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EXISTENCE AND UNIQUENESS FOR A TWO-DIMENSIONAL VENTCEL PROBLEM MODELING THE EQUILIBRIUM OF A PRESTRESSED MEMBRANE

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Abstract. This paper deals with a mixed boundary-value problem of Ventcel type in two variables. The peculiarity of the Ventcel problem lies in the fact that one of the boundary conditions involves second order differentiation along the boundary. Under suitable assumptions on the data, we first give the definition of a weak solution, and then we prove that the problem is uniquely solvable. We also consider a particular case arising in real-world applications and discuss the resulting model.

Keywords: Ventcel boundary condition; Laplace-Beltrami operator; composite Sobolev space; well-posedness

MSC 2020: 35J25, 35M12, 35A01, 35A02

1. Introduction

The Ventcel problem is a boundary-value problem of mixed type, whose peculiarity relies in the fact that the boundary condition involves a second-order differentiation of the solution, which is in contrast with the usual Neumann or even Robin problem, where the boundary condition involves a first-order differentiation only.

The initial motivation of Ventcel (whose name is transliterated in several different ways, we follow [3], [16]) was to find the boundary conditions that restrict a given

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elliptic operator to the infinitesimal generator of a Markov process (see [20]). Much later, Favini, Ruiz Goldstein, Goldstein and Romanelli showed how to solve the parabolic problem with boundary conditions of Ventcel type in a convenient Lebesgue space with weight (see [9], [10] and the references therein).

While the Ventcel boundary condition has received less attention than the classical Dirichlet, Neumann, and Robin conditions, there is by now a pretty extensive literature, which includes results for general second order elliptic equations with variable coefficients (see e.g. [1], [2], [14]).

The questions of well-posedness of the elliptic problem and regularity of the solution, together with the convergence of the finite element approximation, are still under investigation: see, for instance, [3], [6], [13], [16]. Applications are found in the study of thin layers, rough boundaries, heat conduction processes with a heat source on the boundary, fluid-structure interaction problems, and more.

Our interest arises from planar elasticity: indeed, in Section 5 we show that the equilibrium problem of a prestressed membrane whose boundary is composed of rigid and cable elements can be rephrased as a Ventcel problem. The solution was obtained in [21] by means of a numerical strategy, leaving the question of existence and uniqueness open. In the present paper we prove well-posedness of a simplified model.

1.1. Formulation of the problem. Following the notation in [3], [16], for a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$, with Lipschitz boundary $\partial\Omega$, let $\Gamma_{\boldsymbol{\nu}}$ be an open subset of $\partial\Omega$ having positive measure and such that $\Gamma_D = \partial\Omega \setminus \Gamma_{\boldsymbol{\nu}}$ also has positive measure. For a convenient function $f = f(\mathbf{x})$, $\mathbf{x} \in \Omega$, and boundary data $\varphi = \varphi(\mathbf{x})$, $\mathbf{x} \in \Gamma_D$, and $g = g(\mathbf{x})$, $\mathbf{x} \in \Gamma_{\boldsymbol{\nu}}$, as well as for variable coefficients $a_2 = a_2(\mathbf{x})$ and $a_0 = a_0(\mathbf{x})$ with $\mathbf{x} \in \Gamma_{\boldsymbol{\nu}}$, we are interested in the Ventcel mixed boundary-value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = \varphi & \text{on } \Gamma_D, \\ u_{\boldsymbol{\nu}} - a_2 \Delta_{\boldsymbol{\tau}} u + a_0 u = g & \text{on } \Gamma_{\boldsymbol{\nu}}, \end{cases} \tag{1.1}$$

where $\Delta_{\boldsymbol{\tau}}$ is the Laplace-Beltrami operator on $\Gamma_{\boldsymbol{\nu}}$ and the subscript $(\cdot)_{\boldsymbol{\nu}}$ indicates the outward normal derivative on $\Gamma_{\boldsymbol{\nu}}$. Some known results:

- ▷ For $\Gamma_D = \emptyset$, $\Omega \subset \mathbb{R}^d$ with $d \geqslant 2$, $a_0 = \alpha > 0$ (constant), $a_2 = -\beta$ (constant) and $g \equiv 0$, well-posedness has been established independently of the sign of β, see [3] (the notation is taken from there).
- ▷ For $\Gamma_D = \emptyset$, $\Omega \subset \mathbb{R}^d$ with $d \geqslant 2$, $a_0 = \alpha > 0$ (constant) and $a_2 = \beta > 0$ (constant), regularity results and finite element analysis are presented in [13].
- ▷ For $\Omega \subset \mathbb{R}^d$ with $d = 3$, $a_0 = 0$, $a_2 = 1$ (constant) and $\varphi \equiv 0$, regularity results and a priori error analysis in polyhedral domains are obtained in [16].

1.2. Main result. In order to formulate the main claim of our work, we first have to set the following assumption:

Assumption 1.1.

(1) Ω is a bounded Lipschitz domain (an open, connected set) in the plane, whose boundary $\partial\Omega$ properly contains a finite number $N \geqslant 1$ of simple C^1-curves Γ_i, $i = 1, \dots, N$, satisfying $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$.
(2) By a C^1-curve Γ_i we mean the trace of a function $\boldsymbol{r} = \boldsymbol{r}_i(s)$ belonging to the class $C^1([0, L_i], \mathbb{R}^2)$ and satisfying $|\boldsymbol{r}_i'(s)| = 1$ in the interval $[0, L_i]$. Thus, $L_i > 0$ is the length of Γ_i, and the tangent unit vector $\boldsymbol{\tau} = \partial/\partial s$, as well as the outward unit vector $\boldsymbol{\nu}$ may be extended by continuity from the interior of Γ_i to its endpoints.
(3) Each curve Γ_i is simple in the sense that the equality $\boldsymbol{r}(s_1) = \boldsymbol{r}(s_2)$ may hold for $s_1 < s_2$ only if $s_1 = 0$ and $s_2 = L$. We allow the case when $\boldsymbol{r}_i(0) = \boldsymbol{r}_i(L_i)$, i.e., Γ_i is a closed curve, but in such a case we require $\boldsymbol{r}_i'(0) = \boldsymbol{r}_i'(L_i)$; thus, $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ have a unique extension to the (coinciding) endpoints.
(4) The notation Γ_i° stands for the interior of Γ_i, which is defined as follows: $\Gamma_i^\circ = \Gamma_i \setminus \{\boldsymbol{r}_i(0), \boldsymbol{r}_i(L_i)\}$ if $\boldsymbol{r}_i(0) \neq \boldsymbol{r}_i(L_i)$, and $\Gamma_i^\circ = \Gamma_i$ if $\boldsymbol{r}_i(0) = \boldsymbol{r}_i(L_i)$. We also define the open subset $\Gamma_{\boldsymbol{\nu}} = \bigcup_{i=1}^{N} \Gamma_i^\circ \subset \partial\Omega$, and we assume that the closed subset $\Gamma_D = \partial\Omega \setminus \Gamma_{\boldsymbol{\nu}}$ has a positive length.

E x a m p l e 1.2. A domain fulfilling the above structure is, for instance, the annulus $\Omega = B_{R_1}(0) \setminus \overline{B}_{R_0}(0)$ with $0 < R_0 < R_1$. We may let $\Gamma_{\boldsymbol{\nu}} = \partial B_{R_0}(0)$ and $\Gamma_D = \partial B_{R_1}(0)$, as well as $\Gamma_{\boldsymbol{\nu}} = \partial B_{R_1}(0)$ and $\Gamma_D = \partial B_{R_0}(0)$; the last case is considered in [3], Section 2.1.2. Another example is the square $\Omega = (0,1) \times (1,2)$ with $\Gamma_i = [0,1] \times \{i\}$, $i = 1, 2$; in this case we have $\Gamma_i^\circ = (0,1) \times \{i\}$, $i = 1, 2$, hence $\Gamma_{\boldsymbol{\nu}} = \Gamma_1^\circ \cup \Gamma_2^\circ = (0,1) \times \{1,2\}$ and $\Gamma_D = \partial\Omega \setminus \Gamma_{\boldsymbol{\nu}} = \{0,1\} \times [1,2]$.

Our main result is the following:

Theorem 1.3. *Let Assumption* 1.1 *hold and* $\varphi\colon \Gamma_D \to \mathbb{R}$ *satisfy a uniform Lipschitz condition. Then for any* $f \in L^2(\Omega)$, $g \in L^2(\Gamma_{\boldsymbol{\nu}})$, $0 \leqslant a_0 \in L^\infty(\Gamma_{\boldsymbol{\nu}})$ *and* $a_2 \in W^{1,\infty}(\Gamma_{\boldsymbol{\nu}})$ *such that* $\inf_{\Gamma_{\boldsymbol{\nu}}} a_2 > 0$, *problem* (1.1) *admits a unique weak solution.*

R e m a r k 1.4. Taking Assumption 1.1 (2) into account, the tangential derivative $(u(\boldsymbol{r}_i(s)))' = \mathrm{d}u(\boldsymbol{r}_i(s))/\mathrm{d}s$ of a smooth function u is denoted by $\nabla_{\boldsymbol{\tau}} u$, the intrinsic gradient of u on Γ_i, and the second derivative $(u(\boldsymbol{r}_i(s)))'' = \mathrm{d}^2 u(\boldsymbol{r}_i(s))/\,\mathrm{d}s^2$ by $\Delta_{\boldsymbol{\tau}} u$, the Laplace-Beltrami operator on Γ_i. For shortness, we also write u_s and u_{ss} in place of $\nabla_{\boldsymbol{\tau}} u$ and $\Delta_{\boldsymbol{\tau}} u$, respectively. The tangential derivatives and the Laplace-Beltrami operator are defined, for instance, in [13], Section 2.2.

