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GLOBAL REGULARITY FOR THE 3D INHOMOGENEOUS INCOMPRESSIBLE NAVIER-STOKES EQUATIONS WITH DAMPING

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Abstract. This paper is concerned with the 3D inhomogeneous incompressible Navier-Stokes equations with damping. We find a range of parameters to guarantee the existence of global strong solutions of the Cauchy problem for large initial velocity and external force as well as prove the uniqueness of the strong solutions. This is an extension of the theorem for the existence and uniqueness of the 3D incompressible Navier-Stokes equations with damping to inhomogeneous viscous incompressible fluids.

Keywords: inhomogeneous incompressible fluid; Navier-Stokes equations; damping; global regularity

MSC 2020: 35Q30, 76D03, 76D05

1. Introduction

In this paper, we study the following inhomogeneous incompressible Navier-Stokes equations with damping:

$$
(1.1)\qquad \begin{cases} \varrho u_t + (\varrho u\cdot\nabla)u - \nu\Delta u + \alpha|u|^{\beta-1}u + \nabla p = \varrho f, & (x,t)\in\Omega\times(0,\infty),\\ \operatorname{div} u = 0, & (x,t)\in\Omega\times(0,\infty),\\ \varrho_t + u\cdot\nabla\varrho = 0, & (x,t)\in\Omega\times(0,\infty),\\ u|_{\partial\Omega} = 0, & \\ u|_{t=0} = u_0, & \varrho|_{t=0} = \varrho_0, \end{cases}
$$

where $\nu > 0$, $\alpha > 0$ and $\beta \geqslant 1$ are constants. Unknown functions $\varrho(x,t)$, $u(x,t)$ and $p(x,t)$ are the density, velocity, and pressure of the fluid, respectively, and $f = f(x,t)$ is the given external force. $\Omega \subseteq \mathbb{R}^3$ is the whole space or a bounded domain with sufficiently smooth boundary. (When $\Omega = \mathbb{R}^3$, the boundary condition of (1.1) is replaced by $\lim\limits_{|x|\to\infty} |u(x)| = 0, t\in(0,\infty)$.)

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There has been a lot of literature about the regularity theory for the inhomogeneous incompressible Navier-Stokes equations.

When $\alpha = 0$ in (1.1), the global existence of weak solutions of (1.1) has been proved ([14], [17] and others) and it has been well known that there exists a unique strong solution of (1.1) in the case when the initial velocity is small enough (see [3], [6], [13], [20]). Also, global well-posedness for (1.1) with $\alpha = 0$ in critical function spaces was studied under the smallness assumption for u_0 (see [1], [2], [5], [7], [8], [10], [16], [18]).

In the case of $\alpha > 0$, problem (1.1) describes the flow with the resistance to the motion such as porous media flow and drag or friction effects (see [4] and references therein). The term $\alpha|u|^{\beta-1}u$ reflects a resistance to flow of the fluid. From a mathematical viewpoint, (1.1) can be viewed as a modification of the Navier-Stokes equations with the regularizing term $\alpha|u|^{\beta-1}u$.

For problem (1.1) with $\alpha > 0$ and $\varrho \equiv \text{const.}$, it is notable that the uniqueness of weak solutions and the global existence of strong solutions are guaranteed under certain restrictions on α and β, without any smallness assumption for u_0 ([4], [11], [12], [15], [19], [21]).

However, problem (1.1) has not been studied in the case when $\alpha > 0$ and ϱ is not a constant.

In order to extend the regularity results for the Navier-Stokes equations with damping, we study the existence and uniqueness of global strong solutions of (1.1).

The symbols $\|\cdot\|_X$, $\|\cdot\|_q$, $\|\cdot\|_{q,s;T}$ and $\|\cdot\|_{X,s;T}$ denote the norms of the Banach space X, $L^q(\Omega)^n$, $L^s(0,T;L^q(\Omega)^n)$ and $L^s(0,T;X)$, respectively, and $\|\cdot\| = \|\cdot\|_2$, $Q_T := \Omega \times (0,T)$. We define the following function spaces:

$$\begin{aligned} C^\infty_{0,\sigma}(\Omega) &:= \{u \in C^\infty_0(\Omega)^3, \operatorname{div} u = 0\}, \\ V &:= \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{W^{1,2}(\Omega)^3}}, \\ H &:= \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_2}. \end{aligned}$$

Let $P\colon L^2(\Omega)^3 \to H$ be the Helmholtz projection, $A := -P\Delta$ be the Stokes operator with definition domain $D(A) = W^{2,2}(\Omega)^3 \cap W^{1,2}_0(\Omega)^3 \cap H$ and $A^{1/2}$ be the square root of A.

Definition 1.1. Let $\Omega \subseteq \mathbb{R}^3$ be the whole space or a bounded domain with sufficiently smooth boundary and $0 < T < \infty$. Suppose that

$$u_0 \in V \cap L^{\beta+1}(\Omega)^3, \quad \varrho_0 \in W^{1,6}(\Omega) \cap L^3(\Omega), \quad f \in L^1(0,T;L^2(\Omega)^3).$$

A pair of functions (u,ϱ) is called a strong solution of (1.1) on $[0,T]$ if it satisfies

$$u \in L^2(0,T;D(A)) \cap L^\infty(0,T;V \cap L^{\beta+1}(\Omega)^3) \cap L^1(0,T;W^{1,\infty}(\Omega)^3),$$
$$\varrho \in L^\infty(0,T;W^{1,6}(\Omega) \cap L^3(\Omega)),$$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(\varrho u, v) + (\varrho u \otimes u, \nabla v) + \nu(\nabla u, \nabla v) & \\ \qquad\qquad + (\alpha |u|^{\beta-1} u, v) = (\varrho f, v), & v \in C_{0,\sigma}^\infty(\Omega),\ t \in (0,T), \\ (\varrho u(0), v) = (\varrho_0 u_0, v), & v \in C_{0,\sigma}^\infty(\Omega) \end{cases} \tag{1.2}$$

and

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(\varrho, \eta) = (\varrho u, \nabla \eta), & \eta \in C_0^\infty(\Omega),\ t \in (0,T), \\ (\varrho(0), \eta) = (\varrho_0, \eta), & \eta \in C_0^\infty(\Omega). \end{cases} \tag{1.3}$$

Now we state our main result as follows.

