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HERMITIAN-TOEPLITZ DETERMINANTS AND SOME COEFFICIENT FUNCTIONALS FOR THE STARLIKE FUNCTIONS

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Abstract. In this paper, we have determined the sharp lower and upper bounds on the fourth-order Hermitian-Toeplitz determinant for starlike functions with real coefficients. We also obtained the sharp bounds on the Hermitian-Toeplitz determinants of inverse and logarithmic coefficients for starlike functions with complex coefficients. Sharp bounds on the modulus of differences and difference of moduli of logarithmic and inverse coefficients are obtained. In our investigation, it has been found that the bound on the third-order Hermitian-Toeplitz determinant for starlike functions and its inverse coefficients is invariant.

Keywords: starlike function; Hermitian-Toeplitz determinant; logarithmic coefficient; inverse coefficient

MSC 2020: 30C45, 30C50

1. Introduction

The problem of investigating sharp bound on the coefficients of normalized analytic functions satisfying certain geometric properties in the unit disk $\mathbb{D} := \{|z| < 1\colon\ z \in \mathbb{C}\}$ is one of the most studied topic in the field of Geometric Function Theory (GFT). The bounds also useful in deriving geometric properties, for instance, the growth and distortion of an analytic univalent function, can be obtained with the help of sharp bound on the second coefficient in the case of univalent functions [5]. Other quantities related to coefficients of normalized univalent functions like Fekete-Szegő functional, inverse coefficients, logarithmic coefficients, Hankel determinants, Toeplitz determinants and Hermitian-Toeplitz determinants are also helpful in determining the nature of the functions and are among the most studied topics in GFT in recent past [2], [3], [13], [4], [6], [7], [17], [11], [19].

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In what follows, let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \dots \tag{1.1}$$

defined in the unit disk $\mathbb{D}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ which contains univalent functions in $\mathbb{D}$. The class of starlike functions arose during the efforts in searching the proof of the Bieberbach conjecture (1916) and is considered to be an important subclass of $\mathcal{S}$, which gave much hope for the trueness of the Bieberbach conjecture. The class of starlike functions of order $\alpha \in [0,1)$, denoted by $\mathcal{S}^*(\alpha)$, is the collection of functions $f \in \mathcal{S}$ which satisfy

$$\operatorname{Re}\Big(\frac{zf'(z)}{f(z)}\Big) > \alpha, \quad z \in \mathbb{D}.$$

The class $\mathcal{S}^* := \mathcal{S}^*(0)$ is famous as the class of starlike functions.

Ali et al. [1] found sharp bounds for the symmetric Toeplitz determinants for univalent functions and typically real functions. Further, the amount of research work has been done on symmetric Toeplitz and Hankel determinants and their applications in various fields makes them more interesting. For the sequence $\langle a_i \rangle$ of coefficients of the function f given by (1.1) in $\mathcal{A}$ and given natural numbers $q, n \in \mathbb{N}$, the Hermitian-Toeplitz matrix $T_{q,n}(f)$ is given by

$$T_{q,n}(f) := \begin{bmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ \bar{a}_{n+1} & a_n & \dots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ \bar{a}_{n+q-1} & \bar{a}_{n+q-2} & \dots & a_n. \end{bmatrix}$$

Further simplifications give

$$\det(T_{2,1}(f)) = 1 - |a_2|^2 \quad \text{and} \quad \det(T_{3,1}(f)) = 2\operatorname{Re}(a_2^2\bar{a}_3) - 2|a_2|^2 - |a_3|^2 + 1 \tag{1.2}$$

and

$$\begin{aligned} \det(T_{4,1}(f)) = 1 &- 2\operatorname{Re}(a_2^3\bar{a}_4) + 4\operatorname{Re}(a_2^2\bar{a}_3) - 2\operatorname{Re}(a_2\bar{a}_3^2a_4) + 4\operatorname{Re}(a_2a_3\bar{a}_4) \\ &+ |a_2|^4 - 3|a_2|^2 + |a_3|^4 - 2|a_3|^2 + |a_2|^2|a_4|^2 - 2|a_2|^2|a_3|^2 - |a_4|^2. \end{aligned} \tag{1.3}$$

Cudna et al. [4] began the study on Hermitian-Toeplitz determinants and computed sharp lower and upper bounds for the second- and third-order Hermitian-Toeplitz determinants for the classes of starlike and convex functions of order $\alpha \in [0,1)$. In 2020, Kumar et al. [16] generalised the works done by Cudna et al. [4] by investigating the sharp upper and lower bounds for the third-order Hermitian-Toeplitz determinant $\det(T_{3,1}(f))$ for the classes of Janowski type starlike and convex functions, which are a generalization of some recent work. Jastrzębski et al. [8] investigated the sharp upper and lower bounds of the second- and third-order

Hermitian-Toeplitz determinants for some subclasses of close-to-star functions. In 2021, Kumar [12] discussed the sharp upper and lower bounds for several subclasses of close-to-convex functions for the second- and third-order Hermitian-Toeplitz determinants. In 2021, Kumar et al. [14] discussed the sharp upper and lower bounds for the Hermitian-Toeplitz determinant of the third-order $\det(T_{4,1}(f))$ for the classes of strongly starlike functions, lemniscate starlike functions and lune starlike functions. In 2020, Lecko et al. [18] investigated the sharp upper and lower bounds for the Hermitian-Toeplitz determinant of the fourth-order for the class of convex functions and proved that $0 \leqslant \det(T_{4,1}(f)) \leqslant 1$.