Taking $a_2 \in W^{1,\infty}(\Gamma_{\boldsymbol{\nu}})$ gives the third condition in (1.1) a variational structure (see Definition 3.2), namely, the condition becomes

$$u_{\boldsymbol{\nu}} - \nabla_{\boldsymbol{\tau}}(a_2 \nabla_{\boldsymbol{\tau}} u) + \nabla_{\boldsymbol{\tau}} a_2 \nabla_{\boldsymbol{\tau}} u + a_0 u = g \quad \text{on} \quad \Gamma_{\boldsymbol{\nu}}.$$

In the special case when a_2 is constant, the coercivity of the bilinear form associated to the definition of a weak solution is essentially automatic. In contrast, in the non-constant case, coercivity is gained (see Lemma 4.2) by adding a convenient term to both sides of the functional equation, so to obtain a compact resolvent operator, subject to the Fredholm alternative. The weak formulation of the problem also relies on the existence of a sufficient regular lifting of the boundary data to the whole domain; in order to make this paper self-contained, we give details in Lemma 3.1.

1.3. Non-local interpretation. We shortly mention that problem (1.1) can be rephrased in terms of a suitable non-local operator (see, for instance, [3], Section 2.2.1). The idea is that if the domain Ω and the boundary data φ, ψ are sufficiently smooth, then the solution u of the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = \varphi & \text{on } \Gamma_D, \\ u = \psi & \text{on } \Gamma_{\boldsymbol{\nu}}, \end{cases} \tag{1.2}$$

is uniquely determined and differentiable on $\Gamma_{\boldsymbol{\nu}}$. If the boundary value φ on Γ_D is kept fixed and ψ is let to vary, then the outward derivative $u_{\boldsymbol{\nu}}$ along $\Gamma_{\boldsymbol{\nu}}$ can be thought of as the outcome of an operator; namely, the operator $L\colon \psi \mapsto u_{\boldsymbol{\nu}}$, usually called *the Dirichlet-to-Neumann operator* (a related operator is the *Steklov-Poincaré operator* defined in [18], pp. 3–4). The Dirichlet-to-Neumann operator is considered in detail in [5] in the case when the bounded domain $\Omega \subset \mathbb{R}^2$ is replaced by the half-space in $\mathbb{R}^{d-1} \times (0,\infty) \subset \mathbb{R}^d$, $d \geqslant 2$, and $\Gamma_D = \emptyset$. In the present case, instead, the operator L acts on functions defined on the bounded, possibly non-straight and disconnected curve $\Gamma_{\boldsymbol{\nu}}$. Such an operator is non-local in the sense that any modification of the given function ψ in a small neighborhood of any point $\mathbf{x}_0 \in \Gamma_{\boldsymbol{\nu}}$ implies a consequent modification of the outcome $u_{\boldsymbol{\nu}}$ on the whole $\Gamma_{\boldsymbol{\nu}}$. Under convenient assumptions, the Ventcel problem (1.1) is equivalent to the single equation

$$L\psi - a_2 \Delta_{\boldsymbol{\tau}} \psi + a_0 \psi = g \quad \text{on } \Gamma_{\boldsymbol{\nu}}. \tag{1.3}$$

Indeed, if u is any solution of problem (1.1), then its trace $\psi = u|_{\Gamma_{\boldsymbol{\nu}}}$ solves equation (1.3), because $L\psi = u_{\boldsymbol{\nu}}$. Conversely, if ψ is any solution of equation (1.3), then the solution u of the Dirichlet problem (1.2), which is used in the definition of L, is also a solution of the Ventcel problem (1.1). The peculiarity of equation (1.3) lies in the fact that the non-local operator L and the (local) Laplace-Beltrami operator are compared to each other.

2. Poincaré-type inequalities

In this section we establish some estimates that resemble the well-known Poincaré inequality, and play a key role in the subsequent proof of uniqueness of the weak solution to problem (1.1). In particular, in Lemma 2.4 we also prove an interesting Poincaré inequality for the composite Sobolev space V_0 defined in (2.3).

Lemma 2.1 (Poincaré-type inequality in a bounded interval)**.** *Let (a,b) be a bounded interval on the real line, and let $v \in H^1((a,b))$. Since v has a continuous extension to the closed interval $[a,b]$, we may define*

$$v_0 = \min_{[a,b]} v, \quad v_1 = \max_{[a,b]} v, \quad \tilde{I} = \{s \in [a,b];\ v_0 < v(s) < v_1\}.$$

We claim that

$$\int_{\tilde{I}} (v(s) - v_0)^2 \,\mathrm{d}s \leqslant |\tilde{I}|^2 \int_{\tilde{I}} (v'(s))^2 \,\mathrm{d}s, \tag{2.1}$$

where $|\tilde{I}|$ denotes the Lebesgue measure of $\tilde{I}$.

P r o o f. If $\tilde{I} = \emptyset$, i.e., if v is constant, then (2.1) obviously holds. Otherwise, we argue as follows. Let $s_0 \in [a,b]$ be any point such that $v(s) = v_0$. By the fundamental theorem of calculus we get the estimate

$$v(s) - v_0 \leqslant \int_{s_0}^{s} |v'(t)| \,\mathrm{d}t \leqslant \|v'\|_{L^1((a,b))}.$$

By [11], Lemma 7.7 (or [4], Comment 4, p. 314) we have $v' = 0$ almost everywhere in $(a,b) \setminus \tilde{I}$, and hence we may replace $L^1((a,b))$ with $L^1(\tilde{I})$ in the inequality above, thus getting

$$v(s) - \inf_{(a,b)} v \leqslant \|v'\|_{L^1(\tilde{I})}.$$

An application of the Cauchy-Schwarz inequality on the set $\tilde{I}$ yields $\|v'\|_{L^1(\tilde{I})} \leqslant \|1\|_{L^2(\tilde{I})} \|v'\|_{L^2(\tilde{I})} = |\tilde{I}|^{1/2} \|v'\|_{L^2(\tilde{I})}$. This and the preceding inequality yield the pointwise estimate

$$(v(s) - v_0)^2 \leqslant |\tilde{I}| \|v'\|^2_{L^2(\tilde{I})},$$

and (2.1) follows by integration on $\tilde{I}$. $\square$

Corollary 2.2 (Poincaré-type inequality on a simple C^1-curve)**.** *Let Γ be the trace of a function $\boldsymbol{r} = \boldsymbol{r}(s)$ belonging to the class $C^1([0,L],\mathbb{R}^d)$, $L \in (0,\infty)$, $d \geqslant 2$, and satisfying $|\boldsymbol{r}'(s)| = 1$ in $[0,L]$. Assume that Γ is a simple curve satisfying Assumption* 1.1 (3)*. Since each $w \in H^1(\Gamma)$ has a continuous representative, we may define*

$$w_0 = \min_\Gamma w, \quad w_1 = \max_\Gamma w, \quad \widetilde{\Gamma} = \{\mathbf{x} \in \Gamma;\ w_0 < w(\mathbf{x}) < w_1\}.$$

We claim that

$$\int_{\widetilde{\Gamma}} (w(\boldsymbol{r}(s)) - w_0)^2 \,\mathrm{d}s \leqslant |\widetilde{\Gamma}|^2 \int_{\widetilde{\Gamma}} |\nabla_{\boldsymbol{\tau}} w(\boldsymbol{r}(s))|^2 \,\mathrm{d}s, \tag{2.2}$$

where $|\widetilde{\Gamma}|$ denotes the Hausdorff measure of $\widetilde{\Gamma}$.

