Applications of Mathematics

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Applications of Mathematics, Vol. 68 (2023), No. 3, 305–327

Persistent URL: http://dml.cz/dmlcz/151656

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FINITE TIME STABILITY AND RELATIVE CONTROLLABILITY OF SECOND ORDER LINEAR DIFFERENTIAL SYSTEMS WITH PURE DELAY

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Received November 23, 2021. Published online November 30, 2022.

Abstract. We first consider the finite time stability of second order linear differential systems with pure delay via giving a number of properties of delayed matrix functions. We secondly give sufficient and necessary conditions to examine that a linear delay system is relatively controllable. Further, we apply the fixed-point theorem to derive a relatively controllable result for a semilinear system. Finally, some examples are presented to illustrate the validity of the main theorems.

Keywords: finite time stability; relative controllability; second order; delayed matrix function

MSC 2020: 34K05, 93C05

1. Introduction

A delay differential system is an impressive mathematical model and simultaneously provides an available tool to depict various processes in mechanical and technical systems. In addition, it is necessary to make use of such systems to model a number of phenomena in scientific and technological problems. During the past few decades, most previously published studies have focused on asymptotic stability,

The research has been supported by the National Natural Science Foundation of China (12161015;12201148), Guizhou Provincial Science and Technology Projects (No. QKHJC-ZK[2022]YB069), Training Object of High Level and Innovative Talents of Guizhou Province ((2016)4006), Guizhou Data Driven Modeling Learning and Optimization Innovation Team ([2020]5016), the Slovak Research and Development Agency under the Contract No. APVV-18-0308, and the Slovak Grant Agency VEGA No. 1/0358/20 and No. 2/0127/20.

DOI: 10.21136/AM.2022.0249-21

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control problems finite time stability, and the representation of solutions of linear and nonlinear or fractional order delay systems; see for examples [1]–[4], [7], [14], [16], [17], and the references cited therein.

Lyapunov stability, asymptotic stability, and exponential stability concerned with the behavior of systems within an infinite time interval have been extensively studied in [9]–[12], [19] with the help of linear matrix inequality and Gronwall's integral inequality. However, few researchers have been able to draw on systematic research into finite time stability (FTS) concerned with the behavior of systems over a finite intervals. The relative controllability and related problems of linear systems represented by different types of delay systems have been investigated in [8], [15], [20], [13].

In [7], the authors introduced the notations of delayed matrix cosine and delayed matrix sine for solving the following oscillating system with pure delay:

$$(1.1)\qquad \begin{cases} z''(x) + A^2 z(x-\vartheta) = g(x), & x \geqslant 0,\ z(x) \in \mathbb{R}^n, \\ z(x) = \psi(x),\ z'(x) = \psi'(x), & -\vartheta \leqslant x \leqslant 0, \end{cases}$$

providing $\det A \neq 0$. In [13], the authors introduced a delay Grammian matrix, which was used to establish sufficient and necessary conditions of controllability for (1.1).

Very recently, Elshenhab and Wang [3] dropped the invertible condition on the matrix A and studied the explicit representation of solutions of linear systems with pure delay:

$$(1.2)\qquad \begin{cases} z''(x) + A z(x-\vartheta) = g(x), & x \in J := [0,T],\ z(x) \in \mathbb{R}^n, \\ z(x) = \psi(x),\ z'(x) = \psi'(x), & -\vartheta \leqslant x \leqslant 0, \end{cases}$$

where $A \in \mathbb{R}^{n\times n}$ is a constant coefficients matrix, $g \in C(J, \mathbb{R}^n)$ is a given function, $T = k\vartheta$ for a fixed $k \in \mathbb{N} := \{0,1,2,\ldots\}$, ϑ is a fixed delay time, and $\psi \in C^2([-\vartheta,0], \mathbb{R}^n)$ determines initial conditions. The authors introduced the notation of delayed matrix functions $\mathcal{H}_\vartheta(A\cdot)$ and $\mathcal{M}_\vartheta(A\cdot)$ (see [4], Definition 1 or Definition 2.1) for (1.2). The solution z of (1.2) can be given by (see [4], Corollary 1 or 2):

$$(1.3)\qquad z(x) = \begin{cases} \mathcal{H}_\vartheta(Ax)\psi(-\vartheta) + \displaystyle\int_{-\vartheta}^{0} \mathcal{M}_\vartheta(A(x-\vartheta-s))\psi''(s)\,\mathrm{d}s \\ \qquad + \mathcal{M}_\vartheta(Ax)\psi'(-\vartheta) + \displaystyle\int_0^x \mathcal{M}_\vartheta(A(x-\vartheta-s))g(s)\,\mathrm{d}s, & x \in J, \\ \psi(x), \quad -\vartheta \leqslant x \leqslant 0. \end{cases}$$

Motivated by [3], we study the FTS of (1.2) and the relative controllability of control systems with delay governed by:

$$\begin{cases} z''(x) + Az(x-\vartheta) = Bu(x), & x \in J,\ z(x) \in \mathbb{R}^n, \\ z(x) = \psi(x),\ z'(x) = \psi'(x), & -\vartheta \leqslant x \leqslant 0, \end{cases} \tag{1.4}$$

and

$$\begin{cases} z''(x) + Az(x-\vartheta) = f(x, z(x)) + Bu(x), & x \in J,\ z(x) \in \mathbb{R}^n, \\ z(x) = \psi(x),\ z'(x) = \psi'(x), & -\vartheta \leqslant x \leqslant 0, \end{cases} \tag{1.5}$$

where $B \in \mathbb{R}^{n\times n}$, $f\colon\ J \times \mathbb{R}^n \to \mathbb{R}^n$ and $u \in L^2(J, \mathbb{R}^n)$.

The rest of the paper is organized as follows. Firstly, we give some sufficient conditions to guarantee that (1.2) is FTS by researching the estimation of $\mathcal{H}_\vartheta(A\cdot)$ and $\mathcal{M}_\vartheta(A\cdot)$. Secondly, we establish a delayed Grammian matrix and rank criterion to guarantee that (1.4) is relatively controllable. Further, we construct a suitable control function and apply Krasnoselskii's fixed point to derive relatively controllable of (1.5). Finally, some examples are presented to illustrate the validity of the main theorems.

2. Preliminaries

Let $\|z\| = \sum\limits_{i=1}^{n} |z_i|$ and $\|A\| = \max\limits_{1\leqslant j\leqslant n} \sum\limits_{i=1}^{n} |a_{ij}|$ be the vector norm and matrix norm, respectively. Denote by $C(J, \mathbb{R}^n)$ the Banach space of a vector-value continuous function from $J \to \mathbb{R}^n$ endowed with the norm $\|z\| = \max\limits_{s\in[0,T]} \|z(s)\|$. Let X, Y be the Banach spaces, $L_b(X, Y)$ denote the space of all bounded linear operators, and $L^2(J, Y)$ denote the Banach space of all the Bochner-integrable functions endowed with the norm $\|\cdot\|_{L^2(J,Y)}$. In addition, we let $\|\psi\|_C = \max\limits_{s\in[-\vartheta,0]} \|\psi(s)\|$, $\|\psi'\|_C = \max\limits_{s\in[-\vartheta,0]} \|\psi'(s)\|$ and $\|\psi''\|_C = \max\limits_{s\in[-\vartheta,0]} \|\psi''(s)\|$.

Definition 2.1 (see [4], Definition 1)**.** The delayed matrix function $\mathcal{H}_\vartheta(A\cdot)\colon \mathbb{R} \to \mathbb{R}^{n\times n}$ is given by

$$\mathcal{H}_\vartheta(Ax) = \begin{cases} \Theta, \quad -\infty < x < -\vartheta, \\ E, \quad -\vartheta \leqslant x < 0, \\ E - A\dfrac{x^2}{2!}, \quad 0 \leqslant x < \vartheta, \\ E - A\dfrac{x^2}{2!} + \ldots + (-1)^k A^k \dfrac{(x-(k-1)\vartheta)^{2k}}{(2k)!}, \\ (k-1)\vartheta \leqslant x < k\vartheta, \quad k \in \mathbb{N}, \end{cases} \tag{2.1}$$

and $\mathcal{M}_\vartheta(A\cdot)\colon\ \mathbb{R}\to\mathbb{R}^{n\times n}$ is given by

$$\mathcal{M}_\vartheta(Ax)=\begin{cases}\Theta, & -\infty<x<-\vartheta,\\ E(x+\vartheta), & -\vartheta\leqslant x<0,\\ E(x+\vartheta)-A\dfrac{x^3}{3!}, & 0\leqslant x<\vartheta,\\ E(x+\vartheta)-A\dfrac{x^3}{3!}+\ldots+(-1)^kA^k\dfrac{(x-(k-1)\vartheta)^{2k+1}}{(2k+1)!}, & \\ (k-1)\vartheta\leqslant x<k\vartheta, \quad k\in\mathbb{N}, & \end{cases} \tag{2.2}$$

where Θ and E denote the zero and identity matrices, respectively.

Definition 2.2 (see [9]). System (1.2) is FTS with respect to $\{0,J,\vartheta,\delta,\beta\}$ if and only if $\|\psi\|_C<\delta$ implies $\|z(x)\|<\beta$ for all $x\in[0,T]$, where $\delta<\beta$.