The above cited works motivate us to investigate the following results and the paper is arranged as follows: In Section 2, we investigate the sharp upper and lower bounds on the fourth-order Hermitian-Toeplitz determinant $\det(T_{4,1}(f))$ over the class of starlike functions $\mathcal{S}^*$ with real coefficients. In the same section, we find the sharp estimation of the Hermitian-Toeplitz determinants of inverse and logarithmic coefficients for starlike functions with complex coefficients. In Section 3, we derive the sharp bounds on the modulus of differences and difference of moduli of logarithmic and inverse coefficients. In our investigation, it has been found that the bound on the third Hermitian-Toeplitz determinant for starlike functions and its inverse coefficients is invariant.

The class of functions with positive real part, denoted by $\mathcal{P}$, plays a very important role while investigating the bounds on the coefficients. The class $\mathcal{P}$ is the collection of functions of the form $p(z) = 1 + \sum\limits_{n=1}^{\infty} c_n z^n$ for which $\operatorname{Re} p(z) > 0$, $z \in \mathbb{D}$. The following lemma plays a key role in the proof of the main results.

Lemma 1.1 ([20], [21]). *If $p \in \mathcal{P}$ such that $p(z) := 1 + \sum\limits_{n=1}^{\infty} c_n z^n$ with $c_1 \geqslant 0$, then*

$$2c_2 = c_1^2 + (4 - c_1^2)\zeta \quad \text{and} \quad 4c_3 = c_1^3 + (4 - c_1^2)(2 - \zeta)c_1\zeta + 2(4 - c_1^2)(1 - |\zeta|^2)\eta$$

for some $\zeta, \eta \in \bar{\mathbb{D}} := \{z \in \mathbb{C} \colon |z| \leqslant 1\}$.

2. Hermitian-Toeplitz determinants

For the class of convex functions, Cudna et al. [4] proved that $\det(T_{q,1}(f)) \in [0, 1]$, $q = 2, 3$ and conjectured that $\det(T_{q,1}(f)) \in [0, 1]$, $q = 2, 3, 4, \ldots$. Later, Lecko et al. [18] established the above for $q = 4$. However, for starlike functions, Cudna [4] proved that $\det(T_{2,1}(f)) \in [-3, 1]$ and $\det(T_{3,1}(f)) \in [-1, 8]$, see [4], Corollary 3.

Here we observe that the length of the interval in which $\det(T_{q,1}(f))$ belongs gets reduced as q increases but interestingly $[-3,1]$ is not completely contained in $[-1,8]$. This makes it interesting to know the sharp upper and lower bounds on $\det(T_{4,1}(f))$ when f belongs to the class of starlike functions.

2.1. Real coefficients. The following theorem gives the sharp lower and upper bounds on the fourth-order Hermitian-Toeplitz determinant when $f \in \mathcal{S}^*$ and having real coefficients.

Theorem 2.1. *Let $f \in \mathcal{S}^*$ be given by (1.1) with $a_i \in \mathbb{R}$, $i = 2,3,4$. Then the following bounds hold:*

$$-20 \leqslant \det(T_{4,1}(f)) \leqslant 1.$$

The bounds are sharp. Equality in the upper and lower bounds holds in the case of the Koebe function $k(z) = z/(1-z)^2$ and the identity function $f_0(z) = z$, respectively.

Proof. For each function $f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \ldots \in \mathcal{S}^*$, there exists a function $p(z) := 1 + \sum\limits_{n=1}^{\infty} c_n z^n \in \mathcal{P}$ such that

$$zf'(z) = p(z)f(z).$$

Comparing coefficients in the above expression, we get

$$a_2 = c_1, \quad a_3 = \frac{1}{2}(c_2 + c_1^2) \quad \text{and} \quad a_4 = \frac{1}{6}(2c_3 + 3c_1c_2 + c_1^3). \tag{2.1}$$

Since $\mathcal{S}^*$ and $\det(T_{4,1}(f))$ are rotationally invariant, it follows that there is no loss of generality if we set $c_1 = c \in [0,2]$ and $a_2 \in [0,2]$. Using (1.3) and (2.1) on simplification, we get

$$\begin{aligned}144\det(T_{4,1}(f)) = 144 &- 432c^2 + 360c^4 - 76c^6 + c^8 + 144c^2\operatorname{Re} c_2 \\ &- 120c^4\operatorname{Re} c_2 + 36c^4(\operatorname{Re} c_2)^2 + 36c^2|c_2|^2 - 18c^4|c_2|^2 \\ &+ 9|c_2|^4 - 72|c_2|^2 - 12c^4\operatorname{Re} c_2^2 - (8c^5 + 16c^3)\operatorname{Re} c_3 \\ &+ 48c\operatorname{Re}(c_2\bar{c}_3) + 16c^2|c_3|^2 - 16|c_3|^2 - 24c\operatorname{Re}(c_2^2\bar{c}_3).\end{aligned} \tag{2.2}$$

Since a_3 and a_4 are in $\mathbb{R}$, $c_2, c_3 \in \mathbb{R}$ and $\zeta, \eta \in \mathbb{R}$.