P r o o f. Let $(a,b) = (0,L)$ and $v(s) = w(\boldsymbol{r}(s))$. Since $|\boldsymbol{r}'(s)| = 1$, the Hausdorff measure (or total length) of $\widetilde{\Gamma}$ equals the Lebesgue measure of the set $\tilde{I}$ in Lemma 2.1. Furthermore, by the definition of the intrinsic gradient $\nabla_{\boldsymbol{\tau}} w$ (see Remark 1.4) we have $|\nabla_{\boldsymbol{\tau}} w(s)| = |v'(s)|$, and the conclusion is accomplished through inequality (2.1). $\square$

In order to define an appropriate functional setting where facing problem (1.1), it is convenient to introduce the following vector spaces on a domain Ω satisfying Assumption 1.1:

$$V = \{v \in H^1(\Omega);\ v_{|\Gamma_{\boldsymbol{\nu}}} \in H^1(\Gamma_{\boldsymbol{\nu}})\},$$

and

$$V_0 = \{v \in H^1(\Omega);\ v_{|\Gamma_{\boldsymbol{\nu}}} \in H^1_0(\Gamma_{\boldsymbol{\nu}}),\ v_{|\Gamma_D} = 0\}, \tag{2.3}$$

both endowed with the norm

$$\|v\|_V^2 := \int_\Omega |Du|^2 \,\mathrm{d}\mathbf{x} + \int_\Omega u^2 \,\mathrm{d}\mathbf{x} + \int_{\Gamma_{\boldsymbol{\nu}}} |\nabla_{\boldsymbol{\tau}} u|^2 \,\mathrm{d}s + \int_{\Gamma_{\boldsymbol{\nu}}} u^2 \,\mathrm{d}s, \tag{2.4}$$

where $Du = Du(x,y) = (u_x(x,y), u_y(x,y))$ is the gradient of u. We have:

Lemma 2.3 (Poincaré inequality for V_0)**.** *Let Ω and $\Gamma_{\boldsymbol{\nu}}$ be as in Assumption* 1.1, *and let V_0 be given by* (2.3)*. There exists a constant $L > 0$ such that the inequality $\|v\|_{L^2(\Gamma_{\boldsymbol{\nu}})} \leqslant L\|Dv\|_{L^2(\Omega)}$ holds for every $v \in V_0$.*

P r o o f. By the usual Poincaré inequality (see [19], Theorem 7.91), there exists a constant C_P such that $\|v\|_{H^1(\Omega)} \leqslant C_P\|Dv\|_{L^2(\Omega)}$. Furthermore, by the trace inequality [19], Theorem 7.82 we also have $\|v\|_{L^2(\Gamma_{\boldsymbol{\nu}})} \leqslant C\|v\|_{H^1(\Omega)}$ for a convenient C, and the conclusion follows. $\square$

R e m a r k 2.4. Some care is needed when dealing with the Poincaré inequality in the space V_0; in particular, it is not to be expected that $\|w\|_{L^2(\Gamma_{\boldsymbol{\nu}})} \leqslant C\|\nabla_{\boldsymbol{\tau}} w\|_{L^2(\Gamma_{\boldsymbol{\nu}})}$ for a constant C and for all $w \in V_0$. To construct a counterexample, let us consider the annulus $\Omega = B_{R_1}(0) \setminus \overline{B}_{R_0}(0)$ with $\Gamma_{\boldsymbol{\nu}} = \partial B_{R_1}(0)$ and $\Gamma_D = \partial B_{R_0}(0)$ as in Example 1.2. The radial function $w(\mathbf{x}) = (|\mathbf{x}| - R_0)/(R_1 - R_0)$ belongs to V_0 and satisfies $\|w\|_{L^2(\Gamma_{\boldsymbol{\nu}})} = |\Gamma_1|^{1/2}$ as well as $\|\nabla_{\boldsymbol{\tau}} w\|_{L^2(\Gamma_{\boldsymbol{\nu}})} = 0$. Hence, a constant C as above cannot exist.

3. Definition of weak solutions

As a consequence of Lemma 2.3, and by means of the two-dimensional Poincaré inequality, the norm on V_0 (recall the definition in (2.3))

$$\|v\|_{V_0}^2 := \int_\Omega |Du|^2 \,\mathrm{d}\mathbf{x} + \int_{\Gamma_{\boldsymbol{\nu}}} |\nabla_{\boldsymbol{\tau}} u|^2 \,\mathrm{d}s \tag{3.1}$$

is equivalent to (2.4), and standard procedures show that $(V_0, \|\cdot\|_{V_0})$ is a Hilbert space (see, for instance, [3], [6], [16]). On the other hand, we also need that the Dirichlet datum $\varphi\colon \Gamma_D \to \mathbb{R}$ in problem (1.1) has a lifting $\widetilde{\varphi} \in V$, i.e., we need that φ is the trace on Γ_D of some $\widetilde{\varphi} \in V$. Before proceeding further, we give a sufficient condition for φ to satisfy such a requirement:

Lemma 3.1 (Existence of a lifting). *Let Assumption* 1.1 *be complied. If the boundary data $\varphi\colon \Gamma_D \to \mathbb{R}$ satisfy a uniform Lipschitz condition, then there exists $\widetilde{\varphi} \in V$ such that $\varphi = \widetilde{\varphi}|_{\Gamma_D}$.*

P r o o f. By McShane's lemma [15], Theorem 1, there exists a uniformly Lipschitz function $\widetilde{\varphi}\colon \mathbb{R}^2 \to \mathbb{R}$ coinciding with φ on Γ_D. In particular, the restriction $\widetilde{\varphi}|_{\Gamma_{\boldsymbol{\nu}}}$ belongs to the Sobolev space $W^{1,\infty}(\Gamma_{\boldsymbol{\nu}}) \subset H^1(\Gamma_{\boldsymbol{\nu}})$, and therefore $\widetilde{\varphi} \in V$. $\square$

3.1. The inhomogeneous problem. Following [8], (10), p. 314 and [11], (8.3), we allow f and g be given by $f = f_1 + \operatorname{div} \mathbf{f}_2$ and $g = g_1 + \nabla_{\boldsymbol{\tau}} g_2$. Such equalities are intended in the weak sense: namely, every choice of $f_1 \in L^2(\Omega, \mathbb{R})$, $\mathbf{f}_2 \in L^2(\Omega, \mathbb{R}^2)$ and $g_1, g_2 \in L^2(\Gamma_{\boldsymbol{\nu}})$ gives rise to two linear, continuous operators L_f, L_g acting on the function space V_0 through

$$L_f\colon\, w \mapsto \int_\Omega (f_1 w - \mathbf{f}_2 \cdot Dw) \,\mathrm{d}\mathbf{x}, \quad L_g\colon\, w \mapsto \int_{\Gamma_{\boldsymbol{\nu}}} (g_1 w - g_2 \nabla_{\boldsymbol{\tau}} w) \,\mathrm{d}s.$$

For shortness, we let $\mathcal{L}w$ be the sum of the two operators above:

$$\mathcal{L}w = L_f + L_g.$$

If the boundary datum $\varphi\colon\ \Gamma_D \to \mathbb{R}$ has a lifting $\widetilde{\varphi} \in V$, we may define the function space

$$V_\varphi = \{u \in H^1(\Omega);\ u - \widetilde{\varphi} \in V_0\}$$

and give the definition of weak solution of problem (1.1):

Definition 3.2 (Weak solution of the inhomogeneous problem). Let Assumption 1.1 hold and let $f_1 \in L^2(\Omega, \mathbb{R})$, $\mathbf{f}_2 \in L^2(\Omega, \mathbb{R}^2)$, $g_1, g_2 \in L^2(\Gamma_{\boldsymbol{\nu}})$, $a_2 \in W^{1,\infty}(\Gamma_{\boldsymbol{\nu}})$ and $a_0 \in L^\infty(\Gamma_{\boldsymbol{\nu}})$. Suppose that $\varphi\colon\ \Gamma_D \to \mathbb{R}$ has a lifting $\widetilde{\varphi} \in V$. A weak solution of problem (1.1) is a function $u \in V_\varphi$ such that for every $w \in V_0$ the following equality holds:

$$\int_\Omega Du \cdot Dw \,\mathrm{d}\mathbf{x} + \int_{\Gamma_{\boldsymbol{\nu}}} ((a_2 \nabla_{\boldsymbol{\tau}} w + w \nabla_{\boldsymbol{\tau}} a_2) \nabla_{\boldsymbol{\tau}} u + a_0 u w) \,\mathrm{d}s = \mathcal{L}w. \tag{3.2}$$

The definition is motivated by the following property:

Proposition 3.3. *Let Assumption* 1.1 *hold and* $f \in C^0(\Omega)$, $g \in C^0(\Gamma_{\boldsymbol{\nu}})$, $a_2 \in C^1(\Gamma_{\boldsymbol{\nu}})$, $a_0 \in C^0(\Gamma_{\boldsymbol{\nu}})$ *and* $\varphi \in C^1(\Gamma_D)$. *Any function* $u \in C^2(\overline{\Omega})$ *is a weak solution of problem* (1.1) *if and only if the three equalities in* (1.1) *hold pointwise.*

P r o o f. *Step 1.* Let us prepare an identity to be used afterwards. By the product rule, for every $w \in V_0$ we have $\nabla_{\boldsymbol{\tau}}((a_2 w)\nabla_{\boldsymbol{\tau}} u) = \nabla_{\boldsymbol{\tau}}(a_2 w)\nabla_{\boldsymbol{\tau}} u + a_2 w \Delta_{\boldsymbol{\tau}} u$ in the weak sense on Γ_i, $i = 1, \dots, N$, as well as $\nabla_{\boldsymbol{\tau}}(a_2 w) = a_2 \nabla_{\boldsymbol{\tau}} w + w \nabla_{\boldsymbol{\tau}} a_2$; hence

$$\begin{aligned} \int_{\Gamma_i} (a_2 \nabla_{\boldsymbol{\tau}} w + w \nabla_{\boldsymbol{\tau}} a_2) \nabla_{\boldsymbol{\tau}} u \,\mathrm{d}s &= \int_{\Gamma_i} \nabla_{\boldsymbol{\tau}}(a_2 w) \nabla_{\boldsymbol{\tau}} u \,\mathrm{d}s \\ &= \int_{\Gamma_i} \nabla_{\boldsymbol{\tau}}((a_2 w) \nabla_{\boldsymbol{\tau}} u) \,\mathrm{d}s - \int_{\Gamma_i} a_2 w \Delta_{\boldsymbol{\tau}} u \,\mathrm{d}s. \end{aligned} \tag{3.3}$$