Definition 2.3. System (1.4) is called relatively controllable if for an arbitrary initial function $\psi\in C^2$, the terminal state $z_1\in\mathbb{R}^n$ with $z_1>0$, $\exists u^*\in L^2(J,\mathbb{R}^n)$ such that

$$\begin{cases} z''(x)+Az(x-\vartheta)=Bu^*(x), & x\in[0,x_1],\ x_1>0,\\ z(x)=\psi(x),\ z'(x)=\psi'(x), & -\vartheta\leqslant x\leqslant 0,\end{cases} \tag{2.3}$$

there exists a solution $z(x,u^*):=z^*(x)$ satisfying $z^*(x)=\psi(x)$, $-\vartheta\leqslant x\leqslant 0$, and $z^*(x_1)=z_1$.

Definition 2.4. System (1.5) is called relatively controllable if for an arbitrary initial function $\psi\in C^2$, the terminal state $z_1\in\mathbb{R}^n$, $\exists u\in L^2(J,\mathbb{R}^n)$ such that (1.5) there exists a solution $z(x,u):=z(x)$ satisfying $z(x)=\psi(x)$, $-\vartheta\leqslant x\leqslant 0$, and $z(x_1)=z_1$.

Lemma 2.1. *A solution of* (1.4) *has the following form:*

$$z(x)=\begin{cases}\mathcal{H}_\vartheta(Ax)\psi(-\vartheta)+\int_{-\vartheta}^0\mathcal{M}_\vartheta(A(x-\vartheta-s))\psi''(s)\,\mathrm{d}s & \\ \quad+\mathcal{M}_\vartheta(Ax)\psi'(-\vartheta)+\int_0^x\mathcal{M}_\vartheta(A(x-\vartheta-s))Bu(s)\,\mathrm{d}s, & x\geqslant 0,\\ \psi(x), & -\vartheta\leqslant x\leqslant 0,\end{cases} \tag{2.4}$$

Obviously, a solution of (1.5) *has the following form:*

$$z(x)=\begin{cases}\mathcal{H}_\vartheta(Ax)\psi(-\vartheta)+\displaystyle\int_{-\vartheta}^0\mathcal{M}_\vartheta(A(x-\vartheta-s))\psi''(s)\,\mathrm{d}s & \\ \quad+\mathcal{M}_\vartheta(Ax)\psi'(-\vartheta)+\displaystyle\int_0^x\mathcal{M}_\vartheta(A(x-\vartheta-s))f(s,z(s))\,\mathrm{d}s & \\ \quad+\displaystyle\int_0^x\mathcal{M}_\vartheta(A(x-\vartheta-s))Bu(s)\,\mathrm{d}s, & x\geqslant 0,\\ \psi(x), & -\vartheta\leqslant x\leqslant 0,\end{cases}$$

Lemma 2.2. *For any $x \in ((k-1)\vartheta, k\vartheta]$ and $k \in \mathbb{N}$, we have*

$$\|\mathcal{H}_\vartheta(Ax)\| \leqslant \cosh(\sqrt{\|A\|}x), \quad \|\mathcal{M}_\vartheta(Ax)\| \leqslant \Psi_1(x),$$

where

$$\Psi_1(x) = \begin{cases} \vartheta + \sum_{i=0}^{k} \|A\|^i \dfrac{x^{2i+1}}{(2i+1)!}, & \|A\| > 1, \\ \vartheta + \dfrac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x}) = \vartheta + \sinh(x), & 0 < \|A\| \leqslant 1. \end{cases}$$

Proof. By the formula of (2.1) and (2.2), we have

$$\begin{aligned} \|\mathcal{H}_\vartheta(Ax)\| &\leqslant 1 + \|A\|\frac{x^2}{2!} + \ldots + \|A\|^k \frac{(x-(k-1)\vartheta)^{2k}}{(2k)!} \\ &\leqslant 1 + \|A\|\frac{x^2}{2!} + \|A\|^2\frac{x^4}{4!} + \ldots + \|A\|^k\frac{x^{2k}}{(2k)!} \\ &\leqslant \frac{1}{2}(\mathrm{e}^{\sqrt{\|A\|}x} + \mathrm{e}^{-\sqrt{\|A\|}x}) = \cosh(\sqrt{\|A\|}x), \\ \|\mathcal{M}_\vartheta(Ax)\| &\leqslant (x+\vartheta) + \|A\|\frac{x^3}{3!} + \ldots + \|A\|^k\frac{(x-(k-1)\vartheta)^{2k+1}}{(2k+1)!} \\ &\leqslant (x+\vartheta) + \|A\|\frac{x^3}{3!} + \|A\|^2\frac{x^5}{5!} + \ldots + \|A\|^k\frac{x^{2k+1}}{(2k+1)!} \\ &\leqslant \begin{cases} \vartheta + \sum_{i=0}^{k} \|A\|^i \dfrac{x^{2i+1}}{(2i+1)!}, & \|A\| > 1, \\ \vartheta + \dfrac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x}) = \vartheta + \sinh(x), & 0 < \|A\| \leqslant 1. \end{cases} \end{aligned}$$

The proof is completed. □

Lemma 2.3. *For any $x \in ((k-1)\vartheta, k\vartheta]$ and $k \in \mathbb{N}$, we have*

$$\begin{aligned} \int_{-\vartheta}^{0} \|\mathcal{M}_\vartheta(A(x-\vartheta-s))\|\,\mathrm{d}s \leqslant &\sum_{j=1}^{k+1} \frac{\|A\|^{j-1}}{(2j)!}(x-(j-2)\vartheta)^{2j} \\ &- \sum_{j=1}^{k} \frac{\|A\|^{j-1}}{(2j)!}(x-(j-1)\vartheta)^{2j}. \end{aligned}$$

Proof. According to Definition 2.1 and $\mathcal{M}_\vartheta(A\cdot)$, we obtain

$$\begin{aligned}
\int_{-\vartheta}^{0} \|\mathcal{M}_\vartheta(A(x-\vartheta-s))\| \,\mathrm{d}s
&= \int_{-\vartheta}^{x-k\vartheta} \|\mathcal{M}_\vartheta(A(x-\vartheta-s))\| \,\mathrm{d}s + \int_{x-k\vartheta}^{0} \|\mathcal{M}_\vartheta(A(x-\vartheta-s))\| \,\mathrm{d}s \\
&\leqslant \sum_{j=1}^{k} \|A\|^{j-1} \int_{-\vartheta}^{0} \frac{(x-(j-1)\vartheta-s)^{2j-1}}{(2j-1)!} \,\mathrm{d}s \\
&\quad + \|A\|^{k} \int_{-\vartheta}^{x-k\vartheta} \frac{(x-k\vartheta-s)^{2k+1}}{(2k+1)!} \,\mathrm{d}s \\
&\leqslant \sum_{j=1}^{k+1} \frac{\|A\|^{j-1}}{(2j)!}(x-(j-2)\vartheta)^{2j} - \sum_{j=1}^{k} \frac{\|A\|^{j-1}}{(2j)!}(x-(j-1)\vartheta)^{2j}.
\end{aligned}$$

The proof is completed. □

Lemma 2.4. *For any $x \in ((k-1)\vartheta, k\vartheta]$ and $k \in \mathbb{N}$, we have*

$$\begin{aligned}
\int_{-\vartheta}^{0} \|\mathcal{M}_\vartheta(A(x-\vartheta-s))\| \|\psi''(s)\| \,\mathrm{d}s
&\leqslant \sum_{j=1}^{k} \frac{\|A\|^{j-1}}{(2j-1)!}(x-(j-1)\vartheta)^{2j-1} \int_{-\vartheta}^{0} \|\psi''(s)\| \,\mathrm{d}s \\
&\quad + \frac{\|A\|^{k}\|\psi''\|_C}{(2k+2)!}(x-(k-1)\vartheta)^{2k+2}.
\end{aligned}$$

Proof. By Lemma 2.3, we have

$$\begin{aligned}
\int_{-\vartheta}^{0} \|\mathcal{M}_\vartheta(A(x-\vartheta-s))\| \|\psi''(s)\| \,\mathrm{d}s
&\leqslant \sum_{j=1}^{k} \frac{\|A\|^{j-1}}{(2j-1)!}(x-(j-1)\vartheta)^{2j-1} \int_{-\vartheta}^{0} \|\psi''(s)\| \,\mathrm{d}s \\
&\quad + \|A\|^{k} \int_{-\vartheta}^{x-k\vartheta} \frac{(x-k\vartheta-s)^{2k+1}}{(2k+1)!} \|\psi''(s)\| \,\mathrm{d}s \\
&\leqslant \sum_{j=1}^{k} \frac{\|A\|^{j-1}}{(2j-1)!}(x-(j-1)\vartheta)^{2j-1} \int_{-\vartheta}^{0} \|\psi''(s)\| \,\mathrm{d}s \\
&\quad + \frac{\|A\|^{k}\|\psi''\|_C}{(2k+2)!}(x-(k-1)\vartheta)^{2k+2}.
\end{aligned}$$

The proof is completed. □

Define

$$\Psi_2(x) = \sum_{j=1}^{k+1} \frac{\|A\|^{j-1}}{(2j)!}(x-(j-2)\vartheta)^{2j} - \sum_{j=1}^{k} \frac{\|A\|^{j-1}}{(2j)!}(x-(j-1)\vartheta)^{2j},$$

$$\Psi_3(x) = \sum_{j=1}^{k} \frac{\|A\|^{j-1}}{(2j-1)!}(x-(j-1)\vartheta)^{2j-1} \int_{-\vartheta}^{0} \|\psi''(t)\|\,\mathrm{d}t + \frac{\|A\|^{k}\|\psi''\|_C}{(2k+2)!}(x-(k-1)\vartheta)^{2k+2}.$$

3.1. Finite time stability. Now we are ready to give the FTS results.

Theorem 3.1. *If*

$$\cosh(\sqrt{\|A\|}x)\|\psi(-\vartheta)\| + \Psi_1(x)\|\psi'(-\vartheta)\| + \Psi_2(x)\|\psi''\|_C + \|g\|\sum_{j=1}^{k}\frac{\|A\|^{j-1}}{(2j)!}(x-(j-1)\vartheta)^{2j} < \beta \quad \forall\, x \in J, \tag{3.1}$$

then (1.2) *is FTS with respect to* $\{0, J, \vartheta, \delta, \beta\}$.