Now by using Lemma 1.1 and substituting the expressions for c_2 and c_3 into (2.2), after a rigourous computation, we find that

(2.3)

$$\begin{aligned}144\det(T_{4,1}(f)) = {} & \frac{9}{16}c^8 - 126c^6 + 414c^4 - 432c^2 + 144 + c^2(4-c^2)^2\Big(\frac{15}{4}c^2 - 9\Big)\zeta^3 \\ & + (4-c^2)c^2\Big(-\frac{3}{4}c^4 - 36c^2 + 36\Big)\zeta + (4-c^2)\Big(-\frac{61}{4}c^6 + 49c^4\Big)\zeta^2 \\ & + (4-c^2)(8c^6 - 20c^4)\zeta^2 + 2c^4(4-c^2)(c^2-1)\zeta^2 \\ & - 3c^5(4-c^2)(1-\zeta^2)\eta + 4c(c^2-1)(4-c^2)(1-\zeta^2)(2\zeta-\zeta^2)\eta \\ & + (4-c^2)^2(12c-6c^3)(1-\zeta^2)\zeta\eta + 4(c^2-1)(4-c^2)^2(1-\zeta^2)^2\eta^2 \\ & + (4-c^2)^2\Big(\frac{-75}{8}c^4 + 21c^2 - 18\Big)\zeta^2 + (4-c^2)^3\Big(\frac{9}{4} + \frac{15}{16}c^2\Big)\zeta^4 \\ & - 3c(4-c^2)^3(1-\zeta^2)\zeta^2\eta + (c^2-1)(4-c^2)^2c^2(2\zeta-\zeta^2)^2.\end{aligned}$$

We now find the maximum and minimum of $\Psi(\zeta,\eta,c) := \det(T_{4,1}(f))$ in the cuboid

$$\Omega := \{(\zeta,\eta,c) \in \mathbb{R}^3\colon\ 0 \leqslant \zeta \leqslant 1,\ 0 \leqslant \eta \leqslant 1,\ 0 \leqslant c \leqslant 2\}.$$

For this we will proceed as follows:

Case (I): For $c = 0$ we get

$$\det(T_{4,1}(f)) = \frac{1}{144}(144 - 288\zeta^2 + 144\zeta^4 - 64(1-\zeta^2)^2\eta^2), \tag{2.4}$$

where $\zeta \in [0,1]$ and $\eta \in [0,1]$. Then from (2.4) we can write:

$$\det(T_{4,1}(f)) = \frac{1}{144}(144 - 288\zeta^2 + 144\zeta^4 - 64(1-\zeta^2)^2\eta^2) =: h_1(\zeta,\eta).$$

Further computation reveals that the function h has no maximum inside $(0,1)\times(0,1)$, whereas on considering the boundary we find that

$$\max_{\zeta,\eta\in[0,1]} h_1(\zeta,\eta) = 1 = h_1(0,0) \quad \text{and} \quad \min_{\zeta,\eta\in[0,1]} h_1(\zeta,\eta) = 0 = h_1(1,\eta),$$
$$0 \leqslant \det(T_{4,1}(f)) \leqslant 1.$$

Case (II): For $c = 1$, (2.3) gives

$$\det(T_{4,1}(f)) = \frac{1}{256}(1-3\zeta)^2(1 + 2\zeta - 16\eta + \zeta^2(17 + 16\eta)) := h_2(\zeta,\eta).$$

A computation reveals that the function h_2 has no maximum inside $(0,1)\times(0,1)$, whereas on considering the boundary we find that at the point $(1, 1/2)$

$$\max_{\zeta,\eta\in[0,1]} h_2(\zeta,\eta) = \frac{5}{16}.$$

Computation reveals that the function h_2 has no minimum inside $(0,1)\times(0,1)$, whereas on considering the boundary we find that at the point $(0,1)$

$$\min_{\zeta,\eta\in[0,1]} h_2(\zeta,\eta) = -\frac{15}{256}.$$

Therefore, when $c=1$, we have

$$\frac{-15}{256} \leqslant \det(T_{4,1}(f)) \leqslant \frac{5}{16}.$$

Case (III): For c = 2, from (2.3) we get

$$\det(T_{4,1}(f)) = -20.$$

Case (IV): We now consider the vertices, boundary and interior of the cuboid Ω and find the maximum and minimum of $\Psi(\zeta,\eta,c)$ over it.

(a) On the vertices we have:

$$\Psi(0,0,0)=1,\quad \Psi(0,1,0)=\frac{5}{9},\quad \Psi(1,1,0)=0,\quad \Psi(1,0,0)=0,$$
$$\Psi(\zeta,\eta,2)=-20\quad \forall\,(\zeta,\eta)\in[0,1]\times[0,1].$$

(b) Now, critical points on the boundary surface $c=0$ of the cuboid Ω of the function

$$g_1(\zeta,\eta) := \Psi(\zeta,\eta,0) = \frac{1}{144}(144-288\zeta^2+144\zeta^4-64(1-\zeta^2)^2\eta^2)$$

are obtained by solving the equations

$$\frac{\partial g_1}{\partial \zeta} = \frac{\partial g_1}{\partial \eta} = 0.$$

Here we find that the above pair of equations has no solution in $(\zeta,\eta)\in(0,1)\times(0,1)$.