Furthermore, by the fundamental theorem of calculus we also have

$$\begin{aligned} &\int_{\Gamma_i} \nabla_{\boldsymbol{\tau}}((a_2 w) \nabla_{\boldsymbol{\tau}} u) \,\mathrm{d}s \\ &\qquad = a_2(\boldsymbol{r}_i(L_i)) w(\boldsymbol{r}_i(L_i)) \nabla_{\boldsymbol{\tau}} u(\boldsymbol{r}_i(L_i)) - a_2(\boldsymbol{r}_i(0)) w(\boldsymbol{r}_i(0)) \nabla_{\boldsymbol{\tau}} u(\boldsymbol{r}_i(0)) \end{aligned}$$

for every $i = 1, \dots, N$. If $\boldsymbol{r}_i(0) = \boldsymbol{r}_i(L_i)$, then the right-hand side obviously vanishes. If, instead, $\boldsymbol{r}_i(0) \neq \boldsymbol{r}_i(L_i)$, then $w(\boldsymbol{r}_i(0)) = w(\boldsymbol{r}_i(L_i)) = 0$ because $w \in H_0^1(\Gamma_i)$. Consequently, we arrive at

$$\int_{\Gamma_i} \nabla_{\boldsymbol{\tau}}((a_2 w) \nabla_{\boldsymbol{\tau}} u) \,\mathrm{d}s = 0 \quad \text{for every } i = 1, \dots, N.$$

Plugging this into (3.3) and summing over i leads to the identity

$$\int_{\Gamma_{\boldsymbol{\nu}}} (a_2\nabla_{\boldsymbol{\tau}} w + w\nabla_{\boldsymbol{\tau}} a_2)\nabla_{\boldsymbol{\tau}} u \,\mathrm{d}s = -\int_{\Gamma_{\boldsymbol{\nu}}} a_2 w \Delta_{\boldsymbol{\tau}} u \,\mathrm{d}s. \tag{3.4}$$

Step 2. Suppose that $u \in C^2(\overline{\Omega})$ is a weak solution of problem (1.1). Then by the definition of a weak solution we have $u \in V_\varphi$ and therefore $u = \varphi$ pointwise on Γ_D (cf. [12], Corollary 1.5.1.6). Moreover, since $C_c^1(\Omega) \subset V_0$, we may take any $\psi \in C_c^1(\Omega)$ and let $w = \psi$ in (3.2) obtaining

$$\int_\Omega Du \cdot D\psi \,\mathrm{d}\mathbf{x} = \int_\Omega f\psi \,\mathrm{d}\mathbf{x} \quad \forall\, \psi \in C_c^1(\Omega),$$

which implies

$$-\Delta u = f \text{ in } \Omega \tag{3.5}$$

(see, for instance, [4], Step D, p. 293). In order to prove the last equality in (1.1), we start from an arbitrary Lipschitz function $\eta \in W^{1,\infty}(\Gamma_{\boldsymbol{\nu}}) \cap H_0^1(\Gamma_{\boldsymbol{\nu}})$ and we define $w = \eta$ on $\Gamma_{\boldsymbol{\nu}}$, $w = 0$ on Γ_D. Thus, w is Lipschitz continuous on $\partial\Omega$ and therefore, as mentioned in the proof of Lemma 3.1, it has a Lipschitz continuous lifting to the whole plane $\mathbb{R}^2$. We denote such a lifting again by w, for simplicity, and observe that $w \in V_0$. Multiplying both sides of (3.5) by w and integrating by parts we obtain

$$\int_\Omega Du \cdot Dw \,\mathrm{d}\mathbf{x} - \int_{\Gamma_{\boldsymbol{\nu}}} w u_{\boldsymbol{\nu}} \,\mathrm{d}s = \int_\Omega f w \,\mathrm{d}\mathbf{x}. \tag{3.6}$$

Recall that identity (3.2) holds for all $w \in V_0$ by assumption. By comparing (3.2) with the equality above, we deduce that

$$\int_{\Gamma_{\boldsymbol{\nu}}} (w u_{\boldsymbol{\nu}} + (a_2\nabla_{\boldsymbol{\tau}} w + w\nabla_{\boldsymbol{\tau}} a_2)\nabla_{\boldsymbol{\tau}} u + a_0 u w) \,\mathrm{d}s = \int_{\Gamma_{\boldsymbol{\nu}}} g w \,\mathrm{d}s. \tag{3.7}$$

Since $w = \eta$ on $\Gamma_{\boldsymbol{\nu}}$, this and identity (3.4) yield

$$\int_{\Gamma_{\boldsymbol{\nu}}} (\eta u_{\boldsymbol{\nu}} - a_2 \eta \Delta_{\boldsymbol{\tau}} u + a_0 u \eta) \,\mathrm{d}s = \int_{\Gamma_{\boldsymbol{\nu}}} g\eta \,\mathrm{d}s, \tag{3.8}$$

and the last equality in (1.1) follows from the fundamental lemma of the calculus of variations (see [4], Corollary 4.24).

Step 3. Conversely, assume that a function $u \in C^2(\overline{\Omega})$ satisfies the three equalities in (1.1) pointwise. Then $u \in V_\varphi$ (see again [12], Corollary 1.5.1.6, or also Theorem 9.17 and the subsequent Remark 19 in [4]). To proceed further, we multiply (3.5) by an arbitrary test function $w \in V_0$ and integrate by parts obtaining (3.6). On the other side, multiplying the last equality in (1.1) by the same arbitrary w and integrating over $\Gamma_{\boldsymbol{\nu}}$ we get (3.8). This and identity (3.4) produce (3.7), which added to (3.6) leads to (3.2), and the proof is complete. □

3.2. The homogeneous problem. With the aid of the lifting $\widetilde{\varphi} \in V$ of the boundary datum $\varphi\colon \Gamma_D \to \mathbb{R}$, we may reduce the inhomogeneous problem (1.1) to the homogeneous one

$$\begin{cases} -\Delta v = f_{\widetilde{\varphi}} & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma_D, \\ v_{\boldsymbol{\nu}} - a_2 \Delta_{\boldsymbol{\tau}} v + a_0 v = g_{\widetilde{\varphi}} & \text{on } \Gamma_{\boldsymbol{\nu}}, \end{cases} \tag{3.9}$$

where $f_{\widetilde{\varphi}}, g_{\widetilde{\varphi}}$ denote the linear, continuous operators on V_0, respectively defined as

$$\begin{aligned} f_{\widetilde{\varphi}}\colon\ & w \mapsto L_f(w) - \int_\Omega D\widetilde{\varphi} \cdot Dw \,\mathrm{d}\mathbf{x}, \\ g_{\widetilde{\varphi}}\colon\ & w \mapsto L_g(w) - \int_{\Gamma_{\boldsymbol{\nu}}} (a_0 \widetilde{\varphi} w + (a_2 \nabla_{\boldsymbol{\tau}} w + w \nabla_{\boldsymbol{\tau}} a_2) \nabla_{\boldsymbol{\tau}} \widetilde{\varphi}) \,\mathrm{d}s. \end{aligned}$$

Remark 3.4. Any function $u \in V_{\widetilde{\varphi}}$ is a weak solution of problem (1.1) if and only if the function $v = u - \widetilde{\varphi} \in V_0$ is a weak solution of problem (3.9). For simplicity, in the sequel we will show that problem (1.1) is uniquely solvable by proving the existence and uniqueness of a weak solution of problem (3.9).

4. Proof of Theorem 1.3

In this section we will focus on the analysis concerning the existence and uniqueness of solutions to problem (3.9). To this aim we first define, for any positive $a_2 \in W^{1,\infty}(\Gamma_{\boldsymbol{\nu}})$ away from zero, and for every non-negative $a_0 \in L^\infty(\Gamma_{\boldsymbol{\nu}})$, some values which will be used throughout the main proofs. Precisely, we set

$$\begin{cases} \lambda_2 := \inf\limits_{\Gamma_{\boldsymbol{\nu}}} a_2 > 0, \quad \Lambda_2 := \sup\limits_{\Gamma_{\boldsymbol{\nu}}} a_2, \quad M := \sup\limits_{\Gamma_{\boldsymbol{\nu}}} |\nabla_{\boldsymbol{\tau}} a_2|, \\ \lambda_0 := \inf\limits_{\Gamma_{\boldsymbol{\nu}}} a_0 \geqslant 0, \quad \Lambda_0 := \sup\limits_{\Gamma_{\boldsymbol{\nu}}} a_0. \end{cases} \tag{4.1}$$

4.1. Uniqueness of the solution.

Lemma 4.1 (Uniqueness of the solution of the homogeneous problem)**.** *Let Assumption 1.1 be complied. Suppose that $0 \leqslant a_0 \in L^\infty(\Gamma_{\boldsymbol{\nu}})$, and let $a_2 \in W^{1,\infty}(\Gamma_{\boldsymbol{\nu}})$ satisfy $\inf\limits_{\Gamma_{\boldsymbol{\nu}}} a_2 > 0$. If $v \in V_0$ satisfies*

$$\int_\Omega Dv \cdot Dw \,\mathrm{d}\mathbf{x} + \int_{\Gamma_{\boldsymbol{\nu}}} ((a_2 \nabla_{\boldsymbol{\tau}} w + w \nabla_{\boldsymbol{\tau}} a_2) \nabla_{\boldsymbol{\tau}} v + a_0 v w) \,\mathrm{d}s = 0 \tag{4.2}$$

for every $w \in V_0$, then v vanishes almost everywhere in Ω.