Proof. By (1.3), we have

$$z(x) = \mathcal{H}_\vartheta(Ax)\psi(-\vartheta) + \mathcal{M}_\vartheta(Ax)\psi'(-\vartheta) + \int_{-\vartheta}^{0} \mathcal{M}_\vartheta(A(x-\vartheta-s))\psi''(s)\,\mathrm{d}s + \int_{0}^{x} \mathcal{M}_\vartheta(A(x-\vartheta-s))g(s)\,\mathrm{d}s. \tag{3.2}$$

By Lemmas 2.2 and 2.3 via (4.1), we have

$$\begin{aligned}\|z(x)\| &\leqslant \|\mathcal{H}_\vartheta(Ax)\|\|\psi(-\vartheta)\| + \int_{-\vartheta}^{0} \|\mathcal{M}_\vartheta(A(x-\vartheta-s))\|\|\psi''(s)\|\,\mathrm{d}s \\ &\quad + \|\mathcal{M}_\vartheta(Ax)\|\|\psi'(-\vartheta)\| + \int_{0}^{x} \|\mathcal{M}_\vartheta(A(x-\vartheta-s))\|\|g(s)\|\,\mathrm{d}s \\ &\leqslant \cosh(\sqrt{\|A\|}x)\|\psi(-\vartheta)\| + \Psi_1(x)\|\psi'(-\vartheta)\| + \Psi_2(x)\|\psi''\|_C \\ &\quad + \|g\|\sum_{j=1}^{k}\frac{\|A\|^{j-1}}{(2j)!}(x-(j-1)\vartheta)^{2j} \\ &< \beta.\end{aligned}$$

The proof is completed. □

Theorem 3.2. *If*

$$\cosh(\sqrt{\|A\|}x)\|\psi(-\vartheta)\| + \Psi_1(x)\|\psi'(-\vartheta)\| + \Psi_3(x) + \|g\|\sum_{j=1}^{k}\frac{\|A\|^{j-1}}{(2j)!}(x-(j-1)\vartheta)^{2j} < \beta \quad \forall\, x \in J, \tag{3.3}$$

then (1.2) *is FTS with respect to* $\{0, J, \vartheta, \delta, \beta\}$.

P r o o f. By Lemmas 2.2 and 2.4 via (4.3), we have

$$\begin{aligned}\|z(x)\| &\leqslant \|\mathcal{H}_\vartheta(Ax)\|\|\psi(-\vartheta)\| + \int_{-\vartheta}^{0}\|\mathcal{M}_\vartheta(A(x-\vartheta-s))\|\|\psi''(s)\|\,\mathrm{d}s \\ &\quad + \|\mathcal{M}_\vartheta(Ax)\|\|\psi'(-\vartheta)\| + \int_0^x \|\mathcal{M}_\vartheta(A(x-\vartheta-s))\|\|g(s)\|\,\mathrm{d}s \\ &\leqslant \cosh(\sqrt{\|A\|}x)\|\psi(-\vartheta)\| + \Psi_1(x)\|\psi'(-\vartheta)\| + \Psi_3(x) \\ &\quad + \|g\|\sum_{j=1}^{k}\frac{\|A\|^{j-1}}{(2j)!}(x-(j-1)\vartheta)^{2j} \\ &< \beta.\end{aligned}$$

The proof is completed. □

3.2. Relative controllability. We study the relative controllability of (1.4) with $\vartheta > 0$. We introduce the delayed Grammian matrix as follows:

$$W_\vartheta[-\vartheta, x_1] = \int_0^{x_1}\mathcal{M}_\vartheta(A(x_1-\vartheta-s))BB^\top\mathcal{M}_\vartheta(A^\top(x_1-\vartheta-s))\,\mathrm{d}s, \tag{3.4}$$

where $\top$ denotes the transpose of the matrix.

We now will give the delayed Grammian matrix criterion result.

Theorem 3.3. $W_\vartheta[-\vartheta, x_1]$ *is nonsingular matrix if and only if* (1.4) *is relatively controllable.*

P r o o f. *Sufficiency.* Since $W_\vartheta[-\vartheta, x_1]$ is a nonsingular matrix, which guarantees that $W_\vartheta^{-1}[-\vartheta, x_1]$ exists. For an arbitrary $z_1 \in \mathbb{R}^n$, one can select $u \in L^2(J, \mathbb{R}^n)$ such that

$$u(s) = B^\top\mathcal{M}_\vartheta(A^\top(x_1-\vartheta-s))W_\vartheta^{-1}[-\vartheta, x_1]\xi, \tag{3.5}$$

where

$$\xi = z_1 - \mathcal{H}_\vartheta(Ax_1)\psi(-\vartheta) - \mathcal{M}_\vartheta(Ax_1)\psi'(-\vartheta) - \int_{-\vartheta}^{0}\mathcal{M}_\vartheta(A(x_1-\vartheta-s))\psi''(s)\,\mathrm{d}s. \tag{3.6}$$

Inserting (3.5) in (2.4), we have:
(3.7)

$$z(x_1) = \mathcal{H}_\vartheta(Ax_1)\psi(-\vartheta) + \int_{-\vartheta}^{0} \mathcal{M}_\vartheta(A(x_1-\vartheta-s))\psi''(s)\,\mathrm{d}s + \mathcal{M}_\vartheta(Ax_1)\psi'(-\vartheta)$$
$$+ \int_0^{x_1} \mathcal{M}_\vartheta(A(x_1-\vartheta-s))BB^\top \mathcal{M}_\vartheta(A^\top(x_1-\vartheta-s))\,\mathrm{d}s W_\vartheta^{-1}[-\vartheta,x_1]\xi.$$

Linking (3.4) and (3.7) via (3.6), we obtain:

$$\begin{aligned} z(x_1) &= \mathcal{H}_\vartheta(Ax_1)\psi(-\vartheta) + \mathcal{M}_\vartheta(Ax_1)\psi'(-\vartheta) + \int_{-\vartheta}^{0} \mathcal{M}_\vartheta(A(x_1-\vartheta-s))\psi''(s)\,\mathrm{d}s + \xi \\ &= z_1. \end{aligned}$$

According to Definition 2.3, the system (1.4) is relatively controllable.

Necessity. Suppose $W_\vartheta[-\vartheta, x_1]$ is singular. There exists at least one nonzero state $\widehat{z} \in \mathbb{R}^n$ such that $\widehat{z}^\top W_\vartheta[-\vartheta, x_1]\widehat{z} = 0$. Therefore, we have

$$\begin{aligned} 0 = \widehat{z}^\top W_\vartheta[-\vartheta, x_1]\widehat{z} &= \int_0^{x_1} \widehat{z}^\top \mathcal{M}_\vartheta(A(x_1-\vartheta-s))BB^\top \mathcal{M}_\vartheta(A^\top(x_1-\vartheta-s))\widehat{z}\,\mathrm{d}s \\ &= \int_0^{x_1} (\widehat{z}^\top \mathcal{M}_\vartheta(A(x_1-\vartheta-s))B)(\widehat{z}^\top \mathcal{M}_\vartheta(A(x_1-\vartheta-s))B)^\top\,\mathrm{d}s \\ &= \int_0^{x_1} \|\widehat{z}^\top \mathcal{M}_\vartheta(A(x_1-\vartheta-s))B\|^2\,\mathrm{d}s, \end{aligned}$$

which implies that

$$\widehat{z}^\top \mathcal{M}_\vartheta(A(x_1-\vartheta-s))B = (\underbrace{0,\dots,0}_{n}) := \mathbf{0}^\top \quad \forall\, s \in [0, x_1]. \tag{3.8}$$

Note that (1.4) is relatively controllable. Therefore, there exists $\widetilde{u} \in L^2(J, \mathbb{R}^n)$ that drives the initial state to zero at x_1, i.e.,

$$\begin{aligned} z(x_1) &= \mathcal{H}_\vartheta(Ax_1)\psi(-\vartheta) + \int_{-\vartheta}^{0} \mathcal{M}_\vartheta(A(x_1-\vartheta-s))\psi''(s)\,\mathrm{d}s \\ &\quad + \mathcal{M}_\vartheta(Ax_1)\psi'(-\vartheta) + \int_0^{x_1} \mathcal{M}_\vartheta(A(x_1-\vartheta-s))B\widetilde{u}(s)\,\mathrm{d}s \\ &= \mathbf{0}. \end{aligned} \tag{3.9}$$