(c) On the boundary face $c=2$ of the cuboid,

$$g_2(\zeta,\eta) := \Psi(\zeta,\eta,2) = -20.$$

(d) On the boundary face $\zeta=0$ of the cuboid Ω we have

$$g_3(\eta,c) := \Psi(0,\eta,c) = 1-3c^2+\frac{23}{8}c^4-\frac{7}{8}c^6+\frac{1}{256}c^8 + \frac{1}{48}c^5(c^2-4)\eta+\frac{1}{36}(c^2-4)^2(c^2-1)\eta^2.$$

For the critical points we solve the equations

$$\frac{\partial g_3}{\partial \eta} = \frac{\partial g_3}{\partial c} = 0$$

and thus, we find that this pair of equations has only solution $(0.8660, 1.1547)$ in $(\eta, c) \in (0,1) \times (0,2)$ and $g_3(0.8660, 1.1547) = 6.1756 \times 10^{-16}$(nearly zero). The vertices (0, 0), (0, 2), (1, 0), (1, 2) and boundary points (0, 0.921), (1, 0.8330), (1,1.15865) are the critical points on the boundary of the face $\zeta = 0$ of the cuboid and

$$g_3(0,0) = 1, \quad g_3(0,2) = -20, \quad g_3(1,0) = \frac{5}{9}, \quad g_3(1,2) = 1,$$
$$g_3(0, 0.921) = -0.00813, \quad g_3(1, 0.8330) = -0.1094, \quad g_3(1, 1.1586) = 0.00125.$$

(e) On the boundary face $\zeta = 1$ of the cuboid Ω, we have

$$\Psi(1, \eta, c) = \frac{1}{144}c^2(-16 + 144c^2 - 84c^4 + c^6) \leqslant 0.32154 \approx \Psi(1, \eta, 1.0510).$$

(f) On the boundary face $\eta = 0$ of the cuboid Ω, we have

$$\begin{aligned} g_4(\zeta, c) := \Psi(\zeta, 0, c) = {} & (\zeta^2 - 1)^2 + \frac{1}{2304}c^8(-3 - 2\zeta + \zeta^2)^2 \\ & - \frac{1}{48}c^6(42 - 11\zeta - 4\zeta^2 + \zeta^3) \\ & + \frac{1}{24}c^4(69 - 30\zeta - 22\zeta^2 + 6\zeta^3 + \zeta^4) \\ & - \frac{1}{9}c^2(27 - 9\zeta - 26\zeta^2 + 5\zeta^3 + 4\zeta^4). \end{aligned}$$

The critical points of the function $\Psi(\zeta, 0, c)$ are $(0.8869, 0.49771)$, $(0.3504, 1.0388)$, $(0.2164, 0.9658)$, $(0.3553, 0.9644)$, $(0.5, 0.8944)$ and $(0.5, 0.8944)$ in $(\zeta, c) \in (0,1) \times (0,2)$. Further computation reveals that

$$g_4(0.8869, 0.49771) \approx -0.01211 \leqslant g_4(\zeta, c) \leqslant 0.00072 \approx g_4(0.3504, 1.0388).$$

Also, for the points $(0, 0.9217)$, $(0, 1.1675)$, $(1, 1.0510)$ on the boundary of the face $\eta = 0$ of the cuboid Ω, we have

$$g_4(0, 0.9217) = -0.008132, \quad g_4(0, 1.1675) = 0.0499 \quad \text{and} \quad g_4(1, 1.05107) = -0.3215.$$

(g) On the boundary face $\eta = 1$ of the cuboid Ω, we have

$$g_5(\zeta, c) := \Psi(\zeta, 1, c).$$

Now the equations

$$\frac{\partial g_5}{\partial \zeta} = \frac{\partial g_5}{\partial c} = 0$$

have solutions (0.7752, 0.6414), (0.08032, 1.1284), (0.2722, 1.0529), (0.5448, 1.00792) and (0.3627, 0.8846) in $(0,1) \times (0,2)$. Further, the function $g_5(\zeta, c)$ achieves maximum at the point (0.2722, 1.0529) and minimum at the point (0.7752, 0.6414), also

$g_5(0.2722, 1.0529) = 0.001669$ and $g_5(0.7752, 0.6414) = -0.038246$. On the other possible points of extrema, we find that $g_5(0, 0.83305) = -0.109401$, $g_5(0, 1.15865) = 0.00125$, $g_5(1, 0.24197) = -0.00319$, $g_5(1, 0.0510) = -0.00028$.

(h) We now proceed to investigate extrema of the function Ψ inside the domain Ω. For this, we investigate critical points of $\Psi(\zeta, \eta, c)$ in the interior of the cuboid Ω by solving

$$\frac{\partial \Psi(\zeta, \eta, c)}{\partial \zeta} = \frac{\partial \Psi(\zeta, \eta, c)}{\partial \eta} = \frac{\partial \Psi(\zeta, \eta, c)}{\partial c} = 0.$$

Here we find that

$$\frac{\partial f(\zeta, \eta, c)}{\partial \eta} = 0$$

holds for

$$\eta = \eta_0 = \frac{3c^5(\zeta - 1)^2 - 4c^3\zeta(5\zeta - 7) + 4c\zeta(11\zeta - 10)}{8(c^2 - 4)(c^2 - 1)(\zeta^2 - 1)},$$

where $c \neq 1, 2$ and $\zeta \neq 1$. Further, on substituting the value of η_0, we have

$$\frac{\partial f(\zeta, \eta_0, c)}{\partial \zeta} \neq 0,$$

which ascertains that there is no real solution in the interior of the cuboid Ω. Putting the conclusions of all the above cases together, we get the desired bounds.