Theorem 1.1 (Existence of strong solutions)**.** *Let $\Omega = \mathbb{R}^3$ and $0 < T < \infty$. Suppose that*

$$u_0 \in V, \quad \varrho_0 \in W^{1,6}(\Omega) \cap L^3(\Omega), \quad 0 < \underline{\varrho} \leqslant \varrho_0 \leqslant \overline{\varrho} < \infty,$$
$$f \in W^{1,2}(0,T;L^2(\Omega)^3) \cap L^2(0,T;W^{1,2}(\Omega)^3).$$

Also, assume that $5 > \beta > 3$, $\alpha > 0$, $\nu > 0$ or $\beta = 3$, $2\alpha > \overline{\varrho}$, $\nu > \overline{\varrho}$, $2\nu\alpha > \nu\overline{\varrho} + \overline{\varrho}^2$. Then problem (1.1) has a strong solution (u,ϱ) on $[0,T]$ satisfying

$$u_t \in L^2(0,T;H), \quad \sqrt{t}u_t \in L^\infty(0,T;H), \quad \sqrt{t}A^{1/2}u_t \in L^2(0,T;H). \tag{1.4}$$

Theorem 1.2 (Uniqueness of strong solutions)**.** *Let $\Omega \subseteq \mathbb{R}^3$ be the whole space or a bounded domain with sufficiently smooth boundary and $0 < T < \infty$, $\beta \geqslant 1$, $\alpha > 0$, $\nu > 0$. Suppose that*

$$u_0 \in V \cap L^{\beta+1}(\Omega)^3, \quad \varrho_0 \in W^{1,6}(\Omega) \cap L^3(\Omega), \quad f \in L^2(0,T;W^{1,2}(\Omega)^3).$$

Let $(u^{(1)},\varrho^{(1)})$ and $(u^{(2)},\varrho^{(2)})$ be the strong solutions of problem (1.1) satisfying

$$u_t^{(i)} \in L^2(0,T;H), \quad i = 1,2.$$

Then $u^{(1)} = u^{(2)}$, $\varrho^{(1)} = \varrho^{(2)}$.

2. Proof of Theorem 1.1

To prove Theorem 1.1, we show the local existence of strong solutions of (1.1) when Ω is a bounded domain and $3 \leqslant \beta < 5$. (The constant C in the following inequalities is the constant which depends only on α, β, ν, $\underline{\varrho}$ and $\overline{\varrho}$, and $C(c_1, c_2, \ldots, c_n)$ denotes the constant which depends only on α, β, ν, $\underline{\varrho}$, $\overline{\varrho}$ and $c_1, c_2, \ldots, c_n$.)

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with sufficiently smooth boundary and $5 > \beta \geqslant 3$, $\alpha > 0$, $\nu > 0$. Suppose that*

$$u_0 \in V, \quad \varrho_0 \in W^{1,6}(\Omega) \cap L^3(\Omega), \quad 0 < \underline{\varrho} \leqslant \varrho_0 \leqslant \overline{\varrho} < \infty,$$
$$f \in W^{1,2}(0,1; L^2(\Omega)^3) \cap L^2(0,1; W^{1,2}(\Omega)^3).$$

Then there exists an interval $[0,T]$ $(0 < T < 1)$ depending only on α, β, ν, u_0, ϱ_0, f, and the strong solution (u, ϱ) of (1.1) on $[0,T]$ satisfying (1.4).

We use Galerkin's method with the eigenfunctions of the Stokes operator A to prove Theorem 2.1. It is well known that A has denumerable number of positive eigenvalues $\{\lambda_n\}$ clustering at infinity and the corresponding eigenfunctions $\{w_n\}$ form an orthonormal basis in H when Ω is a bounded domain with sufficiently smooth boundary. Let

$$W_m := \operatorname{span}\{w_1, w_2, \ldots, w_m\}, \quad P_m v := \sum_{j=1}^m w_j \int_\Omega w_j v \, \mathrm{d}x, \quad v \in H,$$

and consider the approximations

$$u_m(x,t) := \sum_{j=1}^m g_{jm}(t) w_j, \quad \varrho_m(x,t) := \varrho_0(\Lambda_m(x,t)), \quad (x,t) \in \Omega \times [0,1],$$

satisfying

$$\text{(2.1)} \qquad \begin{cases} (\varrho_m u_{mt}, v) + \nu(\nabla u_m, \nabla v) + \alpha(|u_m|^{\beta-1} u_m, v) \\ \qquad = -((\varrho_m u_m \cdot \nabla) u_m, v) + (\varrho_m f, v) \quad \forall\, v \in W_m, \\ u_m(0) = P_m u_0, \end{cases}$$

$$\text{(2.2)} \qquad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\tau} \Lambda_m(x,\tau) = -u_m(\Lambda_m(x,\tau), t-\tau), \quad 0 < \tau \leqslant t, \\ \Lambda_m(x,0) = x. \end{cases}$$

In fact, (2.1) is an ordinary differential equation with respect to g_{jm} and it has a solution such that $g_{jm} \in C^1[0,1]$.

We have a priori estimates for u_m in the following lemmas.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with sufficiently smooth boundary and α, β, ν, u_0, ϱ_0, f satisfy the same assumptions as in Theorem* 1.1. *Then u_m is estimated by*

$$\|u_m\|_{2,\infty;T} + \|\nabla u_m\|_{2,2;T} + \|u_m\|_{\beta+1,\beta+1;T} \leqslant C(u_0, \varrho_0, f, T) \tag{2.3}$$

for any $0 < T < 1$.

P r o o f. Putting $v = u_m$ in (2.1), we have

$$\begin{aligned}(\varrho_m u_{mt}, u_m) + \nu(\nabla u_m, \nabla u_m) + \alpha(|u_m|^{\beta-1}u_m, u_m) \\ = -((\varrho_m u_m \cdot \nabla)u_m, u_m) + (\varrho_m f, u_m).\end{aligned} \tag{2.4}$$

Simple calculation using $\operatorname{div} u_m = 0$ implies

$$\begin{aligned}((\varrho_m u_m \cdot \nabla)u_m, u_m) &= \int_\Omega \varrho_m u_{mi}\partial_i u_{mj} u_{mj}\,\mathrm{d}x \\ &= -((u_m \cdot \nabla\varrho_m)u_m, u_m) - ((\varrho_m u_m \cdot \nabla)u_m, u_m) \\ &= -\frac{1}{2}((u_m \cdot \nabla\varrho_m)u_m, u_m).\end{aligned} \tag{2.5}$$

Also, $\varrho_{mt} + u_m \cdot \nabla\varrho_m = 0$ leads to

$$\begin{aligned}(\varrho_m u_{mt}, u_m) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\varrho_m u_m, u_m) - \frac{1}{2}(\varrho_{mt} u_m, u_m) \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho_m^{1/2}u_m(t)\|^2 + \frac{1}{2}((u_m \cdot \nabla\varrho_m)u_m, u_m).\end{aligned} \tag{2.6}$$

Combining (2.4), (2.5) and (2.6), we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho_m^{1/2}u_m(t)\|^2 + \nu\|\nabla u_m\|^2 + \alpha\|u_m\|_{\beta+1}^{\beta+1} \\ &\quad\leqslant (\varrho_m f, u_m) \leqslant \|\varrho_m^{1/2}u_m\|\|\varrho_m^{1/2}\|_\infty\|f\| \leqslant \|\varrho_m^{1/2}u_m\|^2 + \frac{1}{4}\|\varrho_0\|_\infty\|f\|^2.\end{aligned} \tag{2.7}$$

By applying Gronwall's lemma to (2.7),

$$\|\varrho_m^{1/2}u_m\|_{2,\infty;T} + \|\nabla u_m\|_{2,2;T} + \|u_m\|_{\beta+1,\beta+1;T} \leqslant C(u_0, \varrho_0, f, T) \tag{2.8}$$

is satisfied for any $0 < T < 1$. The desired inequality (2.3) is obtained from (2.8) and $0 < \underline{\varrho} \leqslant \varrho_m$. □