There exists $\widehat{u} \in L^2(J, \mathbb{R}^n)$ that drives the initial state to $\widehat{z} \neq \mathbf{0}$ at x_1, i.e.,

$$\begin{aligned} z(x_1) &= \mathcal{H}_\vartheta(Ax_1)\psi(-\vartheta) + \int_{-\vartheta}^{0} \mathcal{M}_\vartheta(A(x_1-\vartheta-s))\psi''(s)\,\mathrm{d}s \\ &\quad + \mathcal{M}_\vartheta(Ax_1)\psi'(-\vartheta) + \int_0^{x_1} \mathcal{M}_\vartheta(A(x_1-\vartheta-s))B\widehat{u}(s)\,\mathrm{d}s \\ &= \widehat{z}. \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10) gives

$$\widehat{z} = \int_0^{x_1} \mathcal{M}_\vartheta(A(x_1 - \vartheta - s))B(\widetilde{u}(s) - \widehat{u}(s))\,\mathrm{d}s. \tag{3.11}$$

Multiplying by $\widehat{z}^\top$ of (3.11), we get

$$\widehat{z}^\top \widehat{z} = \int_0^{x_1} \widehat{z}^\top \mathcal{M}_\vartheta(A(x_1 - \vartheta - s))B(\widetilde{u}(s) - \widehat{u}(s))\,\mathrm{d}s.$$

By (3.8), one can obtain $\widehat{z}^\top \widehat{z} = 0$, which conflicts with $\widehat{z} \neq \mathbf{0}$. Thus $W_\vartheta[-\vartheta, x_1]$ is nonsingular. The proof is completed. □

We next will establish the rank criterion result.

Theorem 3.4. *System* (1.4) *is relatively controllable if and only if* $x_1 > (n-1)\vartheta$ *and* $\operatorname{rank} S_n = n$*, where* $S_n = [B, AB, A^2B, \dots, A^{n-1}B]$.

Proof. *Sufficiency.* Let (1.4) be not relatively controllable and $\operatorname{rank} S_n = n$. By Theorem 3.3, $W_\vartheta[-\vartheta, x_1]$ is singular, i.e., there at least exists one nonzero state $\overline{z} \in \mathbb{R}^n$ such that

$$\begin{aligned} 0 = \overline{z}^\top W_\vartheta[-\vartheta, x_1]\overline{z} &= \int_0^{x_1} \overline{z}^\top \mathcal{M}_\vartheta(A(x_1 - \vartheta - s))BB^\top \mathcal{M}_\vartheta(A^\top(x_1 - \vartheta - s))\overline{z}\,\mathrm{d}s \\ &= \int_0^{x_1} (\overline{z}^\top \mathcal{M}_\vartheta(A(x_1 - \vartheta - s))B)(\overline{z}^\top \mathcal{M}_\vartheta(A(x_1 - \vartheta - s))B)^\top\,\mathrm{d}s, \end{aligned}$$

which implies that

$$\overline{z}^\top \mathcal{M}_\vartheta(A(x_1 - \vartheta - s))B = \mathbf{0} \quad \forall\, s \in [0, x_1].$$

Let $x = x_1 - \vartheta - s$ and we have

$$\overline{z}^\top \mathcal{M}_\vartheta(Ax)B = \mathbf{0} \quad \forall\, x \in [-\vartheta, x_1 - \vartheta]. \tag{3.12}$$

By (3.12), taking the derivative $2n$ times and then taking $x = n\vartheta$, we have

$$\overline{z}^\top B = \mathbf{0}, \quad \overline{z}^\top AB = \mathbf{0}, \quad \overline{z}^\top A^2B = \mathbf{0}, \dots, \overline{z}^\top A^{n-1}B = \mathbf{0},$$

that is $\overline{z}^\top[B, AB, A^2B, \dots, A^{n-1}B] = \mathbf{0}$. By using $\overline{z} \neq \mathbf{0}$, we can know that the $\operatorname{rank} S_n < n$. So, (1.4) is relatively controllable.

Necessity. Assume (1.4) is relatively controllable, namely, for arbitrary $\psi(\cdot)$, z_1 and x_1, there exists a control function $u^*(\cdot)$ such that (2.3) has a solution $z(x, u^*) := z^*(x)$ that satisfies $z^*(x) = \psi(x)$, $-\vartheta \leqslant x \leqslant 0$, and $z^*(x_1) = z_1$.

By Lemma 2.1, a solution of (1.4) can be given by

$$z(x)=\begin{cases}\mathcal{H}_\vartheta(Ax)\psi(-\vartheta)+\displaystyle\int_{-\vartheta}^{0}\mathcal{M}_\vartheta(A(x-\vartheta-s))\psi''(s)\,\mathrm{d}s\\ \qquad +\mathcal{M}_\vartheta(Ax)\psi'(-\vartheta)+\int_0^x\mathcal{M}_\vartheta(A(x-\vartheta-s))Bu(s)\,\mathrm{d}s, \quad x\in J,\\ \psi(x), \quad -\vartheta\leqslant x\leqslant 0.\end{cases}$$

Letting $u(\cdot)=u^*(\cdot)$ and $x=x_1$, we have

$$\begin{aligned}z_1=\mathcal{H}_\vartheta(Ax_1)\psi(-\vartheta)&+\int_{-\vartheta}^{0}\mathcal{M}_\vartheta(A(x_1-\vartheta-s))\psi''(s)\,\mathrm{d}s\\ &+\mathcal{M}_\vartheta(Ax_1)\psi'(-\vartheta)+\int_0^{x_1}\mathcal{M}_\vartheta(A(x_1-\vartheta-s))Bu^*(s)\,\mathrm{d}s.\end{aligned}\tag{3.13}$$

Let ξ satisfy the following equation:

$$\xi=z_1-\mathcal{H}_\vartheta(Ax_1)\psi(-\vartheta)-\mathcal{M}_\vartheta(Ax_1)\psi'(-\vartheta)-\int_{-\vartheta}^{0}\mathcal{M}_\vartheta(A(x_1-\vartheta-s))\psi''(s)\,\mathrm{d}s.\tag{3.14}$$

For $x_1\in((k-1)\vartheta,k\vartheta]$ and $k\in\mathbb{N}^+:=\{1,2,\dots\}$, one can get

$$\begin{aligned}&\int_0^{x_1}\mathcal{M}_\vartheta(A(x_1-\vartheta-s))Bu^*(s)\,\mathrm{d}s\\ &\quad=\int_0^{x_1-(k-1)\vartheta}\Big(E(x_1-s)-A\frac{(x_1-\vartheta-s)^3}{3!}+A^2\frac{(x_1-2\vartheta-s)^5}{5!}+\dots\\ &\qquad+(-1)^{k-1}A^{k-1}\frac{(x_1-(k-1)\vartheta-s)^{2k-1}}{(2k-1)!}\Big)Bu^*(s)\,\mathrm{d}s\\ &\qquad+\int_{x_1-(k-1)\vartheta}^{x_1-(k-2)\vartheta}\Big(E(x_1-s)-A\frac{(x_1-\vartheta-s)^3}{3!}+\dots\\ &\qquad+(-1)^{k-2}A^{k-2}\frac{(x_1-(k-2)\vartheta-s)^{2k-3}}{(2k-3)!}\Big)Bu^*(s)\,\mathrm{d}s\\ &\qquad+\int_{x_1-2\vartheta}^{x_1-\vartheta}\Big(E(x_1-s)-A\frac{(x_1-\vartheta-s)^3}{3!}\Big)Bu^*(s)\,\mathrm{d}s\\ &\qquad+\int_{x_1-\vartheta}^{x_1}E(x_1-s)Bu^*(s)\,\mathrm{d}s\\ &\quad=B\int_0^{x_1}(x_1-s)u^*(s)\,\mathrm{d}s-AB\int_0^{x_1-\vartheta}\frac{(x_1-\vartheta-s)^3}{3!}u^*(s)\,\mathrm{d}s+\dots\\ &\qquad+(-1)^{k-1}A^{k-1}B\int_0^{x_1-(k-1)\vartheta}\frac{(x_1-(k-1)\vartheta-s)^{2k-1}}{(2k-1)!}u^*(s)\,\mathrm{d}s.\end{aligned}$$

Set

$$\Phi_1(x_1) = \int_0^{x_1} (x_1 - s)u^*(s)\,\mathrm{d}s,$$
$$\Phi_2(x_1) = \int_0^{x_1-\vartheta} \frac{(x_1 - \vartheta - s)^3}{3!} u^*(s)\,\mathrm{d}s,$$
$$\vdots$$
$$\Phi_k(x_1) = \int_0^{x_1-(k-1)\vartheta} \frac{(x_1 - (k-1)\vartheta - s)^{2k-1}}{(2k-1)!} u^*(s)\,\mathrm{d}s.$$

Noting (3.13) and (3.14), we have

$$B\Phi_1(x_1) - AB\Phi_2(x_1) + \ldots + (-1)^{k-1}A^{k-1}B\Phi_k(x_1) = \xi. \tag{3.15}$$

Because (1.4) is relatively controllable, (3.15) has a solution for an arbitrary ξ. It follows from the Cayley-Hamilton formula that an arbitrary power A^j, $j \geqslant n$ can be expressed as a linear combination of E, A,…, A^{n-1} in [6]. Therefore, for any $k \geqslant n$, (3.15) can be replaced by

$$B\widetilde{\Phi}_1(x_1) - AB\widetilde{\Phi}_2(x_1) + \ldots + (-1)^{n-1}A^{n-1}B\widetilde{\Phi}_n(x_1) = \xi, \tag{3.16}$$

where $\widetilde{\Phi}_i(x_1)$ with $i = 1, 2, \ldots, n$ is a function of x_1. If (3.16) has a unique solution for an arbitrary ξ, this implies that $\operatorname{rank} S_n = n$. The proof is completed. □