To verify the sharpness of the result, consider the Koebe function $k(z) = z/(1 - z)^2 \in \mathcal{S}^*$. For this function, $a_n = n$ $(n = 2, 3, 4, \ldots)$. Substituting $a_n = n$ into (1.3), we find that $\det(T_{4,1}(f)) = -20$. This confirms the sharpness of the lower bound. The upper bound is also sharp as equality holds in the case of the function $f_0(z) = z$. This ends the proof. $\square$

2.2. Inverse coefficients. It is well-known that the univalent function $f \in \mathcal{S}$ has an inverse f^{-1} having the Taylor series expansion of the form

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n \tag{2.5}$$

valid at least in the disk of radius $\frac{1}{4}$ centered at the origin as the inverse of a univalent function need not to be univalent in the whole unit disk $\mathbb{D}$. Löwner [22] found that estimate $|A_n| \leqslant (2n)!/(n!(n+1)!)$ $(n = 2, 3, \ldots)$ is sharp for the classes $\mathcal{S}$ and $\mathcal{S}^*$ with equality in the case of the Koebe function $k(z) = z/(1-z)^2$. For recent development one can refer to [15]. For the inverse coefficients, it would be interesting to see the bound on $\det(T_{3,1}(f^{-1}))$ for starlike function. Note that Cudna [4] proved that $-1 \leqslant \det(T_{3,1}(f)) \leqslant 8$ when $f \in \mathcal{S}^*$.

Theorem 2.2. *Let $f \in \mathcal{S}^*$ be given by* (1.1) *and the inverse f^{-1} be given by* (2.5). *Then the following bounds are sharp:*

$$-1 \leqslant \det(T_{3,1}(f^{-1})) \leqslant 8.$$

Proof. For function $f \in \mathcal{S}^*$, the third-order Hermitian-Toeplitz determinant (1.2) of inverse coefficients is given by

$$\det(T_{3,1}(f^{-1})) = 2\operatorname{Re}(A_2^2\bar{A}_3) - 2|A_2|^2 - |A_3|^2 + 1.$$

On substituting $A_2 = -a_2$ and $A_3 = 2a_2^2 - a_3$ and by using (2.1), on simplification we get

$$\det(T_{3,1}(f^{-1})) = \frac{1}{4}(3c_1^4 + 2c_1^2\operatorname{Re} c_2 - 8c_1^2 - |c_2|^2) + 1.$$

Now by using Lemma 1.1 and rearranging the terms, the above expression simplifies to

$$\det(T_{3,1}(f^{-1})) = \frac{1}{16}(15c^4 - 32c^2 + 2c^2(4-c^2)\operatorname{Re}\zeta - (4-c^2)^2|\zeta|^2) + 1. \tag{2.6}$$

Since $\operatorname{Re}\zeta \leqslant |\zeta| = x \in [0,1]$, it follows that

$$\begin{aligned}\det(T_{3,1}(f^{-1})) &\leqslant \frac{1}{16}(15c^4 - 32c^2 + 2c^2(4-c^2)|\zeta| - (4-c^2)^2|\zeta|^2) + 1\\ &= \frac{1}{16}(15c^4 - 32c^2 + 2c^2(4-c^2)x - (4-c^2)^2x^2) + 1\\ &=: \psi(c,x).\end{aligned}$$

Now we will find $\max\psi$ over the region $[0,2]\times[0,1]$. For this we proceed as follows:

(A) On the boundary of $[0,2]\times[0,1]$, it is easy to arrive at:

$$\begin{gathered}\psi(0,x) \leqslant -x^2 + 1 \leqslant 1, \quad \psi(2,x) \leqslant 8,\\ \psi(c,0) \leqslant \frac{1}{16}(15c^4 - 32c^2) + 1 \leqslant 8, \quad \psi(c,1) \leqslant \frac{3}{4}c^4 - c^2 \leqslant 8.\end{gathered}$$

(B) We now consider the inside of the rectangle $[0,2]\times[0,1]$. Let $(c,x) \in (0,2)\times(0,1)$. Here we see that

$$\frac{\partial\psi(c,x)}{\partial x} = 0 \quad \text{if and only if} \quad x = x_0 = \frac{c^2}{(4-c^2)}.$$

Further,

$$\frac{\partial\psi(c,x_0)}{\partial c} = (c^2-1)^2 = 0$$

for $c = 1$ and the only critical point of ψ in $(0,2) \times (0,1)$ is $(1, 1/3)$ and $\psi(1, 1/3) = -\frac{1}{4}$. Based on the above discussion, we conclude that

$$\det(T_{3,1}(f^{-1})) \leqslant \max\Big\{-\frac{1}{4}, 1, 8, 8, 8\Big\} = 8. \tag{2.7}$$

