma_0'} = f|_{\gamma_0'} = g|_{\gamma_0'}$ and $f \neq g$. Since $P_f \cup P_g = P_{f^{\gamma_0'}}$ and $p \in P_{f^{\gamma_0'}}$, we have either $p \in P_f$ or $p \in P_g$. If $p \in P_f$, we put $f^{\gamma_0} = f$; otherwise, we set $f^{\gamma_0} = g$. Next, we consider the case where γ_0 is a limit ordinal which is not 0. We define $f^{\gamma_0}(\delta) = f^{\delta+1}(\delta)$ for $\delta < \gamma_0$. Finally, we treat the case where $\gamma_0 = 0$. We define $f^{\gamma_0} = \emptyset$. In any case we have $p \in P_{f^{\gamma_0}}$ and $f^{\gamma_0}|_\delta = f^\delta$ for each $\delta < \gamma_0$. Thus, there exists a sequence of functions $f^\gamma \in F_\gamma$, $\gamma < \alpha^+$, such that $p \in P_{f^\gamma}$ for each $\gamma < \alpha^+$ and $f^\gamma|_\delta = f^\delta$ whenever $\delta < \gamma < \alpha^+$.

We are now in a position to define $\{q_\beta\}_{\beta < \alpha^+}$. Fix $\gamma < \alpha^+$. Then, there exist $a \in P_{f^\gamma}$ and $U_\gamma \in \mathcal{V}_\xi(a)$ such that $P_{f^{\gamma+1}} = P_{f^\gamma} \setminus U_\gamma$. We define $q_\gamma = a$.

Let us prove that $q_\gamma \notin \mathrm{adh}_\xi \{q_\delta\}_{\delta > \gamma}$ for each $\gamma < \alpha^+$. Fix $\gamma < \alpha^+$. We have $q_\delta \notin U_\gamma$ for every $\delta > \gamma$, because $q_\delta \in P_{f^\delta} \subset P_{f^{\gamma+1}}$. This, combined with $U_\gamma \in \mathcal{V}_\xi(q_\gamma)$, implies that $\{q_\delta : \delta > \gamma\} \notin \mathcal{V}_\xi(q_\gamma)^\#$; from which and [5], Proposition V.2.12 we obtain $q_\gamma \notin \mathrm{adh}_\xi \{q_\delta\}_{\delta > \gamma}$.

Set $Q = \{q_\gamma : \gamma < \alpha^+\}$. We define a sequence $\{\varrho_\gamma\}_{\gamma < \alpha^+}$ as follows. Fix $\gamma < \alpha^+$. There is a unique $\varrho < (2^\alpha)^+$ for which $q_\gamma = p_\varrho$. We define $\varrho_\gamma = \varrho$. Put

$$F_1 = \{\{q_\delta, q_\gamma\} : \delta < \gamma, \varrho_\delta < \varrho_\gamma\},$$
$$F_2 = \{\{q_\delta, q_\gamma\} : \delta < \gamma, \varrho_\delta > \varrho_\gamma\}.$$

The sets F_1 and F_2 form a partition of $[Q]^2$. We claim that there exists no infinite subset A of Q for which $[A]^2 \subset F_2$, because there is no infinite decreasing sequence of ordinals. This, combined with the Erdős theorem (see [9], Chapter II, (3)), yields that there is a subset A of Q such that card $A = \alpha^+$ and $[A]^2 \subset F_1$. We will demonstrate that $\xi_{|A}$ is a discrete convergence. Put $\Gamma = \{\gamma < \alpha^+ : q_\gamma \in A\}$. Fix $\gamma \in \Gamma$. Since $q_\gamma \notin \mathrm{adh}_\xi \{q_\delta\}_{\delta > \gamma}$, we have $q_\gamma \notin \mathrm{adh}_\xi \{q_\delta\}_{\delta > \gamma, \delta \in \Gamma}$; from which and Lemma 9 we obtain $q_\gamma \notin \mathrm{adh}_{\xi_{|A}} \{q_\delta\}_{\delta > \gamma, \delta \in \Gamma}$. Since $\{q_\delta\}_{\delta \in \Gamma, \delta < \gamma} \subset \{p_\eta\}_{\eta < \varrho_\gamma}$ and since $p_{\varrho_\gamma} \notin \mathrm{adh}_\xi \{p_\eta\}_{\eta < \varrho_\gamma}$, it holds that $q_\gamma \notin \mathrm{adh}_\xi \{q_\delta\}_{\delta \in \Gamma, \delta < \gamma}$. Combining this with Lemma 9, we get $q_\gamma \notin \mathrm{adh}_{\xi_{|A}} \{q_\delta\}_{\delta \in \Gamma, \delta < \gamma}$. Hence, $\mathrm{adh}_{\xi_{|A}}(A \setminus \{q_\gamma\}) \subset A \setminus \{q_\gamma\}$, whereupon $A \setminus \{q_\gamma\}$ is $\xi_{|A}$-closed. Thus $\xi_{|A}$ is a discrete convergence. This is absurd, since card $A = \alpha^+ > s(\xi)$. Therefore, we obtain $d(\xi) \leqslant 2^{s(\xi)}$.

We consider the case where α is finite. It follows from Proposition 7 that $\alpha = \mathrm{card} |\xi|$, from which $d(\xi) \leqslant \alpha < 2^\alpha = 2^{s(\xi)}$. $\qquad\square$

References

[1] *P. S. Alexandroff, P. S. Urysohn*: Über kompakte topologische Räume. Akad. Nauk SSSR, Trudy Mat. Inst. Steklov *31* (1950), 94 pages. (In Russian.) `zbl` `MR`

[2] *E. Čech*: Topological Spaces. Publishing House of the Czechoslovak Academy of Sciences, Prague; John Wiley & Sons, London, 1966. `zbl` `MR`

[3] *G. Choquet*: Convergences. Ann. Univ. Grenoble, Sect. Sci. Math. Phys., II. Ser. *23* (1948), 57–112. `zbl` `MR`

[4] *S. Dolecki, D. Gauld*: Irregularity. Topology Appl. *154* (2007), 1565–1580; Erratum ibid. *159* (2012), 3658–3659. `zbl` `MR` `doi`

[5] *S. Dolecki, F. Mynard*: Convergence Foundations of Topology. World Scientific, Hackensack, 2016. `zbl` `MR` `doi`

[6] *M. Katětov*: Über *H*-abgeschlossene und bikompakte Räume. Čas. Pěst. Mat. Fys. *69* (1940), 36–49. (In German.) `zbl` `MR` `doi`

[7] *M. Katětov*: On *H*-closed extensions of topological spaces. Čas. Pěst. Mat. Fys. *72* (1947), 17–32. `zbl` `MR` `doi`

[8] *J. P. Reynolds*: Hausdorff closedness in the convergence setting. Topol. Proc. *49* (2017), 135–152. `zbl` `MR`

[9] *M. E. Rudin*: Lectures on Set Theoretic Topology. CBMS Regional Conference Series in Mathematics 23. AMS, Providence, 1975. `zbl` `MR` `doi`

Author's address: *Kazushi Yoshitomi*, Department of Mathematical Sciences, Tokyo Metropolitan University, Minamiohsawa 1-1, Hachioji, Tokyo 192-0397, Japan, e-mail: `yositomi@tmu.ac.jp`.