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with sufficiently smooth boundary and α, β, ν, u_0, ϱ_0, f satisfy the same assumptions as in Theorem* 1.1. *Then there exists $0 < T < 1$ depending only on α, β, ν, u_0, ϱ_0, f such that*

$$\|\nabla u_m\|_{2,\infty;T} + \|u_m\|_{\beta+1,\infty;T} + \|u_{mt}\|_{2,2;T} + \|Au_m\|_{2,2;T} \leqslant C(u_0, \varrho_0, f, T). \tag{2.9}$$

Proof. Putting $v = u_{mt}$ in (2.1), we have

$$\begin{aligned} \|\varrho_m^{1/2} u_{mt}\|^2 + \frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\nabla u_m\|^2 + \alpha(|u_m|^{\beta-1}u_m, u_{mt}) \\ = -(\varrho_m(u_m\cdot\nabla)u_m, u_{mt}) + (\varrho_m f, u_{mt}). \end{aligned} \tag{2.10}$$

Use the relations

$$\begin{aligned} \alpha(|u_m|^{\beta-1}u_m, u_{mt}) &= \frac{\alpha}{\beta+1}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m\|_{\beta+1}^{\beta+1}, \\ -(\varrho_m(u_m\cdot\nabla)u_m, u_{mt}) &\leqslant \frac{1}{4}\|\varrho_m^{1/2}u_{mt}\|^2 + \overline{\varrho}\||u_m||\nabla u_m|\|^2, \\ (\varrho_m f, u_{mt}) &\leqslant \frac{1}{4}\|\varrho_m^{1/2}u_{mt}\|^2 + \overline{\varrho}\|f\|^2 \end{aligned}$$

to get

$$\frac{1}{2}\|\varrho_m^{1/2}u_{mt}\|^2 + \frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\nabla u_m\|^2 + \frac{\alpha}{\beta+1}\|u_m\|_{\beta+1}^{\beta+1} \leqslant \overline{\varrho}\||u_m||\nabla u_m|\|^2 + \overline{\varrho}\|f\|^2 \tag{2.11}$$

from (2.10). Next, putting $v = Au_m$ in (2.1) leads to

$$\begin{aligned} (\varrho_m u_{mt}, Au_m) + \nu\|Au_m\|^2 + \alpha(|u_m|^{\beta-1}u_m, Au_m) \\ = -(\varrho_m(u_m\cdot\nabla)u_m, Au_m) + (\varrho_m f, Au_m). \end{aligned} \tag{2.12}$$

The inequalities

$$\begin{aligned} -(\varrho_m u_{mt}, Au_m) &\leqslant \frac{\nu}{4}\|Au_m\|^2 + \frac{\overline{\varrho}}{\nu}\|\varrho_m^{1/2}u_{mt}\|^2, \\ -(\varrho_m(u_m\cdot\nabla)u_m, Au_m) &\leqslant \frac{\nu}{4}\|Au_m\|^2 + \frac{\overline{\varrho}^2}{\nu}\||u_m||\nabla u_m|\|^2, \\ -\alpha(|u_m|^{\beta-1}u_m, Au_m) &\leqslant \frac{\nu}{8}\|Au_m\|^2 + \frac{2\alpha^2}{\nu}\|u_m\|_{2\beta}^{2\beta}, \\ -(\varrho_m f, Au_m) &\leqslant \frac{\nu}{4}\|Au_m\|^2 + \frac{\overline{\varrho}^2}{\nu}\|f\|^2 \end{aligned}$$

and (2.12) imply

$$\frac{\nu}{8}\|Au_m\|^2 \leqslant \frac{\overline{\varrho}}{\nu}\|\varrho_m^{1/2}u_{mt}\|^2 + C(\||u_m||\nabla u_m|\|^2 + \|u_m\|_{2\beta}^{2\beta} + \|f\|^2). \tag{2.13}$$

Multiplying (2.13) by $\nu/(4\overline{\varrho})$ and adding it to (2.11) we have

$$\frac{\nu^2}{32\overline{\varrho}}\|Au_m\|^2+\frac{1}{4}\|\varrho_m^{1/2}u_{mt}\|^2+\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_m(t)\|^2+\frac{\alpha}{\beta+1}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)\|_{\beta+1}^{\beta+1} \leqslant C(\||u_m||\nabla u_m|\|^2+\|u_m\|_{2\beta}^{2\beta}+\|f\|^2). \tag{2.14}$$

By the Gagliardo-Nirenberg inequality $\|\nabla u_m\|_3\leqslant C\|\nabla^2u_m\|^{1/2}\|\nabla u_m\|^{1/2}$,

$$\||u_m||\nabla u_m|\|^2\leqslant\|u_m\|_6^2\|\nabla u_m\|_3^2\leqslant C\|\nabla^2u_m\|\|\nabla u_m\|^3 \tag{2.15}$$

is satisfied. Inserting (2.15) and the Gagliardo-Nirenberg inequality

$$\|u_m\|_{2\beta}^{2\beta}\leqslant C\|\nabla^2u_m\|^{\beta-3}\|\nabla u_m\|^{\beta+3}$$

to (2.14) implies

$$\begin{aligned}\frac{\nu^2}{32\overline{\varrho}}\|Au_m\|^2+\frac{1}{4}\|\varrho_m^{1/2}u_{mt}\|^2+\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_m(t)\|^2+\frac{\alpha}{\beta+1}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)\|_{\beta+1}^{\beta+1}&\\ \leqslant C(\|\nabla^2u_m\|\|\nabla u_m\|^3+\|\nabla^2u_m\|^{\beta-3}\|\nabla u_m\|^{\beta+3}+\|f\|^2)&\\ \leqslant C((\|Au_m\|+\|\nabla u_m\|)\|\nabla u_m\|&\\ +(\|Au_m\|+\|\nabla u_m\|)^{\beta-3}\|\nabla u_m\|^{\beta+3}+\|f\|^2)&\\ \leqslant\frac{\nu^2}{64\overline{\varrho}}\|Au_m\|^2+C(\|\nabla u_m\|^{2(\beta+3)/(5-\beta)}+\|f\|^2+1),&\end{aligned} \tag{2.16}$$

where we have used the inequality $\|\nabla^2u_m\|\leqslant C(\|Au_m\|+\|\nabla u_m\|)$ in Lemma 1 of [9].

By comparison, (2.16) shows that there exists some $0<T<1$ satisfying

$$\|\nabla u_m\|_{2,\infty;T}+\|u_m\|_{\beta+1,\infty;T}+\|\varrho_m^{1/2}u_{mt}\|_{2,2;T}+\|Au_m\|_{2,2;T}\leqslant C(u_0,\varrho_0,f,T), \tag{2.17}$$

which proves (2.9) considering $0<\underline{\varrho}\leqslant\varrho_m$. □

Lemma 2.3. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with sufficiently smooth boundary and α, β, ν, u_0, ϱ_0, f satisfy the same assumptions as in Theorem* 1.1. *Then*

$$\|\sqrt{t}u_{mt}\|_{2,\infty;T}+\|\sqrt{t}A^{1/2}u_{mt}\|_{2,2;T}\leqslant C(u_0,\varrho_0,f,T) \tag{2.18}$$

holds for $0<T<1$, satisfying (2.9).