4. Relative controllability of semilinear systems

We need the following hypothesis:

(H_1) The operator $\Lambda\colon L^2(J, \mathbb{R}^n) \to \mathbb{R}^n$ defined by

$$\Lambda u = \int_0^{x_1} \mathcal{M}_\vartheta(A(x_1 - \vartheta - s))Bu(s)\,\mathrm{d}s$$

has an inverse operator Λ^{-1}, which takes values in $L^2(J, \mathbb{R}^n) \setminus \ker \Lambda$. Then there exists a constant $M > 0$ such that

$$M = \|\Lambda^{-1}\|_{L_b(\mathbb{R}^n, L^2(J,\mathbb{R}^n)\setminus\ker\Lambda)}.$$

Obvious, the operator Λ must be surjective to satisfy (H_1) (see [5], [18]). If Λ is surjective, then we can define $\Lambda^{-1}\colon \mathbb{R}^n \to L^2(J, \mathbb{R}^n) \setminus \ker \Lambda$. Let $(\cdot, \cdot)$ denote the Euclidean scalar product in $\mathbb{R}^n$. Since $L^2(J, \mathbb{R}^n)$ is a Hilbert space, we can use $\ker \Lambda = \operatorname{im} \Lambda^{*\perp}$. We need to look for Λ^*; let $\Gamma(s) = \mathcal{M}_\vartheta(A(x_1 - \vartheta - s))B$, and for an arbitrary $w \in \mathbb{R}^n$ and $u \in L^2(J, \mathbb{R}^n)$, if we have

$$(\Lambda u, w) = \left(\int_0^{x_1} \Gamma(s)u(s)\,\mathrm{d}s, w\right) = \int_0^{x_1} (u(s), \Gamma(s)^\top w)\,\mathrm{d}s,$$

which gives $\Lambda^* w = \Gamma(s)^\top w$. Thus $\ker \Lambda^* = \{\mathbf{0}\}$ if and only if

$$\int_0^{x_1} \|\Gamma(s)^\top w\|^2 \,\mathrm{d}s \neq 0 \quad \text{for any } \mathbf{0} \neq w \in \mathbb{R}^n.$$

By (3.4), if we have

$$\begin{aligned} \int_0^{x_1} \|\Gamma(s)^\top w\|^2 \,\mathrm{d}s &= \int_0^{x_1} (\Gamma(s)^\top w, \Gamma(s)^\top w) \,\mathrm{d}s \\ &= \int_0^{x_1} (\Gamma(s)\Gamma(s)^\top w, w) \,\mathrm{d}s = (W_\vartheta[-\vartheta, x_1]w, w), \end{aligned} \tag{4.1}$$

then the surjectivity of Λ is equivalent to the regularity of $W_\vartheta[-\vartheta, x_1]$, and we assume this.

To solve $\Lambda u = v$, $v \in \ker \Lambda^\top = \operatorname{im} \Lambda^*$, we take $u(s) = \Gamma(s)^\top w$ and we have

$$v = \Lambda(\Gamma(\cdot)^\top w) = \int_0^{x_1} \Gamma(s)\Gamma(s)^\top w \,\mathrm{d}s = W_\vartheta[-\vartheta, x_1]w,$$

which gives $w = W_\vartheta[-\vartheta, x_1]^{-1} v$, and $u(t) = \Lambda^{-1} v = \Gamma(t)^\top W_\vartheta[-\vartheta, x_1]^{-1} v$. Moreover, By (4.1), we have

$$\begin{aligned} \int_0^{x_1} \|u(s)\|^2 \,\mathrm{d}s &= \int_0^{x_1} \|\Gamma(s)^\top W_\vartheta[-\vartheta, x_1]^{-1} v\|^2 \,\mathrm{d}s \\ &= \int_0^{x_1} (\Gamma(s)^\top W_\vartheta[-\vartheta, x_1]^{-1} v, \Gamma(s)^\top W_\vartheta[-\vartheta, x_1]^{-1} v) \,\mathrm{d}s \\ &= \Big((W_\vartheta[-\vartheta, x_1]^{-1})^\top \int_0^{x_1} \Gamma(s)\Gamma(s)^\top \,\mathrm{d}s W_\vartheta[-\vartheta, x_1]^{-1} v, v \Big) \\ &= ((W_\vartheta[-\vartheta, x_1]^{-1})^\top v, v) = (v, W_\vartheta[-\vartheta, x_1]^{-1} v), \end{aligned}$$

which implies that

$$M = \|W_\vartheta[-\vartheta, x_1]^{-1}\|^{1/2}. \tag{4.2}$$

Note that by (4.1), we obtain $\|\Lambda\| = \|\Lambda^*\| = \|W_\vartheta[-\vartheta, x_1]\|^{1/2}$.

(H_2) The function $f\colon J \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and there exists $\varphi \in C(J, \mathbb{R}^+)$ such that

$$\|f(x, z_1) - f(x, z_2)\| \leqslant \varphi(x)\|z_1 - z_2\|, z_1, z_2 \in \mathbb{R}^n, \quad x \in J.$$

According to the hypothesis (H_1), for an arbitrary $z(\cdot) \in C(J, \mathbb{R}^n)$, the control function $u_z(x)$ be given by

$$\begin{aligned} u_z(x) = \Lambda^{-1}\Big(& z_1 - \mathcal{H}_\vartheta(Ax_1)\psi(-\vartheta) - \int_{-\vartheta}^0 \mathcal{M}_\vartheta(A(x_1 - \vartheta - s))\psi''(s) \,\mathrm{d}s \\ & - \mathcal{M}_\vartheta(Ax_1)\psi'(-\vartheta) - \int_0^{x_1} \mathcal{M}_\vartheta(A(x_1 - \vartheta - s)) f(s, z(s)) \,\mathrm{d}s \Big)(x). \end{aligned} \tag{4.3}$$

Next, we apply the fixed-point theorem to prove relative controllability of (1.5). Using (4.3), we will show that for the operator $\mathcal{F}\colon C(J,\mathbb{R}^n)\to C(J,\mathbb{R}^n)$ given by

$$
\begin{aligned}
(4.4)\qquad (\mathcal{F}z)(x) &= \mathcal{H}_\vartheta(Ax)\psi(-\vartheta)+\int_{-\vartheta}^{0}\mathcal{M}_\vartheta(A(x-\vartheta-s))\psi''(s)\,\mathrm{d}s\\
&\quad+\mathcal{M}_\vartheta(Ax)\psi'(-\vartheta)+\int_0^x\mathcal{M}_\vartheta(A(x-\vartheta-s))f(s,z(s))\,\mathrm{d}s\\
&\quad+\int_0^x\mathcal{M}_\vartheta(A(x-\vartheta-s))Bu_z(s)\,\mathrm{d}s
\end{aligned}
$$

there exists a fixed point $z(\cdot)$, which is just a solution of (1.5). Furthermore, we check $(\mathcal{F}z)(x_1)=z_1$ and $(\mathcal{F}z)(0)=\psi(0)$, which implies that the system (1.5) is relatively controllable on $[0,x_1]$. Define

$$
\begin{aligned}
M_1 &= \cosh(\sqrt{\|A\|}x_1)\|\psi(-\vartheta)\|+\Psi_1(x_1)\|\psi'(-\vartheta)\|+\Psi_2(x_1)\|\psi''\|_C,\\
M_2 &= \sum_{j=1}^{k}\frac{\|A\|^{j-1}}{(2j)!}(x_1-(j-1)\vartheta)^{2j};\quad \|\hat f\|=\sup_{x\in J}\|f(x,\mathbf{0})\|,\quad \widehat{M}=\sup_{x\in J}\varphi(x).
\end{aligned}
$$

Theorem 4.1. *Assume that* (H_1) *and* (H_2) *are satisfied. Then system* (1.5) *is relatively controllable provided that*

$$
(4.5)\qquad M\widehat{M}M_2^2\|B\|+\widehat{M}M_2<1.
$$

P r o o f. Consider $\mathcal{F}$ defined in (4.4) on $\mathfrak{B}_r$, where $\mathfrak{B}_r=\{z\in C(J,\mathbb{R}^n)\colon\ \|z\|\leqslant r\}$ and $r>0$. We divide our proof into several steps.