P r o o f. Before proceeding further, observe that if $v \in V_0$ fulfills relation (4.2) for every $w \in V_0$, and if furthermore $\nabla_{\boldsymbol{\tau}} v$ vanishes a.e. on $\Gamma_{\boldsymbol{\nu}}$, then

$$\int_\Omega Dv \cdot Dw \,\mathrm{d}\mathbf{x} + \int_{\Gamma_{\boldsymbol{\nu}}} a_0 vw \,\mathrm{d}s = 0.$$

In such a case, choosing $w = v$ and recalling that $a_0 \geqslant 0$ on $\Gamma_{\boldsymbol{\nu}}$, it follows that $v = 0$. Hence, in order to prove the lemma, we consider an arbitrary $v \in V_0$ satisfying (4.2) for every $w \in V_0$, and show that $\nabla_{\boldsymbol{\tau}} v = 0$ a.e. on $\Gamma_{\boldsymbol{\nu}}$.

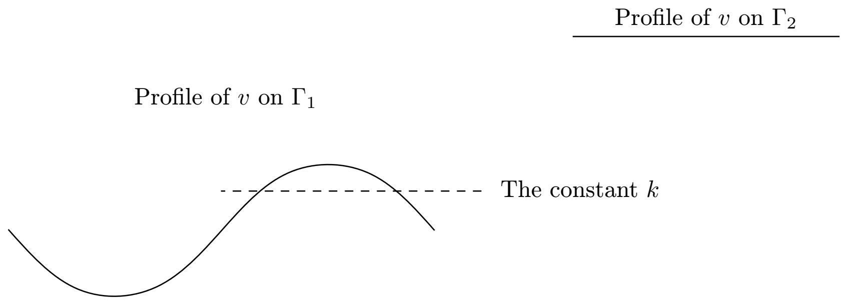

Figure 1. Sample profile of v on the (rectified) curves Γ_1, Γ_2 ($N_1 = 1$).

Suppose, on the contrary, that there exists a positive integer $N_1 \leqslant N$ such that the oscillation ω_i given by

$$\omega_i = \max_{\Gamma_i} v - \min_{\Gamma_i} v$$

is positive if and only if $i \leqslant N_1$ (see Figure 1). Since N_1 is a finite number, there exists a positive ε_0 such that $\omega_i \geqslant \varepsilon_0$ for every $i \leqslant N_1$. Observe that at least one of the following two inequalities must hold:

$$\mu_1 := \max_{1\leqslant i\leqslant N_1} \max_{\Gamma_i} v > 0; \qquad \min_{1\leqslant i\leqslant N_1} \min_{\Gamma_i} v < 0.$$

We consider the first case, the second one being analogous. Furthermore, without loss of generality we assume that μ_1 is attained on Γ_1. By the definition of μ_1 and ε_0, for every $i = 1, \ldots, N_1$ we have

$$\min_{\Gamma_i} v \leqslant \max_{\Gamma_i} v - \varepsilon_0 \leqslant \mu_1 - \varepsilon_0$$

and therefore,

$$\min_{\Gamma_1} v \leqslant \max_{1\leqslant i\leqslant N_1} \min_{\Gamma_i} v =: \mu_0 \leqslant \mu_1 - \varepsilon_0 < \mu_1 = \max_{\Gamma_1} v. \tag{4.3}$$

Arguing as in [11], Theorem 8.1 and in [17], Theorem 3.2.1, let us define $\mu_0^+ = \max\{\mu_0, 0\}$ and choose a real number k in the open interval (μ_0^+, μ_1). Consider the function $w = (v-k)^+$. Since k is positive, we have $w \in V_0$. Apart from a negligible set, the gradient Dw in Ω is given by

$$Dw(\mathbf{x}) = \begin{cases} Dv(\mathbf{x}) & \text{if } v(\mathbf{x}) > k, \\ 0 & \text{if } v(\mathbf{x}) \leqslant k, \end{cases} \tag{4.4}$$

and a similar representation holds for $\nabla_{\boldsymbol{\tau}} w$ on $\Gamma_{\boldsymbol{\nu}}$. In particular, we have

$$\nabla_{\boldsymbol{\tau}} w \nabla_{\boldsymbol{\tau}} v = |\nabla_{\boldsymbol{\tau}} w|^2, \quad w\nabla_{\boldsymbol{\tau}} v = w\nabla_{\boldsymbol{\tau}} w \quad \text{a.e. on } \Gamma_{\boldsymbol{\nu}}.$$

As a consequence of position (4.4), the first integral in (4.2) is non-negative, and hence,

$$\int_{\Gamma_{\boldsymbol{\nu}}} (a_2 \nabla_{\boldsymbol{\tau}} w + w\nabla_{\boldsymbol{\tau}} a_2)\nabla_{\boldsymbol{\tau}} v \,\mathrm{d}s + \int_{\Gamma_{\boldsymbol{\nu}}} a_0 v w \,\mathrm{d}s \leqslant 0. \tag{4.5}$$

Let us focus on the product vw. By definition, the function $w = (v-k)^+$ is either positive or zero. When w vanishes, the product vw obviously vanishes. When w is positive, instead, $v - k$ is also positive and therefore $v > k > 0$. In conclusion, we have $vw \geqslant 0$ on $\Gamma_{\boldsymbol{\nu}}$. Since $a_0 \geqslant 0$, it follows that the last integral in (4.5) is non-negative, and therefore we arrive at

$$\int_{\Gamma_{\boldsymbol{\nu}}} a_2 |\nabla_{\boldsymbol{\tau}} w|^2 \,\mathrm{d}s \leqslant \int_{\Gamma_{\boldsymbol{\nu}}} |\nabla_{\boldsymbol{\tau}} a_2||\nabla_{\boldsymbol{\tau}} w| w \,\mathrm{d}s.$$

Let $C = \max\{1, M/\lambda_2\}$, where M, λ_2 are as in (4.1). In particular, we assume $\lambda_2 > 0$ (ellipticity). With this notation we may write

$$\int_{\Gamma_{\boldsymbol{\nu}}} |\nabla_{\boldsymbol{\tau}} w|^2 \,\mathrm{d}s \leqslant C \int_{\Gamma_{\boldsymbol{\nu}}} |\nabla_{\boldsymbol{\tau}} w| w \,\mathrm{d}s. \tag{4.6}$$

We aim to replace the domain of integration $\Gamma_{\boldsymbol{\nu}}$ by the set $\widetilde{\Gamma} = \bigcup_{i=1}^{N_1} \widetilde{\Gamma}_i$, where

$$\widetilde{\Gamma}_i = \Big\{\mathbf{x} \in \Gamma_i;\ 0 < w(\mathbf{x}) < \max_{\Gamma_i} w\Big\} = \Big\{\mathbf{x} \in \Gamma_i;\ k < v(\mathbf{x}) < \max_{\Gamma_i} v\Big\},\ i = 1, \dots, N_1.$$

By definition, $\widetilde{\Gamma}_i$ is a (possibly empty) relatively open subset of Γ_i. Furthermore, the equality $\|\nabla_{\boldsymbol{\tau}} w\|_{L^2(\widetilde{\Gamma}_i)} = \|\nabla_{\boldsymbol{\tau}} w\|_{L^2(\Gamma_i)}$ holds, because $\nabla_{\boldsymbol{\tau}} w$ vanishes almost everywhere in $\Gamma_i \setminus \widetilde{\Gamma}_i$ (see [11], Lemma 7.7). Since $\nabla_{\boldsymbol{\tau}} w = 0$ on Γ_i for $i > N_1$, inequality (4.6) implies

$$\int_{\widetilde{\Gamma}} |\nabla_{\boldsymbol{\tau}} w|^2 \,\mathrm{d}s \leqslant C \int_{\widetilde{\Gamma}} |\nabla_{\boldsymbol{\tau}} w| w \,\mathrm{d}s. \tag{4.7}$$

In order to estimate the last integral, using the Cauchy-Schwarz inequality in $\widetilde{\Gamma}$ we get

$$\int_{\widetilde{\Gamma}} |\nabla_{\boldsymbol{\tau}} w| w \,\mathrm{d}s \leqslant \|\nabla_{\boldsymbol{\tau}} w\|_{L^2(\widetilde{\Gamma})} \|w\|_{L^2(\widetilde{\Gamma})}.$$

By (4.3), $\nabla_{\boldsymbol{\tau}} w$ cannot vanish identically in $\widetilde{\Gamma}_1$, hence $\|\nabla_{\boldsymbol{\tau}} w\|_{L^2(\widetilde{\Gamma})} > 0$. Thus, the inequality above and (4.7) imply

$$\|\nabla_{\boldsymbol{\tau}} w\|_{L^2(\widetilde{\Gamma})} \leqslant C\|w\|_{L^2(\widetilde{\Gamma})}. \tag{4.8}$$