P r o o f. Differentiating (2.1) with respect to t and putting $v = u_{mt}$, we get

$$(2.19)\qquad \begin{aligned}((\varrho_m u_{mt})_t, u_{mt}) + \alpha((|u_m|^{\beta-1}u_m)_t, u_{mt}) - \nu(\Delta u_{mt}, u_{mt})\\ = -(((\varrho_m u_m\cdot)u_m)_t, u_{mt}) + ((\varrho_m f)_t, u_{mt})\\ = -((\varrho_{mt} u_m\cdot\nabla)u_m, u_{mt}) - ((\varrho_m u_{mt}\cdot\nabla)u_m, u_{mt})\\ -((\varrho_m u_m\cdot\nabla)u_{mt}, u_{mt}) + ((\varrho_m f)_t, u_{mt}).\end{aligned}$$

Considering the estimates

$$(2.20)\qquad \begin{aligned}((\varrho_m u_{mt})_t, u_{mt}) &= (\varrho_m u_{mtt}, u_{mt}) + (\varrho_{mt}u_{mt}, u_{mt})\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho_m^{1/2}u_{mt}\|^2 + \frac{1}{2}(\varrho_{mt}u_{mt}, u_{mt})\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho_m^{1/2}u_{mt}\|^2 - \frac{1}{2}(u_m\cdot\nabla\varrho_m u_{mt}, u_{mt})\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho_m^{1/2}u_{mt}\|^2 + ((\varrho_m u_m\cdot\nabla)u_{mt}, u_{mt}),\end{aligned}$$

$$(2.21)\qquad \begin{aligned}\alpha((|u_m|^{\beta-1}u_m)_t, u_{mt}) &= \alpha\||u_m|^{(\beta-1)/2}u_{mt}\|^2\\ &\quad + \frac{\alpha(\beta-1)}{4}\int_\Omega |u_m|^{\beta-3}\Big|\frac{\partial}{\partial t}|u_m|^2\Big|^2\,\mathrm{d}x\\ &\geqslant \alpha\||u_m|^{(\beta-1)/2}u_{mt}\|^2,\end{aligned}$$

equation (2.19) becomes

$$(2.22)\qquad \begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho_m^{1/2}u_{mt}(t)\|^2 + \alpha\||u_m|^{(\beta-1)/2}u_{mt}\|^2 + \nu\|\nabla u_{mt}\|^2\\ \leqslant -((\varrho_{mt}u_m\cdot\nabla)u_m, u_{mt}) - ((\varrho_m u_{mt}\cdot\nabla)u_m, u_{mt}\\ -2((\varrho_m u_m\cdot\nabla)u_{mt}, u_{mt}) + ((\varrho_m f)_t, u_{mt}).\end{aligned}$$

Now, estimate the right-hand-side terms of (2.22):

$$(2.23)\qquad \begin{aligned}&-((\varrho_{mt}u_m\cdot\nabla)u_m, u_{mt})\\ &= (((u_m\cdot\nabla\varrho_m)u_m\cdot\nabla)u_m, u_{mt})\\ &\leqslant \int_\Omega \varrho_m|u_m||u_{mt}||\nabla u_m|^2\,\mathrm{d}x + \int_\Omega \varrho_m|u_m|^2|\nabla u_{mt}||\nabla u_m|\,\mathrm{d}x\\ &\quad + \int_\Omega \varrho_m|u_m|^2|u_{mt}||\nabla^2 u_m|\,\mathrm{d}x\\ &\leqslant C\|\varrho_m\|_\infty(\|u_m\|_6\|u_{mt}\|_6\|\nabla u_m\|_3^2\\ &\quad + \|u_m\|_\infty^2\|\nabla u_{mt}\|\|\nabla u_m\| + \|u_m\|_6^2\|u_{mt}\|_6\|\nabla^2 u_m\|)\\ &\leqslant C\|\nabla u_{mt}\|\|\nabla u_m\|^2\|\nabla^2 u_m\| \leqslant \frac{\nu}{8}\|\nabla u_{mt}\|^2 + C\|\nabla^2 u_m\|^2\|\nabla u_m\|^4,\end{aligned}$$

$$(2.24)\qquad \begin{aligned}-((\varrho_m u_{mt}\cdot\nabla)u_m, u_{mt}) &\leqslant \|\varrho_m\|_\infty^{1/2}\|\varrho_m^{1/2}u_{mt}\|\|\nabla u_m\|_3\|u_{mt}\|_6\\ &\leqslant \frac{\nu}{8}\|\nabla u_{mt}\|^2 + C\|\varrho_m^{1/2}u_{mt}\|^2\|\nabla^2 u_m\|\|\nabla u_m\|,\end{aligned}$$

$$(2.25)\quad -2((\varrho_m u_m\cdot u)u_{mt},u_{mt})\leqslant 2\|\varrho_m\|_\infty^{1/2}\|\nabla u_{mt}\|\|\varrho_m^{1/2}u_{mt}\|\|u_m\|_\infty \leqslant \frac{\nu}{8}\|\nabla u_{mt}\|^2+C\|\varrho_m^{1/2}u_{mt}\|^2\|\nabla^2 u_m\|\|\nabla u_m\|,$$

$$\begin{aligned}(2.26)\quad ((\varrho_m f)_t,u_{mt}) &= (\varrho_{mt}f,u_{mt})+(\varrho_m f_t,u_{mt})\\ &= -((u_m\cdot\nabla\varrho_m)f,u_{mt})+(\varrho_m f_t,u_{mt})\\ &= ((\varrho_m u_m\cdot\nabla)f,u_{mt})+((\varrho_m u_m\cdot\nabla)u_{mt},f)+(\varrho_m f_t,u_{mt})\\ &\leqslant \|\varrho_m\|_\infty^{1/2}\|\varrho_m^{1/2}u_{mt}\|\|u_m\|_\infty\|\nabla f\|+\|\varrho_m\|_\infty\|\nabla u_{mt}\|\|f\|\\ &\quad+\|\varrho_m\|_\infty^{1/2}\|f_t\|\|\varrho_m^{1/2}u_{mt}\|\\ &\leqslant \frac{\nu}{8}\|\nabla u_{mt}\|^2+C(\|\varrho_m^{1/2}u_{mt}\|^2\|\nabla^2u_m\|\|\nabla u_m\|+\|\varrho_m^{1/2}u_{mt}\|^2\\ &\quad+\|\nabla^2u_m\|\|\nabla u_m\|\|f\|^2+\|\nabla f\|^2+\|f_t\|^2),\end{aligned}$$

where we have used the inequalities

$$\|u_m\|_\infty\leqslant C\|\nabla^2u_m\|^{1/2}\|\nabla u_m\|^{1/2},\quad \|\nabla u_m\|_3\leqslant C\|\nabla^2u_m\|^{1/2}\|\nabla u_m\|^{1/2}.$$