Step 1. We show that $\mathcal{F}(\mathfrak{B}_r)\subseteq\mathfrak{B}_r$. For any $z\in\mathfrak{B}_r$, by Lemmas 2.2 and 2.3, we have

$$
\begin{aligned}
(4.6)\qquad &\int_0^x\|\mathcal{M}_\vartheta(A(x-\vartheta-s))\|(\|f(s,z(s))-f(s,\mathbf{0})\|+\|f(s,\mathbf{0})\|)\,\mathrm{d}s\\
&\leqslant(\widehat{M}\|z\|+\|\hat f\|)\int_0^x\|\mathcal{M}_\vartheta(A(x-\vartheta-s))\|\,\mathrm{d}s\leqslant(\widehat{M}\|z\|+\|\hat f\|)M_2,
\end{aligned}
$$

and

(4.7)

$$
\begin{aligned}
&\|u_z\|_{L^2(J,\mathbb{R}^n)\setminus\ker\Lambda}\\
&\leqslant\|\Lambda^{-1}\|_{L_b(\mathbb{R}^n,L^2(J,\mathbb{R}^n)\setminus\ker\Lambda)}\bigg\|z_1-\mathcal{H}_\vartheta(Ax_1)\psi(-\vartheta)-\mathcal{M}_\vartheta(Ax_1)\psi'(-\vartheta)\\
&\quad-\int_{-\vartheta}^{0}\mathcal{M}_\vartheta(A(x_1-\vartheta-s))\psi''(s)\,\mathrm{d}s-\int_0^{x_1}\mathcal{M}_\vartheta(A(x_1-\vartheta-s))f(s,z(s))\,\mathrm{d}s\bigg\|\\
&\leqslant M(\|z_1\|+\cosh(\sqrt{\|A\|}x_1)\|\psi(-\vartheta)\|+\Psi_1(x_1)\|\psi'(-\vartheta)\|\\
&\quad+\Psi_2(x_1)\|\psi''\|_C+(\widehat{M}\|z\|+\|\hat f\|)M_2).
\end{aligned}
$$

Noting (4.6) and (4.7), we have

$$
\begin{aligned}
(4.8)\quad \|(\mathcal{F}z)(x)\| &\leqslant \|\mathcal{H}_\vartheta(Ax)\psi(-\vartheta)\| + \int_{-\vartheta}^{0} \|\mathcal{M}_\vartheta(A(x-\vartheta-s))\psi''(s)\|\,\mathrm{d}s \\
&\quad + \|\mathcal{M}_\vartheta(Ax)\psi'(-\vartheta)\| + \int_0^x \|\mathcal{M}_\vartheta(A(x-\vartheta-s))f(s,z(s))\|\,\mathrm{d}s \\
&\quad + \int_0^x \|\mathcal{M}_\vartheta(A(x-\vartheta-s))Bu_z(s)\|\,\mathrm{d}s \\
&\leqslant \cosh(\sqrt{\|A\|}x)\|\psi(-\vartheta)\| + \Psi_1(x)\|\psi'(-\vartheta)\| + \Psi_2(x)\|\psi''\|_C \\
&\quad + \int_0^x \|\mathcal{M}_\vartheta(A(x-\vartheta-s))\|(\|f(s,z(s)) - f(s,\mathbf{0})\| + \|f(s,\mathbf{0})\|)\,\mathrm{d}s \\
&\quad + \int_0^x \|\mathcal{M}_\vartheta(A(x-\vartheta-s))\|\|B\|\|u_z(s)\|\,\mathrm{d}s \\
&\leqslant M_1 + \|\hat{f}\|M_2 + MM_2\|B\|(\|z_1\| + M_1 + M_2\|\hat{f}\|) \\
&\quad + (MM_2^2\widehat{M}\|B\| + \widehat{M}M_2)r \\
&\leqslant r,
\end{aligned}
$$

where

$$
r = \frac{M_1 + \|\hat{f}\|M_2 + MM_2\|B\|(\|z_1\| + M_1 + M_2\|\hat{f}\|)}{1 - MM_2^2\widehat{M}\|B\| + \widehat{M}M_2}.
$$

So $\mathcal{F}(\mathfrak{B}_r) \subseteq \mathfrak{B}_r$ for $x \in J$.

Next, we subdivide the operator $\mathcal{F}$ into two operators $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathfrak{B}_r$ as follows:

$$
\begin{aligned}
(\mathcal{F}_1 z)(x) &= \mathcal{H}_\vartheta(Ax)\psi(-\vartheta) + \int_{-\vartheta}^{0} \mathcal{M}_\vartheta(A(x-\vartheta-s))\psi''(s)\,\mathrm{d}s \\
&\quad + \mathcal{M}_\vartheta(Ax)\psi'(-\vartheta) + \int_0^x \mathcal{M}_\vartheta(A(x-\vartheta-s))Bu_z(s)\,\mathrm{d}s, \quad x \in J, \\
(\mathcal{F}_2 z)(x) &= \int_0^x \mathcal{M}_\vartheta(A(x-\vartheta-s))f(s,z(s))\,\mathrm{d}s, \quad x \in J.
\end{aligned}
$$

Step 2. We show that $\mathcal{F}_1$ is a contraction mapping. In viewing of (H_1) and (H_2), for any $\widetilde{z}(\cdot), \widehat{z}(\cdot) \in \mathfrak{B}_r$ and $x \in J$, we have

$$
\begin{aligned}
&\|u_{\widehat{z}} - u_{\widetilde{z}}\|_{L^2(J,\mathbb{R}^n)\setminus\ker\Lambda} \\
&\leqslant \|\Lambda^{-1}\|_{L_b(\mathbb{R}^n, L^2(J,\mathbb{R}^n)\setminus\ker\Lambda)} \left\| \int_0^{x_1} \mathcal{M}_\vartheta(A(x_1-\vartheta-s))(f(s,\widehat{z}(s)) - f(s,\widetilde{z}(s)))\,\mathrm{d}s \right\| \\
&\leqslant M \int_0^{x_1} \|\mathcal{M}_\vartheta(A(x_1-\vartheta-s))\|\varphi(s)\|\widehat{z}(s) - \widetilde{z}(s)\|\,\mathrm{d}s \leqslant MM_2\widehat{M}\|\widehat{z} - \widetilde{z}\|,
\end{aligned}
$$

and

$$\begin{aligned}\|(\mathcal{F}_1\widehat{z})(x)-(\mathcal{F}_1\widetilde{z})(x)\| &\leqslant \int_0^x \|\mathcal{M}_\vartheta(A(x-\vartheta-s))\|B\|\|u_{\widehat{z}}(s)-u_{\widetilde{z}}(s)\|\,\mathrm{d}s\\ &\leqslant M\widehat{M}M_2^2\|B\|\|\widehat{z}-\widetilde{z}\|,\end{aligned}$$

which gives that

$$\|(\mathcal{F}_1\widehat{z})-(\mathcal{F}_1\widetilde{z})\|\leqslant L\|\widehat{z}-\widetilde{z}\|,\quad L:=M\widehat{M}M_2^2\|B\|.$$

By (4.5), the operator $\mathcal{F}_1$ is a contraction.

Step 3. We show that $\mathcal{F}_2$ is a continuous and compact operator. Owing to the function $f\in C(J,\mathbb{R}^n)$, the operator $\mathcal{F}_2$ is continuous on $\mathfrak{B}_r$. In order to check the compactness of $\mathcal{F}_2$, we prove that $\mathcal{F}_2(\mathfrak{B}_r)$ is uniformly bounded and equicontinuous. By Step 1, it follows that $\mathcal{F}_2(\mathfrak{B}_r)$ is uniformly bounded.

For any $z\in\mathfrak{B}_r$ and $0<x\leqslant x+h\leqslant x_1$, we have

$$\begin{aligned}&\|(\mathcal{F}_2z)(x+h)-(\mathcal{F}_2z)(x)\|\\ &=\left\|\int_0^{x+h}\mathcal{M}_\vartheta(A(x+h-\vartheta-s))f(s,z(s))\,\mathrm{d}s-\int_0^x\mathcal{M}_\vartheta(A(x-\vartheta-s))f(s,z(s))\,\mathrm{d}s\right\|\\ &\leqslant I_1+I_2,\end{aligned}$$

where

$$\begin{aligned}I_1&=\int_0^x\|(\mathcal{M}_\vartheta(A(x+h-\vartheta-s))-\mathcal{M}_\vartheta(A(x-\vartheta-s)))\|\|f(s,z(s))\|\,\mathrm{d}s,\\ I_2&=\int_x^{x+h}\|\mathcal{M}_\vartheta(A(x+h-\vartheta-s))\|\|f(s,z(s))\|\,\mathrm{d}s.\end{aligned}$$

Without loss of generality, letting $(k-1)\vartheta<x\leqslant x+h<k\vartheta$ as $h\to0$ and $k\in\mathbb{N}^+$, we obtain

$$\begin{aligned}I_1&\leqslant\int_0^x\|(\mathcal{M}_\vartheta(A(x+h-\vartheta-s))-\mathcal{M}_\vartheta(A(x-\vartheta-s)))\|\|f(s,z(s))\|\,\mathrm{d}s\\ &\leqslant(\widehat{M}r+\|\hat{f}\|)\int_0^x\|(\mathcal{M}_\vartheta(A(x+h-\vartheta-s))-\mathcal{M}_\vartheta(A(x-\vartheta-s)))\|\,\mathrm{d}s.\end{aligned}$$

Letting $h\to0$ and taking any fixed point ϑ, we have

$$\mathcal{M}_\vartheta(A(x+h-\vartheta-s))\to\mathcal{M}_\vartheta(A(x-\vartheta-s))\quad\text{as }h\to0,$$

which implies that $I_1\to0$ as $h\to0$.