Let us estimate the last term. For every $i = 1, \dots, N_1$ we have $k > \mu_0^+ \geqslant \min_{\Gamma_i} v$, and consequently $\min_{\Gamma_i} w = 0$. Hence, by using relation (2.2) we obtain $\|w\|^2_{L^2(\widetilde{\Gamma}_i)} \leqslant |\widetilde{\Gamma}_i|^2 \|\nabla_{\boldsymbol{\tau}} w\|^2_{L^2(\widetilde{\Gamma}_i)}$, and by summation over $i = 1, \dots, N_1$

$$\|w\|_{L^2(\widetilde{\Gamma})} \leqslant |\widetilde{\Gamma}| \|\nabla_{\boldsymbol{\tau}} w\|_{L^2(\widetilde{\Gamma})}.$$

This and (4.8) imply $C|\widetilde{\Gamma}| \geqslant 1$. In contrast, we now check that $|\widetilde{\Gamma}| \searrow 0$ as $k \nearrow \mu_1$. To see this, it suffices to observe that when k increases, the corresponding set $\widetilde{\Gamma} = \widetilde{\Gamma}(k)$ describes a decreasing family of open subsets of $\Gamma_{\boldsymbol{\nu}}$ with finite Hausdorff measure, and by the continuity of the measure we have

$$\lim_{k \nearrow \mu_1} |\widetilde{\Gamma}(k)| = \Bigg| \bigcap_{k \in (\mu_0^+, \mu_1)} \widetilde{\Gamma}(k) \Bigg| = |\emptyset| = 0,$$

which is a contradiction. Thereafter, the unique function $v \in V_0$ satisfying (4.2) for every $w \in V_0$ is the null function. $\square$

4.2. Existence of a solution. Let us now turn our attention to the question concerning the existence of solutions to the homogeneous problem (3.9). To this aim, we have to recall that the composite Sobolev space V_0 introduced in (2.3) was endowed with the norm (3.1). We start with the following lemma.

Lemma 4.2. *Let Assumption* 1.1 *be complied. Suppose that $a_0 \in L^\infty(\Gamma_{\boldsymbol{\nu}})$ and let $a_2 \in W^{1,\infty}(\Gamma_{\boldsymbol{\nu}})$ satisfy $\inf_{\Gamma_{\boldsymbol{\nu}}} a_2 > 0$. For V_0 as in* (2.3) *let us consider the bilinear form $\mathfrak{B}\colon V_0 \times V_0 \to \mathbb{R}$ given by*

$$\mathfrak{B}(v, w) = \int_\Omega Dv \cdot Dw \,\mathrm{d}\mathbf{x} + \int_{\Gamma_{\boldsymbol{\nu}}} a_2 \nabla_{\boldsymbol{\tau}} v \nabla_{\boldsymbol{\tau}} w \,\mathrm{d}s + \int_{\Gamma_{\boldsymbol{\nu}}} (\nabla_{\boldsymbol{\tau}} a_2 \nabla_{\boldsymbol{\tau}} v) w \,\mathrm{d}s + \int_{\Gamma_{\boldsymbol{\nu}}} a_0 v w \,\mathrm{d}s.$$

Then for λ_0, λ_2 and M defined in (4.1), *the bilinear form*

$$\mathfrak{B}_{\sigma_0}(v, w) := \begin{cases} \mathfrak{B}(v, w) + \sigma_0 \displaystyle\int_{\Gamma_{\boldsymbol{\nu}}} v w \,\mathrm{d}s & \text{if } \sigma_0 > 0, \\ \mathfrak{B}(v, w) & \text{if } \sigma_0 \leqslant 0, \end{cases} \quad \text{where} \quad \sigma_0 = \frac{M^2}{2\lambda_2} - \lambda_0,$$

is continuous and coercive.

P r o o f. *Part 1.* Continuity. From the definition of $\mathfrak{B}$, we directly have by means of the Cauchy-Schwarz inequality and (4.1)

$$
\begin{aligned}
(4.9)\qquad |\mathfrak{B}(v,w)| &\leqslant \int_\Omega |Dv\cdot Dw|\,\mathrm{d}\mathbf{x} + \Lambda_2\int_{\Gamma_{\boldsymbol\nu}} |\nabla_{\boldsymbol\tau} v||\nabla_{\boldsymbol\tau} w|\,\mathrm{d}s\\
&\quad + M\int_{\Gamma_{\boldsymbol\nu}} |\nabla_{\boldsymbol\tau} v||w|\,\mathrm{d}s + \Lambda_0\int_{\Gamma_{\boldsymbol\nu}} |v||w|\,\mathrm{d}s\\
&\leqslant \|Dv\|_{L^2(\Omega)}\|Dw\|_{L^2(\Omega)} + \Lambda_2\|\nabla_{\boldsymbol\tau} v\|_{L^2(\Gamma_{\boldsymbol\nu})}\|\nabla_{\boldsymbol\tau} w\|_{L^2(\Gamma_{\boldsymbol\nu})}\\
&\quad + M\|\nabla_{\boldsymbol\tau} v\|_{L^2(\Gamma_{\boldsymbol\nu})}\|w\|_{L^2(\Gamma_{\boldsymbol\nu})} + \Lambda_0\|v\|_{L^2(\Gamma_{\boldsymbol\nu})}\|w\|_{L^2(\Gamma_{\boldsymbol\nu})}.
\end{aligned}
$$

On the other hand, applying Lemma 2.3 leads to $\|v\|_{L^2(\Gamma_{\boldsymbol\nu})} \leqslant L\|Dv\|_{L^2(\Omega)}$ and $\|w\|_{L^2(\Gamma_{\boldsymbol\nu})} \leqslant L\|Dw\|_{L^2(\Omega)}$. These inequalities in conjunction with (4.9) yield for a positive k

$$|\mathfrak{B}(v,w)| \leqslant k\|v\|_{V_0}\|w\|_{V_0}.$$

Further, for all $v,w\in V_0$ we similarly have

$$\int_{\Gamma_{\boldsymbol\nu}} vw\,\mathrm{d}s \leqslant \|v\|_{L^2(\Gamma_{\boldsymbol\nu})}\|w\|_{L^2(\Gamma_{\boldsymbol\nu})}.$$

As an immediate consequence of this, by invoking again Lemma 2.3, as well as recalling definition (3.1), it holds that

$$|\mathfrak{B}_{\sigma_0}(v,w)| \leqslant |\mathfrak{B}(v,w)| + \Bigl|\sigma_0\int_{\Gamma_{\boldsymbol\nu}} vw\,\mathrm{d}s\Bigr| \leqslant L_1\|v\|_{V_0}\|w\|_{V_0}\quad \forall\, v,w\in V_0,$$

where L_1 is a positive constant. Hence, $\mathfrak{B}_{\sigma_0}$ is continuous.

Part 2. Coercivity. Again in light of positions (4.1) we directly have

$$
\begin{aligned}
(4.10)\qquad &\int_{\Gamma_{\boldsymbol\nu}} a_2|\nabla_{\boldsymbol\tau} v|^2\,\mathrm{d}s + \int_{\Gamma_{\boldsymbol\nu}} (\nabla_{\boldsymbol\tau} a_2\nabla_{\boldsymbol\tau} v)v\,\mathrm{d}s + \int_{\Gamma_{\boldsymbol\nu}} a_0v^2\,\mathrm{d}s\\
&\geqslant \lambda_2\int_{\Gamma_{\boldsymbol\nu}} |\nabla_{\boldsymbol\tau} v|^2\,\mathrm{d}s - M\int_{\Gamma_{\boldsymbol\nu}} |\nabla_{\boldsymbol\tau} v||v|\,\mathrm{d}s + \lambda_0\int_{\Gamma_{\boldsymbol\nu}} v^2\,\mathrm{d}s.
\end{aligned}
$$

Now if $M=0$, then $\sigma_0\leqslant 0$ and therefore,

$$\mathfrak{B}_{\sigma_0}(v,v) = \mathfrak{B}(v,v) \geqslant \int_\Omega |Dv|^2\,\mathrm{d}\mathbf{x} + \lambda_2\int_{\Gamma_{\boldsymbol\nu}} |\nabla_{\boldsymbol\tau} v|^2\,\mathrm{d}s \geqslant L_2\|v\|^2_{V_0},$$

where $L_2=\min\{1,\lambda_2\}>0$, hence $\mathfrak{B}_{\sigma_0}$ is coercive. If, instead, $M>0$, then we let $\varepsilon=\lambda_2/(2M)$ in the Young inequality and obtain $|\nabla_{\boldsymbol\tau} v||v| \leqslant \varepsilon|\nabla_{\boldsymbol\tau} v|^2 + v^2/(4\varepsilon)$. By plugging such an estimate into (4.10), we arrive at

$$\mathfrak{B}(v,v) \geqslant \int_\Omega |Dv|^2\,\mathrm{d}\mathbf{x} + \frac{\lambda_2}{2}\int_{\Gamma_{\boldsymbol\nu}} |\nabla_{\boldsymbol\tau} v|^2\,\mathrm{d}s - \Bigl(\frac{M^2}{2\lambda_2} - \lambda_0\Bigr)\int_{\Gamma_{\boldsymbol\nu}} v^2\,\mathrm{d}s$$

and for $L_3 = \min\{1, \lambda_2/2\} > 0$ we may write

$$\mathfrak{B}_{\sigma_0}(v, v) \geqslant \int_\Omega |Dv|^2 \,\mathrm{d}\mathbf{x} + \frac{\lambda_2}{2} \int_{\Gamma_{\boldsymbol{\nu}}} |\nabla_{\boldsymbol{\tau}} v|^2 \,\mathrm{d}s \geqslant L_3 \|v\|^2_{V_0}.$$