To calculate the lower bound, we use the fact that $-|\zeta| \leqslant \operatorname{Re} \zeta$ and (2.6). Therefore, with the setting $|\zeta| = x \in [0,1]$ we have

$$\begin{aligned}\det(T_{3,1}(f^{-1})) &\geqslant \frac{1}{16}(15c^4 - 32c^2 - 2c^2(4-c^2)|\zeta| - (4-c^2)^2|\zeta|^2) + 1 \\ &= \frac{1}{16}(15c^4 - 32c^2 - 2c^2(4-c^2)x - (4-c^2)^2x^2) + 1 \\ &=: \varphi(c,x).\end{aligned}$$

(A1) On the boundary of $[0,2] \times [0,1]$, it is easy to arrive at:

$$\begin{gathered}\varphi(0,x) \geqslant -x^2 + 1 \geqslant 0, \quad \varphi(2,x) \geqslant 8, \\ \varphi(c,0) \geqslant \frac{1}{16}(15c^4 - 32c^2) + 1 \geqslant -\frac{1}{15}, \quad \varphi(c,1) \geqslant c^2(c^2-2) \geqslant -1.\end{gathered}$$

(B1) Now consider the interior of $[0,2] \times [0,1]$ and let $(c,x) \in (0,2) \times (0,1)$. Further computation reveals that the function φ has no minimum inside the domain $(0,2) \times (0,1)$. The above discussion reveals that

$$\det(T_{3,1}(f^{-1})) \geqslant \min\{0, 8, -1/15, -1\} = -1. \tag{2.8}$$

Therefore, putting together the estimates in (2.7) and (2.8), we get the required result.

The lower bound is sharp as equality holds for the function h given by

$$h(z) = z \exp\left(\int_0^z \frac{p(t)-1}{t}\,\mathrm{d}t\right), \quad p(z) = \frac{1-z^2}{1-z+z^2} = z + z^2 - z^4 + \dots$$

and equality in the upper bound occurs in the case of the Koebe function $k(z) = z/(1-z)^2$. This completes the proof. $\square$

2.3. Logarithmic coefficients. The logarithmic coefficients and the Milin's conjecture, which implies the Bieberbach conjecture, are very much related. For univalent function $f \in \mathcal{S}$, the logarithmic coefficients γ_i are related as

$$F_f(z) := \log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n, \quad z \in \mathbb{D}. \tag{2.9}$$

The sharp estimates on the initial two logarithmic coefficients are known for the class of univalent functions $\mathcal{S}$. Thomas [24] obtained that the estimate $|\gamma_n| \leqslant 1/n$ is sharp for the class of starlike functions with equality in the case of the Koebe function $k(z) = z/(1-z)^2$. For recent development one may refer to [15].

Inspired by the work done and the importance of the logarithmic coefficient in proving the Bieberbach conjecture, Kowalczyk and Lecko [9], [10] investigated the sharp bound on the second-order Hermitian-Toeplitz determinant. In the similar concept, we consider the second-order Hermitian-Toeplitz determinant of logarithmic coefficients for univalent functions. For a univalent function $f \in \mathcal{S}$, the logarithmic coefficients γ_i are related as given in (2.9). The second-order Hermitian-Toeplitz matrix $T_{2,1}(F_f/2)$ is given by

$$T_{2,1}\Big(\frac{F_f}{2}\Big) := \begin{bmatrix} \gamma_1 & \gamma_2 \\ \overline{\gamma}_2 & \gamma_1 \end{bmatrix},$$

and therefore, $\det(T_{2,1}(F_f/2)) = \gamma_1^2 - |\gamma_2|^2$. The next theorem gives the sharp lower and upper bounds on $\det(T_{2,1}(F_f/2))$.

Theorem 2.3. *Let $f \in \mathcal{S}^*$ be given by* (1.1). *Then the following bounds are sharp:*

$$-\frac{1}{4} \leqslant \det\Big(T_{2,1}\Big(\frac{F_f}{2}\Big)\Big) \leqslant \frac{3}{4}.$$

Proof. Let $f \in \mathcal{S}^*$. Then from (2.9) we have $\gamma_1 = a_2/2$ and $\gamma_2 = (a_3 - a_2^2/2)/2$. Using these in $\det(T_{2,1}(F_f/2)) = \gamma_1^2 - |\gamma_2|^2$ and using (2.1), on simplification we get

$$\det\Big(T_{2,1}\Big(\frac{F_f}{2}\Big)\Big) = \frac{1}{16}(4c_1^2 - |c_2|^2).$$

Now by using Lemma 1.1 and rearranging the terms, the above expression simplifies to

$$\det\Big(T_{2,1}\Big(\frac{F_f}{2}\Big)\Big) = \frac{1}{4}c_1^2 - \frac{1}{64}(c_1^4 + (4-c_1^2)^2|\zeta|^2 + 2c_1^2(4-c_1^2)\operatorname{Re}\zeta).$$

Since the class $\mathcal{S}^*$ and the functional $\det(T_{2,1}(F_f/2))$ are rotationally invariant, there is no harm in considering $c_1 =: c \in [0,2]$. Since $\operatorname{Re}\zeta \geqslant -|\zeta|$, $|\zeta| = x \in [0,1]$, it follows that

$$\det\Big(T_{2,1}\Big(\frac{F_f}{2}\Big)\Big) \leqslant \frac{1}{4}c^2 - \frac{1}{64}(c^4 + (4-c^2)^2x^2 - 2c^2(4-c^2)x) =: \psi_1(c,x).$$

Now we will find $\max\psi_1$ over the region $[0,2]\times[0,1]$. For this we proceed as follows:

(C1) On the boundary of $[0,2]\times[0,1]$, it is easy to arrive at:

$$\psi_1(0,x)\leqslant -\frac{1}{4}x^2\leqslant 0,\quad \psi_1(2,x)\leqslant \frac{3}{4},$$
$$\psi_1(c,0)\leqslant \frac{1}{4}c^2-\frac{1}{64}c^4\leqslant\frac{3}{4},\quad \psi_1(c,1)\leqslant\frac{1}{16}(-c^4+8c^2-4)\leqslant\frac{3}{4}.$$

Here we note that the upper bounds on the right-hand side of the later two expressions uses the fact that they are increasing functions in $c\in[0,2]$.