For I_2, by Lemma 2.2, we get

$$\begin{aligned}I_2&\leqslant(\widehat{M}r+\|\hat{f}\|)\int_x^{x+h}\|\mathcal{M}_\vartheta(A(x+h-\vartheta-s))\|\,\mathrm{d}s\\ &\leqslant(\widehat{M}r+\|\hat{f}\|)\Psi_1(x_1)h\to0\quad\text{as }h\to0.\end{aligned}$$

From the above we obtain $\|(\mathcal{F}_2 z)(x+h)-(\mathcal{F}_2 z)(x)\|\to 0$ as $h\to 0$ and $\mathcal{F}_2$ is an equicontinuous operator. Hence, $\mathcal{F}_2$ is relatively compact on $\mathfrak{B}_r$ by the Arzela-Ascoli theorem. Then Krasnoselskii's fixed-point theorem gives that $\mathcal{F}$ has a fixed point $z\in\mathfrak{B}_r$. Apparently, z is a solution of (1.5) satisfying $z(x)=\psi(x)$, $-\vartheta\leqslant x\leqslant 0$, and $z(x_1)=z_1$. The proof is completed. □

5. Examples

Example 5.1. Set $\vartheta=0.4$, $k=5$, $J=[0,2]$ and $T=2$. Consider

$$\begin{cases} z''(x)+Az(x-\vartheta)=g(x), & x\in J,\\ z(x)=\psi(x),\ z'(x)=\psi'(x), & x\in[-0.4,0], \end{cases} \tag{5.1}$$

where

$$z(x)=\begin{pmatrix} z_1(x)\\ z_2(x)\end{pmatrix},\quad A=\begin{pmatrix} 0.2 & 0.4\\ 0.3 & 0.2\end{pmatrix},\quad g(x)=\begin{pmatrix}\cos 2x\\ \sin^2 x\end{pmatrix},\quad \psi(x)=\begin{pmatrix}\frac14 x^2\\ x^2\end{pmatrix}.$$

By (1.3), for $x\in J$, we have

$$\begin{aligned} z(x)&=\mathcal{H}_\vartheta(Ax)\psi(-\vartheta)+\mathcal{M}_\vartheta(Ax)\psi'(-\vartheta)\\ &\quad+\int_{-\vartheta}^0\mathcal{M}_\vartheta(A(x-\vartheta-t))\psi''(t)\,\mathrm{d}t+\int_0^x\mathcal{M}_\vartheta(A(x-\vartheta-t))g(t)\,\mathrm{d}t, \end{aligned}$$

where

$$\mathcal{H}_\vartheta(Ax)=\begin{cases} E-A\dfrac{x^2}{2!}, & 0\leqslant x<0.4,\\ E-A\dfrac{x^2}{2!}+A^2\dfrac{(x-0.4)^4}{4!}, & 0.4\leqslant x<0.8,\\ E-A\dfrac{x^2}{2!}+A^2\dfrac{(x-0.4)^4}{4!}-A^3\dfrac{(x-0.8)^6}{6!}, & 0.8\leqslant x<1.2,\\ E-A\dfrac{x^2}{2!}+A^2\dfrac{(x-0.4)^4}{4!}-A^3\dfrac{(x-0.8)^6}{6!} & \\ \quad+A^4\dfrac{(x-1.2)^8}{8!}, & 1.2\leqslant x<1.6,\\ E-A\dfrac{x^2}{2!}+A^2\dfrac{(x-0.4)^4}{4!}-A^3\dfrac{(x-0.8)^6}{6!} & \\ \quad+A^4\dfrac{(x-1.2)^8}{8!}-A^5\dfrac{(x-1.6)^{10}}{10!}, & 1.6\leqslant x\leqslant 2, \end{cases}$$

and

$$
\mathcal{M}_\vartheta(Ax) = \begin{cases} E(x+0.4) - A\dfrac{x^3}{3!}, & 0 \leqslant x < 0.4, \\ E(x+0.4) - A\dfrac{x^3}{3!} + A^2\dfrac{(x-0.4)^5}{5!}, & 0.4 \leqslant x < 0.8, \\ E(x+0.4) - A\dfrac{x^3}{3!} + A^2\dfrac{(x-0.4)^5}{5!} - A^3\dfrac{(x-0.8)^7}{7!}, & 0.8 \leqslant x < 1.2, \\ E(x+0.4) - A\dfrac{x^3}{3!} + A^2\dfrac{(x-0.4)^5}{5!} & \\ \quad -A^3\dfrac{(x-0.8)^7}{7!} + A^4\dfrac{(x-1.2)^9}{9!}, & 1.2 \leqslant x < 1.6, \\ E(x+0.4) - A\dfrac{x^3}{3!} + A^2\dfrac{(x-0.4)^5}{5!} - A^3\dfrac{(x-0.8)^7}{7!} & \\ \quad +A^4\dfrac{(x-1.2)^9}{9!} - A^5\dfrac{(x-1.6)^{11}}{11!}, & 1.6 \leqslant x \leqslant 2. \end{cases}
$$

By calculation, one has $\|\psi\|_C = 0.2$, $\|\psi(-0.4)\| = 0.2$, $\|\psi'(-0.4)\| = 0.3$, $\|\psi''\|_C = 2.5$, $\Psi_1(2) = 4.03$, $\Psi_2(2) = 1.12$, $\Psi_3(2) = 2.42$,

$$
\sum_{i=1}^{5} \frac{0.6^{i-1}}{(2i)!}(2-0.4(i-1))^{2i} = 2.16,
$$

$\cosh(2\sqrt{0.6}) = 2.46$ and $\|g\| = 1$. We present finite time stable results of (5.1) in Table 1.

Theorem	$\|\psi\|_C$	J	ϑ	$\|z\|$	δ	β	FTS
3.1	0.2	[0,2]	0.4	7.81	0.3	7.9(optimal)	Yes
3.2	0.2	[0,2]	0.4	8.93	0.3	8.94	Yes

Table 1. Finite time stable results of (5.1) with $T = 2$.

By Definition 2.2, we seek a suitable β such that $\|z\|$ of (1.2) does not exceed β on J. On the one hand, we can use the explicit formula of solution to (5.1) via numerical simulation to find a corresponding $\beta = 7.9$ for a fixed $T = 2$ (see Figure 1).

By checking the conditions in Theorems 3.1 and 3.2 for J, one can choose a better value $\beta = 7.9$ by comparing with the value of β in Table 1.

Example 5.2. Set $\vartheta = 0.5$, $k = 5$, $J = [0, x_1]$ and $x_1 = 2.5$. Consider

$$
(5.2) \qquad \begin{cases} z''(x) + Az(x-\vartheta) = Bu(x), & x \in J, \\ z(x) = \psi(x), z'(x) = \psi'(x), & -0.5 \leqslant x \leqslant 0, \end{cases}
$$

where $u \in L^2(J, \mathbb{R}^2)$ and

$$
z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}, \quad A = \begin{pmatrix} 0.5 & 0.6 \\ 0.4 & 0.3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0.4 \\ 0.5 & 1 \end{pmatrix}, \quad \psi(x) = \begin{pmatrix} x \\ x^2 \end{pmatrix}.
$$

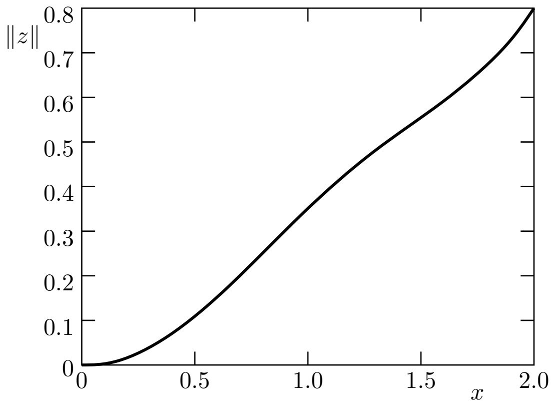

Figure 1. The norm of the state vector of (5.2) when $T = 2$.

By calculation, the matrix $W_{0.5}[-0.5, 2.5]$ of (5.2) via (3.4) can be given by:

$$\begin{aligned} W_{0.5}[-0.5, 2.5] &= \int_0^{2.5} \mathcal{M}_{0.5}(A(2-t))BB^\top \mathcal{M}_{0.5}(A^\top(2-t))\,\mathrm{d}t \\ &= W_1 + W_2 + W_3 + W_4 + W_5, \end{aligned}$$

where

$$\begin{aligned} W_1 = &\int_0^{0.5} \Big(E(2.5-t) - A\frac{(2-t)^3}{3!} + A^2\frac{(1.5-t)^5}{5!} - A^3\frac{(1-t)^7}{7!} + A^4\frac{(0.5-t)^9}{9!}\Big) \\ &\times BB^\top\Big(E(2.5-t) - A^\top\frac{(2-t)^3}{3!} + (A^\top)^2\frac{(1.5-t)^5}{5!} \\ &- (A^\top)^3\frac{(1-t)^7}{7!} + (A^\top)^4\frac{(0.5-t)^9}{9!}\Big)\,\mathrm{d}t, \\ W_2 = &\int_{0.5}^{1} \Big(E(2.5-t) - A\frac{(2-t)^3}{3!} + A^2\frac{(1.5-t)^5}{5!} - A^3\frac{(1-t)^7}{7!}\Big) \\ &\times BB^\top\Big(E(2.5-t) - A^\top\frac{(2-t)^3}{3!} + (A^\top)^2\frac{(1.5-t)^5}{5!} - (A^\top)^3\frac{(1-t)^7}{7!}\Big)\,\mathrm{d}t, \\ W_3 = &\int_{1}^{1.5} \Big(E(2.5-t) - A\frac{(2-t)^3}{3!} + A^2\frac{(1.5-t)^5}{5!}\Big) \\ &\times BB^\top\Big(E(2.5-t) - A^\top\frac{(2-t)^3}{3!} + (A^\top)^2\frac{(1.5-t)^5}{5!}\Big)\,\mathrm{d}t, \\ W_4 = &\int_{1.5}^{2} \Big(E(2.5-t) - A\frac{(2-t)^3}{3!}\Big)BB^\top\Big(E(2.5-t) - A^\top\frac{(2-t)^3}{3!}\Big)\,\mathrm{d}t, \\ W_5 = &\int_{2}^{2.5} (E(2.5-t))BB^\top(E(2.5-t))\,\mathrm{d}t. \end{aligned}$$