Inserting (2.23)–(2.26) to (2.22), we have

$$\begin{aligned}(2.27)\qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho_m^{1/2}u_{mt}(t)\|^2+\frac{\nu}{4}\|\nabla u_{mt}\|^2+\alpha\||u_m|^{(\beta-1)/2}u_{mt}\|^2 &\leqslant C(\|\varrho_m^{1/2}u_{mt}\|^2\|\nabla^2u_m\|\|\nabla u_m\|+\|\varrho_m^{1/2}u_{mt}\|^2\\ &\quad+\|\nabla^2u_m\|\|\nabla u_m\|\|f\|^2+\|\nabla f\|^2+\|f_t\|^2)\end{aligned}$$

and multiplying (2.27) by t, we get

$$\begin{aligned}(2.28)\qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{t}\varrho_m^{1/2}u_{mt}(t)\|^2+\frac{\nu}{4}\|\sqrt{t}\nabla u_{mt}\|^2+\alpha\|\sqrt{t}|u_m|^{(\beta-1)/2}u_{mt}\|^2 &\leqslant C\|\sqrt{t}\varrho_m^{1/2}u_{mt}\|^2\|\nabla^2u_m\|\|\nabla u_m\|\\ &\quad+Ct(\|\varrho_m^{1/2}u_{mt}\|^2+\|\nabla^2u_m\|\|\nabla u_m\|\|f\|^2+\|\nabla f\|^2+\|f_t\|^2).\end{aligned}$$

Apply Gronwall's lemma to (2.28), considering (2.9), the inequality $\|\nabla^2u_m\|\leqslant C(\|Au_m\|+\|\nabla u_m\|)$ in Lemma 1 of [9] and the restriction on f, then (2.18) holds for $0<T<1$, satisfying (2.9). □

P r o o f of Theorem 2.1. Let $0<T<1$ satisfy (2.9). Then using the boundedness of $\{u_m\}$ from Lemma 2.1 and 2.2, we can prove that $\{u_m\}$ and $\{\varrho_m\}$ converge to some $u\in L^\infty(0,T;V)\cap L^2(0,T;D(A))$ and $\varrho\in L^\infty(0,T;L^\infty(\Omega)\cap L^3(\Omega))$, respectively, that satisfy (1.2) and (1.3). Here we use the fact that $\|\varrho_m\|_{L^\infty(Q_T)}\leqslant\overline{\varrho}$ from (2.2) as well as $\|\varrho_{mt}\|_{W^{-1,\infty}(\Omega),4;T}\leqslant C(u_0,\varrho_0,f,T)$ from the estimate

$$|(\varrho_{mt},\eta)|=|(\varrho_m u_m,\nabla\eta)|\leqslant C\|Au_m\|^{1/2}\|\nabla u_m\|^{1/2}\|\nabla\eta\|_1,\quad \eta\in W_0^{1,1}(\Omega).$$

Furthermore, u_t fulfills

$$u_t \in L^2(0,T;H), \quad \sqrt{t}u_t \in L^\infty(0,T;H), \quad \sqrt{t}A^{1/2}u_t \in L^2(0,T;H)$$

by Lemma 2.3. In order to prove the convergence of the damping term $|u_m|^{\beta-1}u_m$ to $|u|^{\beta-1}u$, we need the strong convergence of u_m, which is obtained by applying the compactness theorem with the boundedness of u_{mt}. Note that the restriction $0 < \underline{\varrho} \leqslant \varrho_0$ is essential to get the boundedness of u_{mt}.

Now, it remains to prove $u \in L^1(0,T;W^{1,\infty}(\Omega)^3)$ and $\nabla\varrho \in L^\infty(0,T;L^6(\Omega))$. Let $3 < p < 6$. From the first equation of (1.1), $\|Au\|_p$ is estimated by

$$\begin{aligned}\|Au\|_p &\leqslant C(\|P(\varrho u_t)\|_p + \|P((\varrho u\cdot\nabla)u)\|_p + \|P(|u|^{\beta-1}u)\|_p + \|P(\varrho f)\|_p) \\ &\leqslant C(\|\varrho u_t\|_p + \|(\varrho u\cdot\nabla)u\|_p + \|u\|_{\beta p}^\beta + \|\varrho f\|_p).\end{aligned} \tag{2.29}$$

Estimate the right-hand-side terms of (2.29) as follows:

$$\begin{aligned}\|\varrho u_t\|_{p,1;T} &\leqslant \overline{\varrho}\int_0^T \|u_t\|_p\,\mathrm{d}t \leqslant C\int_0^T \|u_t\|^{(6-p)/(2p)}\|u_t\|_6^{(3p-6)/(2p)}\,\mathrm{d}t \\ &= C\int_0^T t^{-1/2}(t^{(6-p)/(4p)}\|u_t\|^{(6-p)/(2p)}) \\ &\quad\times (t^{(3p-6)/(4p)}\|\nabla u_t\|^{(3p-6)/(2p)})\,\mathrm{d}t \\ &\leqslant C\|\sqrt{t}u_t\|_{2,\infty;T}^{(6-p)/(2p)} \\ &\quad\times \|\sqrt{t}A^{1/2}u_t\|_{2,2;T}^{(3p-6)/(2p)}\|t^{-1/2}\|_{\infty,4p/(p+6);T} \\ &\leqslant C\|\sqrt{t}u_t\|_{2,\infty;T}^{(6-p)/(2p)}\|\sqrt{t}A^{1/2}u_t\|_{2,2;T}^{(3p-6)/(2p)}T^{(6-p)/(4p)} \\ &< \infty,\end{aligned} \tag{2.30}$$

$$\begin{aligned}\|(\varrho u\cdot\nabla)u\|_{p,1;T} &\leqslant C\overline{\varrho}\int_0^T \|u\|_\infty\|\nabla u\|_p\,\mathrm{d}t \\ &\leqslant C\int_0^T \|\nabla^2 u\|^{(2p-3)/p}\|\nabla u\|^{3/p} + \|\nabla^2 u\|^{1/2}\|\nabla u\|^{3/2}\,\mathrm{d}t \\ &\leqslant C\int_0^T (\|u\|_{D(A)}^2 + 1)(\|u\|_V^{3/2} + 1)\,\mathrm{d}t \\ &\leqslant C(\|u\|_{V,\infty;T}^{3/2} + 1)(\|u\|_{D(A),2;T}^2 + T) \\ &< \infty,\end{aligned} \tag{2.31}$$

$$\begin{aligned}\int_0^T \|u\|_{\beta p}^\beta\,\mathrm{d}t &\leqslant C\int_0^T \|\nabla^2 u\|^{(p\beta-6)/(2p)}\|\nabla u\|^{(p\beta+6)/(2p)}\,\mathrm{d}t \\ &\leqslant C\int_0^T (\|u\|_{D(A)}^2 + 1)(\|u\|_V^4 + 1)\,\mathrm{d}t \\ &\leqslant C(\|u\|_{V,\infty;T}^4 + 1)(\|u\|_{D(A),2;T}^2 + T) \\ &< \infty,\end{aligned} \tag{2.32}$$

where we have used the Gagliardo-Nirenberg inequalities

$$\begin{aligned}(2.33)\qquad \|u\|_\infty &\leqslant C\|\nabla^2 u\|^{1/2}\|\nabla u\|^{1/2},\\ \|\nabla u\|_p &\leqslant C(\|\nabla^2 u\|^{3(p-2)/(2p)}\|\nabla u\|^{(6-p)/(2p)} + \|\nabla u\|),\\ \|u\|_{\beta p}^\beta &\leqslant C\|\nabla^2 u\|^{(p\beta-6)/(2p)}\|\nabla u\|^{(p\beta+6)/(2p)},\end{aligned}$$

$$\int_0^T \|\varrho f\|_p\,\mathrm{d}t \leqslant \overline{\varrho}\int_0^T \|f\|_p\,\mathrm{d}t \leqslant C\|f\|_{W^{1,2}(\Omega)^3,1;T} < \infty.$$