Hence, $\mathfrak{B}_{\sigma_0}$ is coercive also in this case, and the proof is complete. $\square$

Lemma 4.3 (Unique solvability of the homogeneous problem)**.** *Let Assumption* 1.1 *be complied. Suppose* $0 \leqslant a_0 \in L^\infty(\Gamma_{\boldsymbol{\nu}})$ *and let* $a_2 \in W^{1,\infty}(\Gamma_{\boldsymbol{\nu}})$ *satisfy* $\inf\limits_{\Gamma_{\boldsymbol{\nu}}} a_2 > 0$. *Then problem* (3.9) *admits a unique weak solution.*

P r o o f. We rely on the properties of the bilinear form $\mathfrak{B}_{\sigma_0}(v, w)$ introduced in Lemma 4.2. For $\sigma_0 \leqslant 0$, equation (3.2) may be rewritten as $\mathfrak{B}_{\sigma_0}(v, w) = \mathcal{L}w$, and the existence and uniqueness of the solution v follow immediately from the Lax-Milgram theorem. If, instead, $\sigma_0 > 0$, then we invoke the Fredholm alternative [11], Theorem 5.3 (see also [8], Section D.5). To be precise, solving equation (3.2) is equivalent to finding $v \in V_0$ such that

$$\mathfrak{B}_{\sigma_0}(v, w) = \mathcal{L}w + \sigma_0 \int_{\Gamma_{\boldsymbol{\nu}}} vw \,\mathrm{d}s \quad \forall\, w \in V_0. \tag{4.11}$$

By the Lax-Milgram theorem we may define the linear, continuous operator $T\colon V_0^* \to V_0$ given by $T(L) = v_L$, where v_L is the unique solution of the equation $\mathfrak{B}_{\sigma_0}(v, w) = Lw$ in the unknown v. For every $v \in V_0$ we also define the linear, continuous operator $I_v \in V_0^*$ given by

$$I_v w = \int_{\Gamma_{\boldsymbol{\nu}}} vw \,\mathrm{d}s.$$

Consequently, relation (4.11) can be rewritten as

$$v = T(\mathcal{L}) + \sigma_0 T(I_v).$$

The existence and uniqueness of a solution v follow from the Fredholm alternative provided we ensure that

(i) the homogeneous equation $v = \sigma_0 T(I_v)$ has only the trivial solution;
(ii) the operator $K\colon\ v \mapsto T(I_v)$ is compact.

Condition (i) was already established in Lemma 4.1. To check (ii), take a bounded sequence (v_n) in V_0. By Lemma 2.3, the traces $v_{n|\Gamma_{\boldsymbol{\nu}}}$ are uniformly bounded in $H^1_0(\Gamma_{\boldsymbol{\nu}})$, hence there exists a subsequence of functions still denoted by v_n, whose traces on $\Gamma_{\boldsymbol{\nu}}$ converge to some v_∞ in $L^2(\Gamma_{\boldsymbol{\nu}})$. Henceforth, the operators I_{v_n} converge to I_{v_∞} in the norm of the dual space V_0^*. Since the operator T is continuous, the sequence of functions $K(v_n) = T(I_{v_n})$ converges in V_0. In conclusion, K is a compact operator, and the lemma follows from the Fredholm theorem. $\square$

As a consequence of all the above preparations, we finally can prove our main statement:

P r o o f of Theorem 1.3. The assumption on φ implies by means of Lemma 3.1 the existence of a lifting $\widetilde{\varphi} \in V$. Hence, the considerations presented in Remark 3.4 convert problem (1.1) into a homogeneous problem having the form (3.9). Finally, Lemma 4.3 ensures its unique solvability. □

5. Equilibrium of a prestressed membrane

As announced in Section 1, let us attend to the interplay between *Equilibrium Problem* 1 in [21], Section 3.1 and problem (1.1), exactly by presenting the formulation of the former in terms of the nomenclature employed in the present paper. We consider a plane domain Ω satisfying Assumption 1.1, and we restrict ourselves to the case when $\Gamma_{\boldsymbol{\nu}}$ is a curve parametrized by a vector-valued function $\boldsymbol{r}(t) = (x(t), y(t))$ belonging to $C^2((t_0,t_1)) \cap C^1([t_0,t_1])$ and satisfying $\boldsymbol{r}'(t) \neq \mathbf{0}$ for all t in a bounded interval $[t_0,t_1]$ on the real line, with $t_0 < t_1$.

Expression of the Laplace-Beltrami operator. Let us consider a function $u \in C^2(\Omega \cup \Gamma_{\boldsymbol{\nu}})$, and recall that the arc length along the curve $\Gamma_{\boldsymbol{\nu}}$ is given by

$$s(t) = \int_{t_0}^{t} \sqrt{(x'(\xi))^2 + (y'(\xi))^2}\,\mathrm{d}\xi.$$

Since $\boldsymbol{r}'(t) \neq \mathbf{0}$, the function $s(t)$ admits a smooth inverse $t = t(s)$, so yields (recall Remark 1.4) the following representation of the Laplace-Beltrami operator of u along the one-dimensional manifold $\Gamma_{\boldsymbol{\nu}}$:

$$\Delta_{\boldsymbol{\tau}} u = u_{ss} = \frac{\mathrm{d}^2}{\mathrm{d}s^2} u(x(t(s)), y(t(s))). \tag{5.1}$$

It is essential for our purposes to express $\Delta_{\boldsymbol{\tau}} u$ in terms of the partial derivatives of u with respect to the Cartesian coordinates x, y, the outward derivative $u_{\boldsymbol{\nu}}$, and the curvature κ of the curve $\Gamma_{\boldsymbol{\nu}}$, given by

$$\kappa(t) = \frac{x'(t)y''(t) - y'(t)x''(t)}{((x'(t))^2 + (y'(t))^2)^{3/2}}, \quad t \in (t_0, t_1). \tag{5.2}$$

To this purpose, we will make use of the obvious identity

$$\frac{\mathrm{d}}{\mathrm{d}t} = \sqrt{(x'(t))^2 + (y'(t))^2}\,\frac{\mathrm{d}}{\mathrm{d}s}. \tag{5.3}$$

Furthermore, for every point $(x,y) = (x(t),y(t)) \in \Gamma_{\boldsymbol{\nu}}$, and by introducing the tangent unit vector to $\Gamma_{\boldsymbol{\nu}}$

$$\boldsymbol{\tau} := \boldsymbol{\tau}(t) = \frac{(x'(t),y'(t))}{\sqrt{(x'(t))^2+(y'(t))^2}}, \quad t \in (t_0,t_1), \tag{5.4}$$

we let

$$\begin{aligned} u_{\boldsymbol{\tau}} &= \boldsymbol{\tau}\cdot Du(x(t),y(t)) \\ &= \frac{x'(t)u_x(x(t),y(t)) + y'(t)u_y(x(t),y(t))}{\sqrt{(x'(t))^2+(y'(t))^2}}, \quad t \in (t_0,t_1), \end{aligned} \tag{5.5}$$

$$\begin{aligned} u_{\boldsymbol{\tau\tau}} &= \boldsymbol{\tau}\cdot D^2u(x(t),y(t))\boldsymbol{\tau}^\top \\ &= \frac{(x'(t))^2u_{xx}(x(t),y(t)) + 2x'(t)y'(t)u_{xy}(x(t),y(t))}{(x'(t))^2+(y'(t))^2} \\ &\quad + \frac{(y'(t))^2u_{yy}(x(t),y(t))}{(x'(t))^2+(y'(t))^2}, \quad t \in (t_0,t_1), \end{aligned} \tag{5.6}$$

where $Du(x,y) = (u_x(x,y),u_y(x,y))$ is the gradient of u, $D^2u(x,y)$ the Hessian matrix and $\boldsymbol{\tau}^\top$ the transpose of $\boldsymbol{\tau}$. The following lemma shows the relation between u_{ss}, $u_{\boldsymbol{\tau\tau}}$ and the outward derivative of u on $\Gamma_{\boldsymbol{\nu}}$, explicitly given by

$$u_{\boldsymbol{\nu}} = \frac{x'(t)u_y(x(t),y(t)) - y'(t)u_x(x(t),y(t))}{\sqrt{(x'(t))^2+(y'(t))^2}}, \quad t \in (t_0,t_1). \tag{5.7}$$

Lemma 5.1. *If $u \in C^2(\Omega\cup\Gamma_{\boldsymbol{\nu}})$, then $u_{ss} = u_{\boldsymbol{\tau\tau}} + \kappa(t)u_{\boldsymbol{\nu}}$ on $\Gamma_{\boldsymbol{\nu}}$.*