We now consider the interior of the rectangle $[0,2]\times[0,1]$. Let $(c,x)\in(0,2)\times(0,1)$. Then the equations

$$\frac{\partial\psi_1(c,x)}{\partial x}=0\quad\text{and}\quad\frac{\partial\psi_1(c,x)}{\partial c}=0$$

have no simultaneous solution in the considered domain. Therefore,

$$\det\Bigl(T_{2,1}\Bigl(\frac{F_f}{2}\Bigr)\Bigr)\leqslant\max\Bigl\{0,\frac{3}{4},\frac{3}{4},\frac{3}{4},0\Bigr\}=\frac{3}{4}.$$

(C2) We now proceed to find the minimum for $\det(T_{2,1}(F_f/2))$. With the notation $c_1=:c\in[0,2]$ and the fact that $\operatorname{Re}\zeta\leqslant|\zeta|=x\in[0,1]$, we have

$$\det\Bigl(T_{2,1}\Bigl(\frac{F_f}{2}\Bigr)\Bigr)\geqslant\frac{1}{4}c^2-\frac{1}{64}(c^4+(4-c^2)^2x^2+2c^2(4-c^2)x)=:\psi_2(c,x).$$

Now we will find $\min\psi_2$ over the region $[0,2]\times[0,1]$. For this we proceed as follows:

On the boundary of $[0,2]\times[0,1]$, for $(c,x)\in[0,2]\times[0,1]$, it is easy to arrive at:

$$\psi_2(0,x)\geqslant-\frac{1}{4}x^2\geqslant-\frac{1}{4},\quad\psi_2(2,x)\geqslant\frac{3}{4},$$
$$\psi_2(c,0)\geqslant\frac{1}{4}c^2-\frac{1}{64}c^4\geqslant 0,\quad\psi_2(c,1)\geqslant\frac{1}{4}(c^2-1)\geqslant-\frac{1}{4}.$$

The lower bounds on the right-hand side of the later two expressions uses the fact that they are increasing functions in $c\in[0,2]$.

We now consider the inside of the rectangle $[0,2]\times[0,1]$. Let $(c,x)\in(0,2)\times(0,1)$. As before, here we find that the function $\psi_2(c,x)$ has no critical point inside the domain $(0,2)\times(0,1)$, and therefore, putting all the conclusions together, we conclude that

$$-\frac{1}{4}\leqslant\det\Bigl(T_{2,1}\Bigl(\frac{F_f}{2}\Bigr)\Bigr)\leqslant\frac{3}{4}.$$

The equality in the upper bound holds in the case of the Koebe function $k(z)=z/(1-z)^2$ and in the lower bound it holds for the function $\tilde{f}_2$ defined by

$$\tilde{f}_2(z)=z\exp\Bigl(\int_0^z\frac{q(t)-1}{t}\,\mathrm{d}t\Bigr),\quad q(z)=\frac{1+z^2}{1-z^2},\tag{2.10}$$

that is, for the function $\tilde{f}_2(z)=z+z^3+z^5\ldots$ This ends the proof. □

3. Difference of logarithmic and inverse coefficients

The following lemma is due to Sim and Thomas [23] and is going to play the main role in investigation of the results in this section.

Lemma 3.1 ([23], Proposition 1, p. 5). *Let B_1, B_2 and B_3 be numbers such that $B_1 \geqslant 0$, $B_2 \in \mathbb{C}$, $B_3 \in \mathbb{R}$ and $B_4 = |4B_2 + 2B_3|$. Let $p \in \mathcal{P}$ as in Lemma 1.1. Define $\Psi_+(c_1, c_2)$ and $\Psi_-(c_1, c_2)$ by*

$$\Psi_+(c_1, c_2) = |B_2 c_1^2 + B_3 c_2| - |B_1 c_1| \quad \textit{and} \quad \Psi_-(c_1, c_2) = -\Psi_+(c_1, c_2).$$

Then

$$\Psi_+(c_1, c_2) \leqslant \begin{cases} |4B_2 + 2B_3| - 2B_1 & \textit{if } |2B_2 + B_3| \geqslant |B_3| + B_1, \\ 2|B_3| & \textit{otherwise}, \end{cases}$$

$$\Psi_-(c_1, c_2) \leqslant \begin{cases} 2B_1 - B_4 & \textit{if } B_1 \geqslant 2|B_3| + B_4, \\ 2B_1\sqrt{\dfrac{2|B_3|}{B_4 + 2|B_3|}} & \textit{if } B_1^2 \leqslant 2|B_3|(B_4 + 2|B_3|), \\ 2|B_3| + \dfrac{B_1^2}{B_4 + 2|B_3|} & \textit{otherwise.} \end{cases}$$