Inequalities (2.30)–(2.33) with (2.29) show that $\|Au\|_{p,1;T} < \infty$, which proves $\|\nabla u\|_{\infty,1;T} < \infty$ with the Gagliardo-Nirenberg inequality

$$\|\nabla u\|_\infty \leqslant C\|Au\|_p^{5p/(7p-6)}\|u\|^{(2p+6)/(7p-6)}.$$

Finally, to prove $\|\nabla\varrho\|_{6,\infty;T} < \infty$, apply the gradient operator to the both sides of $\varrho_t = -u\cdot\nabla\varrho$, multiply by $|\nabla\varrho|^4\nabla\varrho$, and integrate on Ω. We have

$$\begin{aligned}(2.34)\qquad \frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\varrho(t)\|_6^6 &= -(\nabla(u\cdot\nabla\varrho), |\nabla\varrho|^4\nabla\varrho) = -\int_\Omega \partial_k(u_j\partial_j\varrho)|\nabla\varrho|^4\partial_k\varrho\,\mathrm{d}x\\ &= \int_\Omega \partial_k u_j\partial_j\varrho|\nabla\varrho|^4\partial_k\varrho\,\mathrm{d}x - \frac{1}{5}\int_\Omega \operatorname{div} u|\nabla\varrho|^5\,\mathrm{d}x\\ &\leqslant C\int_\Omega |\nabla u||\nabla\varrho|^6\,\mathrm{d}x \leqslant C\|\nabla u\|_\infty\|\nabla\varrho\|_6^6.\end{aligned}$$

Applying Gronwall's lemma to (2.34) considering $\|\nabla u\|_{\infty,1;T} < \infty$, proves $\nabla\varrho \in L^\infty(0,T;L^6(\Omega))$. □

P r o o f of Theorem 1.1. By the local existence of strong solutions of (1.1) satisfying (1.4) in bounded domains (Theorem 2.1), we can prove that there exists a local strong solution (u,ϱ) on $[0,T_1]$ $(0 < T_1 \leqslant T)$ satisfying (1.4) when $\Omega = \mathbb{R}^3$ and $5 > \beta \geqslant 3$, $\alpha > 0$, $\nu > 0$. (Follow the same argument of the proof as for Theorem 2 in [9]). Let

$$T^* := \sup\{T_1 \leqslant T' \leqslant T;\ \text{there exists a strong solution } (u,\varrho) \text{ of (1.1) on } [0,T'],\ \text{satisfying (1.4)}\}.$$

If $T^* = T$, then the existence of strong solutions satisfying (1.4) is proved, so let us assume that $T^* < T$.

Let $T_1 \leqslant T' < T$ be sufficiently close to T and (u,ϱ) be a strong solution of (1.1) on $[0,T']$ satisfying (1.4). Now, we have a priori estimates for u. First, multiply the first equation of (1.1) by u and integrate on Ω,

$$(2.35)\qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho^{1/2}u(t)\|^2 + \nu\|\nabla u\|^2 + \alpha\|u\|_{\beta+1}^{\beta+1} \leqslant \|\varrho^{1/2}u\|^2 + C\|f\|^2,\quad 0 < t \leqslant T',$$

holds (see (2.7)), and applying Gronwall's lemma to (2.35), we get

$$\|u\|_{2,\infty,T'} + \|\nabla u\|_{2,2;T'} + \|u\|_{\beta+1,\beta+1;T'} \leqslant C(u_0, \varrho_0, f, T^*). \tag{2.36}$$

Next, multiply the first equation of (1.1) by u_t and integrate on Ω, then

$$\begin{aligned} &\|\varrho^{1/2}u_t\|^2 + \nu(A^{1/2}u, A^{1/2}u_t) + \alpha(|u|^{\beta-1}u, u_t) \\ &\quad = -(\varrho(u\cdot\nabla)u, u_t) + (\varrho f, u_t), \quad 0 < t \leqslant T' \end{aligned} \tag{2.37}$$

is obtained. The regularity of u

$$\begin{aligned} A^{1/2}u &\in L^2(0,T';V)(u \in L^2(0,T';D(A))), \\ A^{1/2}u_t &\in L^2(0,T';V')(u_t \in L^2(0,T';H)), \end{aligned}$$

implies

$$(A^{1/2}u, A^{1/2}u_t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A^{1/2}u(t)\|^2, \quad 0 < t \leqslant T'. \tag{2.38}$$

Also, $|u|^{(\beta-1)/2}u \in L^1(0,T';L^2(\Omega)^3)$ and $(|u|^{(\beta-1)/2}u)_t \in L^1(0,T';L^2(\Omega)^3)$ lead to

$$\begin{aligned} (|u|^{\beta-1}u, u_t) &= \frac{2}{\beta+1}(|u|^{(\beta-1)/2}u, (|u|^{(\beta-1)/2}u)_t) \\ &= \frac{1}{\beta+1}\frac{\mathrm{d}}{\mathrm{d}t}\||u|^{(\beta-1)/2}u\|^2 = \frac{1}{\beta+1}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\beta+1}^{\beta+1}. \end{aligned} \tag{2.39}$$

In fact, $|u|^{(\beta-1)/2}u \in L^1(0,T';L^2(\Omega)^3)$ is got by

$$\||u|^{(\beta-1)/2}u\|_{2,1;T'} = \int_0^{T'} \|u\|_{\beta+1}^{(\beta+1)/2}\,\mathrm{d}t \leqslant C\|u\|_{V,(\beta+1)/2;T'}^{(\beta+1)/2} < \infty$$

and $(|u|^{(\beta-1)/2}u)_t \in L^1(0,T;L^2(\Omega)^3)$ is obtained from

$$\begin{aligned} \|(|u|^{(\beta-1)/2}u)_t\|_{2,1;T'} &\leqslant C\||u|^{(\beta-1)/2}u_t\|_{2,1;T'} \\ &\leqslant C\|u_t\|_{2,2;T'}\||u|^{(\beta-1)/2}\|_{\infty,2;T'} \\ &\leqslant C\|u_t\|_{2,2;T'}\|u\|_{\infty,\beta-1;T'}^{(\beta-1)/2} \\ &\leqslant C\|u_t\|_{2,2;T'}\|A^{1/2}u\|_{2,\infty;T'}^{(\beta-1)/4}\|Au\|_{2,(\beta-1)/2;T'}^{(\beta-1)/4} \\ &\leqslant C\|u_t\|_{2,2;T'}\|u\|_{V,\infty;T'}^{(\beta-1)/4}(\|u\|_{D(A),2;T'}^{(\beta-1)/4} + T') \\ &< \infty. \end{aligned}$$

Inserting (2.38), (2.39), and the inequalities

$$-(\varrho(u\cdot\nabla)u, u_t) \leqslant \varepsilon_1\|\varrho^{1/2}u_t\|^2 + \frac{\overline{\varrho}}{4\varepsilon_1}\||\nabla u|u\|^2, \tag{2.40}$$

$$(\varrho f, u_t) \leqslant \varepsilon_2\|\varrho_{1/2}u_t\|^2 + C(\varepsilon_2)\|f\|^2, \tag{2.41}$$

where $\varepsilon_1, \varepsilon_2 > 0$, to (2.37), we have

$$(2.42)\qquad (1-\varepsilon_1-\varepsilon_2)\|\varrho^{1/2}u_t\|^2 + \frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A^{1/2}u\| + \frac{\alpha}{\beta+1}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\beta+1}^{\beta+1} \leqslant \frac{\overline{\varrho}}{4\varepsilon_1}\||\nabla u|u\|^2 + C(\varepsilon_2)\|f\|^2, \quad 0 < t \leqslant T'.$$