Proof. By computing the derivative on the right-hand side of the equality $u_s = \frac{\mathrm{d}}{\mathrm{d}s}(u(x(t(s)),y(t(s)))$ and taking (5.3) into account, we find for all $t \in (t_0,t_1)$

$$\sqrt{(x'(t))^2+(y'(t))^2}\,u_s = x'(t)u_x(x(t),y(t)) + y'(t)u_y(x(t),y(t)), \tag{5.8}$$

which in particular shows due to (5.5) that $u_s = u_{\boldsymbol{\tau}}$. Differentiating both sides of (5.8) with respect to t yields

$$\begin{aligned} &\frac{x'(t)x''(t)+y'(t)y''(t)}{\sqrt{(x'(t))^2+(y'(t))^2}}u_s + ((x'(t))^2+(y'(t))^2)u_{ss} \\ &\quad = (x'(t))^2u_{xx}(x(t),y(t)) + 2x'(t)y'(t)u_{xy}(x(t),y(t)) + (y'(t))^2u_{yy}(x(t),y(t)) \\ &\qquad + x''(t)u_x(x(t),y(t)) + y''(t)u_y(x(t),y(t)), \quad t \in (t_0,t_1). \end{aligned}$$

By dividing both sides by $(x'(t))^2+(y'(t))^2$ and using relation (5.6), the last equality becomes, on (t_0,t_1),

$$\frac{x'(t)x''(t)+y'(t)y''(t)}{((x'(t))^2+(y'(t))^2)^{3/2}}u_s + u_{ss} = u_{\boldsymbol{\tau\tau}} + \frac{x''(t)u_x(x(t),y(t)) + y''(t)u_y(x(t),y(t))}{(x'(t))^2+(y'(t))^2}.$$

Finally, taking (5.2) and (5.8) into consideration, we arrive at

$$u_{ss} = u_{\boldsymbol{\tau}\boldsymbol{\tau}} + \kappa(t)\frac{x'(t)u_y(x(t),y(t)) - y'(t)u_x(x(t),y(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}}, \quad t \in (t_0, t_1),$$

and the lemma follows thanks to (5.7). □

Setting the domain. The definition of a convex plane curve is quite standard: see, for instance, [7], p. 37. In view of the example below we assume strict convexity of the curve $\Gamma_{\boldsymbol{\nu}}$, in the sense that $\kappa(t) > 0$ for all $t \in (t_0, t_1)$. We also suppose

$$x(t_0) < 0, \quad x(t_1) > 0 \quad \text{and} \quad x'(t), y(t) > 0 \quad \forall\, t \in (t_0, t_1).$$

By setting

$$\text{(5.9)} \qquad \begin{cases} \Gamma_{\boldsymbol{\nu}} = \{(x(t), y(t)) \in \mathbb{R}^2;\ t \in (t_0, t_1)\}, \\ \Gamma_D = \{(x(t_0), t) \in \mathbb{R}^2;\ 0 \leqslant t \leqslant y(t_0)\} \\ \qquad\quad \cup\{(t, 0)) \in \mathbb{R}^2;\ t \in (x(t_0), x(t_1))\} \\ \qquad\quad \cup\{(x(t_1), t) \in \mathbb{R}^2;\ 0 \leqslant t \leqslant y(t_1)\}, \end{cases}$$

we consider the planar, non-empty, open and bounded domain Ω, uniquely determined by the condition $\partial\Omega = \Gamma_{\boldsymbol{\nu}} \cup \Gamma_D$. In these circumstances, the equilibrium of a prestressed membrane obeys an elliptic equation in Ω for the unknown $u\colon \Omega \to \mathbb{R}$ (whose graph represents the shape of the membrane) endowed with a classical Dirichlet condition on Γ_D (rigid boundaries with their own stiffness), which basically fixes the shape of the membrane on the portion Γ_D, and a Ventcel-type one on $\Gamma_{\boldsymbol{\nu}}$ (non-rigid boundaries, *without any stiffness*), idealizing the physical equilibrium for cable elements.

The mixed boundary-value problem. As discussed in [21], Section 2.2.2, the introduction of a cable boundary as a structural element providing equilibrium to a membrane requires a restriction on its shape, which is modeled by the convexity assumption $\kappa(t) > 0$, and the so-called *cable-membrane compatibility equation* $u_{\boldsymbol{\tau}\boldsymbol{\tau}} = 0$ along $\Gamma_{\boldsymbol{\nu}}$ (see [21], relation (5) for a particular parametrization of $\Gamma_{\boldsymbol{\nu}}$). Recalling Lemma 5.1 and relation (5.1), we have

$$u_{\boldsymbol{\tau}\boldsymbol{\tau}} = 0 \Leftrightarrow u_{\boldsymbol{\nu}} - \frac{1}{\kappa(x)}\Delta_{\boldsymbol{\tau}} u = 0.$$

Now consider the most simplified case of *Equilibrium Problem* 1 in [21], Section 3.1, which is obtained when $\boldsymbol{\sigma} = c\mathbf{1}$. The problem corresponds to the equilibrium of a membrane tensioned with the same stress $c > 0$ in the two main orthogonal directions and not exposed to external loads. The comments above make such a problem read as follows: fixed a sufficiently regular function $\varphi\colon \Gamma_D \to \mathbb{R}$, find u such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \Gamma_D, \\ u_{\boldsymbol{\nu}} - \dfrac{1}{\kappa(x)} \Delta_{\boldsymbol{\tau}} u = 0 & \text{on } \Gamma_{\boldsymbol{\nu}}. \end{cases}$$

This is manifestly a special case of problem (1.1), which is well-posed by virtue of Theorem 1.3.

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References

[1] *D. E. Apushkinskaya, A. I. Nazarov*: A survey of results on nonlinear Venttsel problems. Appl. Math., Praha *45* (2010), 69–80. zbl MR doi

[2] *D. E. Apushkinskaya, A. I. Nazarov, D. K. Palagachev, L. G. Softova*: Venttsel boundary value problems with discontinuous data. SIAM J. Math. Anal. *53* (2021), 221–252. zbl MR doi

[3] *V. Bonnaillie-Noël, M. Dambrine, F. Hérau, G. Vial*: On generalized Ventcel's type boundary conditions for Laplace operator in a bounded domain. SIAM J. Math. Anal. *42* (2010), 931–945. zbl MR doi

[4] *H. Brezis*: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Universitext. Springer, New York, 2011. zbl MR doi

[5] *L. Caffarelli, L. Silvestre*: An extension problem related to the fractional Laplacian. Commun. Partial Differ. Equations *32* (2007), 1245–1260. zbl MR doi

[6] *S. Creo, M. R. Lancia, A. Nazarov, P. Vernole*: On two-dimensional nonlocal Venttsel' problems in piecewise smooth domains. Discrete Contin. Dyn. Syst., Ser. S. *12* (2019), 57–64. zbl MR doi

[7] *M. P. do Carmo*: Differential Geometry of Curves and Surfaces. Prentice-Hall, Englewood Cliffs, 1976. zbl MR

[8] *L. C. Evans*: Partial Differential Equations. Graduate Studies in Mathematics 19. American Mathematical Society, Providence, 2010. zbl MR doi

[9] *A. Favini, G. Ruiz Goldstein, J. A. Goldstein, S. Romanelli*: The heat equation with generalized Wentzell boundary condition. J. Evol. Equ. *2* (2002), 1–19. zbl MR doi

[10] *A. Favini, G. Ruiz Goldstein, J. A. Goldstein, S. Romanelli*: The heat equation with nonlinear general Wentzell boundary condition. Adv. Differ. Equ. *11* (2006), 481–510. zbl MR

[11] *D. Gilbarg, N. S. Trudinger*: Elliptic Partial Differential Equations of Second Order. Grundlehren der Mathematischen Wissenschaften 224. Springer, Berlin, 1983. zbl MR doi

[12] *P. Grisvard*: Elliptic Problems in Nonsmooth Domains. Monographs and Studies in Mathematics 24. Pitman, Boston, 1985. zbl MR doi

[13] *T. Kashiwabara, C. M. Colciago, L. Dedè, A. Quarteroni*: Well-posedness, regularity, and convergence analysis of the finite element approximation of a generalized Robin boundary value problem. SIAM J. Numer. Anal. *53* (2015), 105–126. zbl MR doi

[14] *Y. Luo, N. S. Trudinger*: Quasilinear second order elliptic equations with Venttsel boundary conditions. Potential Anal. *3* (1994), 219–243. zbl MR doi

[15] *E. J. McShane*: Extension of range of functions. Bull. Am. Math. Soc. *40* (1934), 837–842. zbl MR doi
[16] *S. Nicaise, H. Li, A. Mazzucato*: Regularity and a priori error analysis of a Ventcel problem in polyhedral domains. Math. Methods Appl. Sci. *40* (2017), 1625–1636. zbl MR doi
[17] *P. Pucci, J. Serrin*: The Maximum Principle. Progress in Nonlinear Differential Equations and Their Applications 73. Birkhäuser, Basel, 2007. zbl MR doi
[18] *A. Quarteroni, A. Valli*: Domain Decomposition Methods for Partial Differential Equations. Numerical Mathematics and Scientific Computation. Clarendon Press, New York, 1999. zbl MR
[19] *S. Salsa*: Partial Differential Equations in Action: From Modelling to Theory. Unitext 99. Springer, Cham, 2016. zbl MR doi
[20] *A. D. Venttsel'*: On boundary conditions for multidimensional diffusion processes. Theor. Probab. Appl. *4* (1959), 164–177. zbl MR doi
[21] *G. Viglialoro, Á. González, J. Murcia*: A mixed finite-element finite-difference method to solve the equilibrium equations of a prestressed membrane having boundary cables. Int. J. Comput. Math. *94* (2017), 933–945. zbl MR doi

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