Theorem 3.2. *Let $f \in \mathcal{S}^*$ be given by* (1.1) *and the corresponding inverse and logarithmic coefficients are given by* (2.5) *and* (2.9)*, respectively. Then the following bounds, except the lower bound in* (iv)*, are sharp.*

(i) $|A_2 - \gamma_1| \leqslant 3$ *and* $0 \leqslant |A_2| - |\gamma_1| \leqslant 1$,
(ii) $-\frac{\sqrt{6}}{6} \leqslant |A_3| - |\gamma_1| \leqslant 4$,
(iii) $-\frac{1}{2} \leqslant |A_2| - |\gamma_2| \leqslant \frac{3}{2}$,
(iv) $-\frac{3}{2} \leqslant |A_3| - |\gamma_2| \leqslant \frac{9}{2}$.

Proof. Let $f \in \mathcal{S}^*$. Then from (2.5) and (2.9) we have the following relationships:

$$A_2 = -a_2, \quad A_3 = 2a_2^2 - a_3, \quad \gamma_1 = \frac{a_2}{2}, \quad \gamma_2 = \frac{1}{2}\Big(a_3 - \frac{1}{2}a_2^2\Big)$$

and

$$\gamma_3 = \frac{1}{2}\Big(\frac{a_2^3}{3} - a_2 a_3 + a_4\Big).$$

(i) The fact that $|a_n| \leqslant 2$, $n = 2, 3, 4, \ldots$, straightforwardly results in the following inequalities

$$|A_2 - \gamma_1| = \frac{3}{2}|a_2| \leqslant 3 \quad \text{and} \quad |A_2| - |\gamma_1| = \frac{1}{2}|a_2| \leqslant 1.$$

The bounds are sharp in the case of the Koebe function $k(z) = z/(1-z)^2$.

(ii) Using (2.1), we find that

$$|A_3|-|\gamma_1| = |2a_2^2-a_3|-\frac{1}{2}|a_2| = \left|2c_1^2-\frac{1}{2}(c_2+c_1^2)\right|-\frac{1}{2}|c_1| = \frac{1}{2}(|B_2c_1^2+B_3c_2|-|B_1c_1|),$$

where $B_1 = 1$, $B_2 = 3$ and $B_3 = -1$. Now by Lemma 3.1, we obtain that

$$-\frac{\sqrt{6}}{6} \leqslant |A_3| - |\gamma_1| \leqslant 4.$$

The upper bound is sharp in the case of the Koebe function $k(z) = z/(1-z)^2$. However, the equality in the case of the lower bound holds for the function $\tilde{f}_3$ defined by

$$\tilde{f}_3(z) = z\exp\left(\int_0^z \frac{q(t)-1}{t}\,\mathrm{d}t\right), \quad q(z) = \frac{1+(2/\sqrt{6})z+z^2}{1-z^2},$$

which on simplification gives

$$\tilde{f}_3(z) = z + \sqrt{\frac{2}{3}}z^2 + \frac{4}{3}z^3 + \frac{13}{9}\sqrt{\frac{2}{3}}z^4 + \frac{85}{54}z^5 + \dots.$$

(iii) Proceeding as before, we find that

$$|A_2| - |\gamma_2| = |-a_2| - \left|\frac{1}{2}\Big(a_3 - \frac{1}{2}a_2^2\Big)\right| = \frac{1}{2}\Big(2|a_2| - |a_3 - \frac{1}{2}a_2^2|\Big) = |c_1| - \left|\frac{1}{4}c_2\right|.$$

Therefore, by using Lemma 3.1, we get

$$-\frac{1}{2} \leqslant |A_2| - |\gamma_2| \leqslant \frac{3}{2}.$$

The sharpness in the upper bound can be confirmed by taking the Koebe function $k(z) = z/(1-z)^2$ and the equality in the lower bound occurs in the case of the function $\tilde{f}_2$ defined by (2.10).

(iv) Now using (2.1) we find that

$$|A_3|-|\gamma_2| = |2a_2^2-a_3|-\left|\frac{1}{2}\Big(a_3-\frac{1}{2}a_2^2\Big)\right| \leqslant \frac{1}{2}\left|3c_1^2-\frac{3}{2}c_2\right| = \frac{1}{2}(|B_2c_1^2+B_3c_2|-|B_1c_1|),$$

where $B_1 = 0$, $B_2 = 3$ and $B_3 = -3/2$. Now by using Lemma 3.1, we obtain

$$|A_3| - |\gamma_2| \leqslant \frac{9}{2}.$$

For the lower bound we have

$$|A_3| - |\gamma_2| \geqslant \frac{1}{2}\Big(|3c_1^2 - c_2| - \frac{1}{2}|c_2|\Big) \geqslant \frac{3}{2}|c_1|^2 - \frac{3}{4}|c_2| \geqslant -\frac{3}{2}.$$

The sharpness in the upper bound can be confirmed by taking the Koebe function $k(z) = z/(1-z)^2$. This completes the proof. $\square$

R e m a r k 3.3. It would be interesting to evaluate how close γ_m and A_n are, that is, to evaluate the bounds on $|\gamma_m| - |A_n|$ and on $|\gamma_m - A_n|$.

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