Finally, multiply the first equation of (1.1) by $-\Delta u$ and integrate on Ω to get

$$(2.43)\qquad \nu\|\Delta u\|^2 - \alpha(|u|^{\beta-1}u, \Delta u) = (\varrho u_t, \Delta u) - (\varrho(u\cdot\nabla)u, \Delta u) - (\varrho f, \Delta u).$$

Using the inequalities

$$\begin{aligned}
-\alpha(|u|^{\beta-1}u, \Delta u) &= \alpha(|u|^{\beta-1}\nabla u, \nabla u) + \frac{\alpha(\beta-1)}{4}\int_\Omega |u|^{\beta-3}|\nabla|u|^2|^2\,\mathrm{d}x \\
&\geqslant \alpha\||u|^{(\beta-1)/2}|\nabla u|\|^2, \\
-(\varrho(u\cdot\nabla)u, \Delta u) &\leqslant \varepsilon_3\nu\overline{\varrho}\|\Delta u\|^2 + \frac{\overline{\varrho}}{4\varepsilon_3\nu}\||\nabla u|u\|^2, \\
-(\varrho f, \Delta u) &\leqslant \varepsilon_4\nu\overline{\varrho}\|\Delta u\|^2 + C(\varepsilon_4)\|f\|^2, \\
(\varrho u_t, \Delta u) &\leqslant \varepsilon_5\nu\overline{\varrho}\|\Delta u\|^2 + \frac{1}{4\varepsilon_5\nu}\|\varrho^{1/2}u_t\|^2,
\end{aligned}$$

with (2.43), leads to

$$(2.44)\qquad \nu(1-(\varepsilon_3+\varepsilon_4+\varepsilon_5)\overline{\varrho})\|\Delta u\|^2 + \alpha\||u|^{(\beta-1)/2}|\nabla u|\|^2 \leqslant \frac{\overline{\varrho}}{4\varepsilon_3\nu}\||u||\nabla u|\|^2 + \frac{1}{4\varepsilon_5\nu}\|\varrho^{1/2}u_t\|^2 + C(\varepsilon_4)\|f\|^2.$$

In the case of $\beta > 3$, $\alpha > 0$, $\nu > 0$, taking $\varepsilon_1 = \varepsilon_2 = \frac{1}{4}$, $\varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 1/(4\overline{\varrho})$, multiply (2.44) by $\nu/(4\overline{\varrho})$ and add it to (2.42) to obtain

$$(2.45)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\|A^{1/2}u(t)\| \leqslant C(\|\nabla u\|^2 + \|f\|^2), \quad 0 < t \leqslant T'.$$

In the case of $\beta = 3$, $2\alpha > \overline{\varrho}$, $\nu > \overline{\varrho}$, $2\nu\alpha > \nu\overline{\varrho} + \overline{\varrho}^2$, take $\varepsilon_1 = \frac{1}{2}$ and ε_2, ε_4 sufficiently small, choosing ε_3, ε_5 so that they satisfy

$$\varepsilon_3 = \varepsilon_5, \quad \varepsilon_3 > \frac{1}{2\nu}, \quad \varepsilon_3 < \frac{1}{2\overline{\varrho}}, \quad \varepsilon_3 > \frac{\overline{\varrho}}{(2\alpha-\overline{\varrho})\nu},$$

then, adding (2.42) to (2.44), we have (2.45).

Applying Gronwall's lemma to (2.45) implies

$$(2.46)\qquad \|A^{1/2}u\|_{2,\infty;T'} \leqslant C(u_0, \varrho_0, f, T^*).$$

Estimates (2.36) and (2.46) show that $\|u(T')\|_V \leqslant C(u_0, \varrho_0, f, T^*)$. Thus, there exists a local strong solution of (1.1) on $[T', T'+\delta](T^* - T' < \delta < T - T')$ with initial condition $u(T')$, which means that there exists a strong solution of (1.1) on $[0, T'+\delta]$ satisfying (1.4) and contradicting the definition of T^*. □

3. Proof of Theorem 1.2

Let $(u^{(1)}, \varrho^{(1)})$ and $(u^{(2)}, \varrho^{(2)})$ be the strong solutions of (1.1) satisfying $u_t^{(i)} \in L^2(0,T;H)$, $i=1,2$.

Step 1. Estimate $\varrho^{(1)}-\varrho^{(2)}$. Multiply the both of sides of the equality

$$(\varrho^{(1)}-\varrho^{(2)})_t = -(u^{(1)}-u^{(2)})\cdot\nabla\varrho^{(1)} - u^{(2)}\cdot\nabla(\varrho^{(1)}-\varrho^{(2)})$$

by $3|\varrho^{(1)}-\varrho^{(2)}|(\varrho^{(1)}-\varrho^{(2)})$ and integrate by parts on Ω to get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\|(\varrho^{(1)}-\varrho^{(2)})(t)\|_3^3 &\leqslant 3\int_\Omega |u^{(1)}-u^{(2)}||\nabla\varrho^{(1)}||\varrho^{(1)}-\varrho^{(2)}|^2\,\mathrm{d}x\\ &\leqslant 3\|\varrho^{(1)}-\varrho^{(2)}\|_3^2\Big(\int_\Omega |u^{(1)}-u^{(2)}|^3|\nabla\varrho^{(1)}|^3\,\mathrm{d}x\Big)^{1/3},\end{aligned}$$

therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\|(\varrho^{(1)}-\varrho^{(2)})(t)\|_3 &\leqslant \Big(\int_\Omega |u^{(1)}-u^{(2)}|^3|\nabla\varrho^{(1)}|^3\,\mathrm{d}x\Big)^{1/3}\\ &\leqslant \|\nabla(u^{(1)}-u^{(2)})\|\|\nabla\varrho^{(1)}\|_6.\end{aligned} \tag{3.1}$$

Integrate (3.1) considering $\|\nabla\varrho^{(1)}\|_{6,\infty;T}<\infty$ to get

$$\|\varrho^{(1)}(t)-\varrho^{(2)}(t)\|_3^2 \leqslant C(T)\|\nabla\varrho^{(1)}\|_{6,\infty;T}^2\int_0^T \|\nabla(u^{(1)}(t)-u^{(2)}(t))\|^2\,\mathrm{dt}. \tag{3.2}$$

Step 2. Prove $u^{(1)}=u^{(2)}$. It is obvious that

$$\begin{aligned}&(\varrho^{(1)}u_t^{(1)}-\varrho^{(2)}u_t^{(2)}) + ((\varrho^{(1)}u^{(1)}\cdot\nabla)u^{(1)}\\ &\quad-(\varrho^{(2)}u^{(2)}\cdot\nabla)u^{(2)}) + \nabla p^{(1)} - \nabla p^{(2)}\\ &\quad=\nu\Delta(u^{(1)}-u^{(2)}) - \alpha(|u^{(1)}|^{\beta-1}u^{(1)} - |u^{(2)}|^{\beta-1}u^{(2)}) + (\varrho^{(1)}-\varrho^{(2)})f,\\ &\varrho^{(1)}(u_t^{(1)}-u_t^{(2)}) + ((\varrho^{(1)}u^{(1)}\cdot\nabla)u^{(1)} - (\varrho^{(1)}u^{(2)}\cdot\nabla)u^{(2)})\\ &\quad-\nu\Delta(u^{(1)}-u^{(2)}) + \alpha(|u^{(1)}|^{\beta-1}u^{(1)} - |u^{(2)}|^{\beta-1}u^{(2)})\\ &\quad= -(\nabla p^{(1)}-\nabla p^{(2)}) + (\varrho^{(1)}-\varrho^{(2)})f - J,\\ &J := (\varrho^{(1)}-\varrho^{(2)})u_t^{(2)} + ((\varrho^{(1)}-\varrho^{(2)})u^{(2)}\cdot\nabla)u^{(2)}.\end{aligned} \tag{3.3}$$

Multiply (3.3) by $z:=u^{(1)}-u^{(2)}$ and integrate on Ω, considering

$$\begin{aligned}(\varrho^{(1)}z_t,z) &= \frac12\frac{\mathrm{d}}{\mathrm{dt}}\|\sqrt{\varrho^{(1)}}z\|^2 - \frac12(\varrho_t^{(1)}z,z) = \frac12\frac{\mathrm{d}}{\mathrm{dt}}\|\sqrt{\varrho^{(1)}}z\|^2 + \frac12((u^{(1)}\cdot\nabla\varrho^{(1)})z,z)\\ &= \frac12\frac{\mathrm{d}}{\mathrm{dt}}\|\sqrt{\varrho^{(1)}}z\|^2 - \frac12((\varrho^{(1)}z\cdot\nabla)u^{(1)},z) - ((\varrho^{(1)}u^{(1)}\cdot\nabla)z,z)\end{aligned}$$

and

$$\begin{aligned}
((\varrho^{(1)}u^{(1)}\cdot\nabla)u^{(1)}&-(\varrho^{(1)}u^{(2)}\cdot\nabla)u^{(2)},z)\\
&=((\varrho^{(1)}(u^{(1)}-u^{(2)})\cdot\nabla)u^{(1)}+(\varrho^{(1)}u^{(2)}\cdot\nabla)(u^{(1)}-u^{(2)}),z)\\
&=((\varrho^{(1)}z\cdot\nabla)u^{(1)},z)+((\varrho^{(1)}u^{(2)}\cdot\nabla)z,z),\\
|((\varrho^{(1)}u^{(i)}\cdot\nabla)z,z)|&\leqslant\frac{\nu}{8}\|\nabla z\|^2+C\|\sqrt{\varrho^{(1)}}z\|^2\|u^{(i)}\|_\infty^2\\
&\leqslant\frac{\nu}{8}\|\nabla z\|^2+C\|\sqrt{\varrho^{(1)}}z\|^2(\|u^{(i)}\|_{D(A)}^2+1),\quad i=1,2,\\
((\varrho^{(1)}z\cdot\nabla)u^{(1)},z)&\leqslant\sqrt{\overline{\varrho}}\|\sqrt{\varrho^{(1)}}z\|\|\nabla u^{(1)}\|_3\|z\|_6\\
&\leqslant C\|\sqrt{\varrho^{(1)}}z\|\|\nabla u^{(1)}\|_3\|\nabla z\|\\
&\leqslant\frac{\nu}{8}\|\nabla z\|^2+C\|\sqrt{\varrho^{(1)}}z\|^2(\|u^{(1)}\|_{D(A)}^2+1),\\
\alpha(|u^{(1)}|^{\beta-1}u^{(1)}&-|u^{(2)}|^{\beta-1}u^{(2)},u^{(1)}-u^{(2)})\geqslant 0.
\end{aligned}$$

Then we have

$$\begin{aligned}
(3.4)\qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\varrho^{(1)}}z(t)\|^2+\frac{\nu}{2}\|\nabla z\|^2\leqslant C\|\sqrt{\varrho^{(1)}}z\|^2(\|u^{(1)}\|_{D(A)}^2+\|u^{(1)}\|_{D(A)}^2+1)\\
+((\varrho^{(1)}-\varrho^{(2)})f,z)-(J,z).
\end{aligned}$$

Combining the estimates

$$\begin{aligned}
((\varrho^{(1)}-\varrho^{(2)})f,z)&\leqslant C\|z\|_6\|\varrho^{(1)}-\varrho^{(2)}\|_3\|f\|_6\\
&\leqslant\frac{\nu}{8}\|\nabla z\|^2+C\|\varrho^{(1)}-\varrho^{(2)}\|_3^2\|f\|_{W^{1,2}(\Omega)}^2,\\
-(J,z)&=((\varrho^{(1)}-\varrho^{(2)})u_t^{(2)},z)+(((\varrho^{(1)}-\varrho^{(2)})u^{(2)}\cdot\nabla)u^{(2)},z),\\
((\varrho^{(1)}-\varrho^{(2)})u_t^{(2)},z)&\leqslant\|z\|_6\|\varrho^{(1)}-\varrho^{(2)}\|_3\|u_t^{(2)}\|_2\\
&\leqslant\frac{\nu}{4}\|\nabla z\|^2+C\|\varrho^{(1)}-\varrho^{(2)}\|_3\|u_t^{(2)}\|,\\
(((\varrho^{(1)}-\varrho^{(2)})u^{(2)}\cdot\nabla)u^{(2)},z)&\leqslant C\|\sqrt{\varrho^{(1)}}z\|\|\varrho^{(1)}-\varrho^{(2)}\|_3\|\nabla u^{(2)}\|\|u^{(2)}\|_\infty\\
&\leqslant C\|\sqrt{\varrho^{(1)}}z\|\|\nabla u^{(2)}\|^2\|u^{(2)}\|_\infty^2+\|\varrho^{(1)}-\varrho^{(2)}\|_3^2\\
&\leqslant C\|\sqrt{\varrho^{(1)}}z\|\|\nabla u^{(2)}\|^2(\|u^{(2)}\|_{D(A)}^2+1)+\|\varrho^{(1)}-\varrho^{(2)}\|_3^2.
\end{aligned}$$

with (3.4) and (3.2) leads to

$$\begin{aligned}
(3.5)\qquad &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\varrho^{(1)}}z(t)\|^2+\frac{\nu}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\|\nabla z\|^2\,\mathrm{d}s\\
&\leqslant C\|\sqrt{\varrho^{(1)}}z\|^2(\|\nabla u^{(2)}\|^2+1)(\|u^{(1)}\|_{D(A)}^2+\|u^{(2)}\|_{D(A)}^2+1)\\
&\quad+C(T)\|\nabla\varrho^{(1)}\|_{6,\infty;T}^2\int_0^t\|\nabla z\|^2ds(\|f\|_{W^{1,2}(\Omega)^3}+\|u_t^{(2)}\|^2+1).
\end{aligned}$$

Applying Gronwall's lemma to (3.5) considering $u_t^{(2)}\in L^2(0,T;H)$ implies $u^{(1)}=u^{(2)}$. And $u^{(1)}=u^{(2)}$ leads $\varrho^{(1)}=\varrho^{(2)}$ with (3.2). □

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