By computation, one can get

$$W_1 = \begin{pmatrix} 1.213 & 0.664 \\ 0.664 & 1.905 \end{pmatrix}, \ W_2 = \begin{pmatrix} 1.192 & 0.854 \\ 0.854 & 1.515 \end{pmatrix}, \ W_3 = \begin{pmatrix} 0.807 & 0.616 \\ 0.616 & 0.916 \end{pmatrix},$$
$$W_4 = \begin{pmatrix} 0.333 & 0.258 \\ 0.258 & 0.361 \end{pmatrix}, \ W_5 = \begin{pmatrix} 0.048 & 0.036 \\ 0.036 & 0.052 \end{pmatrix}.$$

and

$$W_{0.5}[-0.5, 2.5] = \begin{pmatrix} 3.593 & 2.428 \\ 2.428 & 4.749 \end{pmatrix}, \quad W_{0.5}^{-1}[-0.5, 2.5] = \begin{pmatrix} 0.425 & -0.217 \\ -0.217 & 0.3217 \end{pmatrix}.$$

Set $z(x_1) = (z_1, z_2)^\top$. By (3.5), one can construct $u \in L^2(J, \mathbb{R}^2)$ as

$$\text{(5.3)} \qquad \begin{aligned} u(t) &= B^\top \mathcal{M}_{0.5}(A^\top(2-t))W_{0.5}^{-1}[-0.5, 2.5]\xi \\ &= \begin{cases} B^\top\Big(E(2.5-t) - A^\top\dfrac{(2-t)^3}{3!} + A^2\dfrac{(1.5-t)^5}{5!} - A^3\dfrac{(1-t)^7}{7!} \\ \quad +A^4\dfrac{(0.5-t)^9}{9!}\Big)W_{0.5}^{-1}[-0.5, 2.5]\xi, \quad 0 \leqslant x < 0.5, \\ B^\top\Big(E(2.5-t) - A^\top\frac{(2-t)^3}{3!} + A^2\frac{(1.5-t)^5}{5!} - A^3\frac{(1-t)^7}{7!}\Big) \\ \quad \times W_{0.5}^{-1}[-0.5, 2.5]\xi, \quad 0.5 \leqslant x < 1, \\ B^\top\Big(E(2.5-t) - A^\top\dfrac{(2-t)^3}{3!} + A^2\dfrac{(1.5-t)^5}{5!}\Big) \\ \quad \times W_{0.5}^{-1}[-0.5, 2.5]\xi, \quad 1 \leqslant x < 1.5, \\ B^\top\Big(E(2.5-t) - A^\top\dfrac{(2-t)^3}{3!}\Big)W_{0.5}^{-1}[-0.5, 2.5]\xi, \quad 1.5 \leqslant t < 2, \\ B^\top(E(2.5-t))W_{0.5}^{-1}[-0.5, 2.5]\xi, \quad 2 \leqslant t < 2.5. \end{cases} \end{aligned}$$

where

$$\begin{aligned} \xi &= z(2.5) - \mathcal{H}_{0.5}(2.5A)\psi(-0.5) - \mathcal{M}_{0.5}(2.5A)\psi'(-0.5) \\ &\quad - \int_{-0.5}^{0} \mathcal{M}_{0.5}(A(2-t))\psi''(t)\,\mathrm{d}t \\ &= \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - \Big(E - A\frac{2.5^2}{2!} + A^2\frac{2^4}{4!} - A^3\frac{1.5^6}{6!} + A^4\frac{1^8}{8!} - A^5\frac{0.5^{10}}{10!}\Big)\begin{pmatrix} -0.5 \\ 0.25 \end{pmatrix} \\ &\quad - \Big(3E - A\frac{2.5^3}{3!} + A^2\frac{2^5}{5!} - A^3\frac{1.5^7}{7!} + A^4\frac{1^9}{9!} - A^5\frac{0.5^{11}}{11!}\Big)\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &\quad - \int_{-0.5}^{0}\Big(E(2.5-t) - A\frac{(2-t)^3}{3!} + A^2\frac{(1.5-t)^5}{5!} - A^3\frac{(1-t)^7}{7!} \\ &\quad + A^4\frac{(0.5-t)^9}{9!} - A^5\frac{(-t)^{11}}{11!}\Big)\begin{pmatrix} 0 \\ 2 \end{pmatrix}\,\mathrm{d}t \\ &= \begin{pmatrix} z_1 - 1.911 \\ z_2 + 0.452 \end{pmatrix}. \end{aligned}$$

Consider $S_n = \{B, AB\} = \begin{pmatrix} 1 & 0.4 & 0.8 & 0.8 \\ 0.5 & 1 & 0.55 & 0.46 \end{pmatrix}$. Obviously, $x_1 = 2.5 > (2-1)\vartheta$ and $\operatorname{rank} S_n = 2$.

By Theorems 3.3 and 3.4, (5.2) is relatively controllable. Figures 2 and 3 show the state $z(x)$ of (5.2) when we set $z = (z_1, z_2)^\top = (1,1)^\top$ and $z = (z_1, z_2)^\top = (0.5, 2)^\top$. Clearly, we can see the state of system (5.2) is relatively consistent with the achieved states.

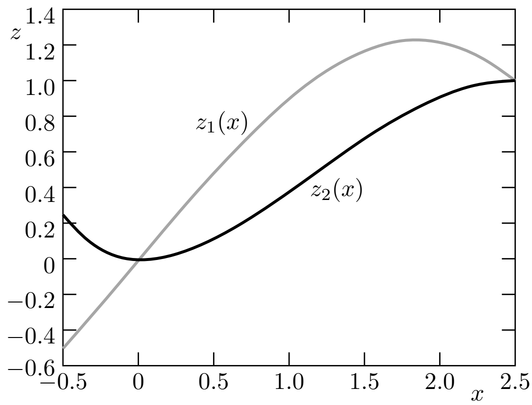

Figure 2. The state $z(t)$ of system (5.2) when we set $z = (z_1, z_2)^\top = (1,1)^\top$.

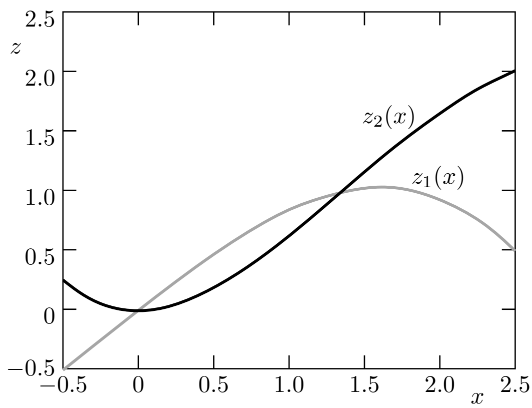

Figure 3. The state $z(x)$ of system (5.2) when we set $z = (z_1, z_2)^\top = (0.5, 2)^\top$.

Example 5.3. Set $\vartheta = 0.5$, $k = 5$, $J = [0, x_1]$ and $x_1 = 2.5$. Consider

$$
\begin{cases} z''(x) + Az(x-\vartheta) = f(x, z(x)) + Bu(x), & x \in J, \\ z(x) = \psi(x), \quad z'(x) = \psi'(x), & -0.5 \leqslant x \leqslant 0, \end{cases} \tag{5.4}
$$

where $u \in L^2(J, \mathbb{R}^2)$ and

$$z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}, \quad A = \begin{pmatrix} 0.5 & 0.6 \\ 0.4 & 0.3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0.4 \\ 0.5 & 1 \end{pmatrix},$$
$$\psi(x) = \begin{pmatrix} x \\ x^2 \end{pmatrix}, \quad f(x, z(x)) = \begin{pmatrix} \frac{1}{60}(x+0.1)z_1(x) \\ \frac{1}{60}(x+0.1)z_2(x) \end{pmatrix}.$$

We now use (4.2) to estimate M. By Example 5.2, we have

$$W_{0.5}[-0.5, 2.5] = \begin{pmatrix} 3.593 & 2.428 \\ 2.428 & 4.749 \end{pmatrix}, \quad W_{0.5}^{-1}[-0.5, 2.5] = \begin{pmatrix} 0.425 & -0.217 \\ -0.217 & 0.3217 \end{pmatrix}.$$

Consequently, one can get $M = \|W_\vartheta[-\vartheta, x_1]^{-1}\|^{1/2} = \sqrt{0.642}$. Hence, Λ satisfies assumption (H_1).

Further, it is easy to verify that for any $\widetilde{z}(x), \widehat{z}(x) \in \mathbb{R}^2$ and $x \in J$,

$$\begin{aligned} \|f(x, \widetilde{z}(x)) - f(x, \widehat{z}(x))\| &\leqslant \frac{1}{60}(x+0.1)(|\widetilde{z}_1(x) - \widehat{z}_1(x)| + |\widetilde{z}_2(x) - \widehat{z}_2(x)|) \\ &\leqslant \frac{1}{60}(x+0.1)\|\widetilde{z} - \widehat{z}\|. \end{aligned}$$

Hence, f satisfies the assumption (H_2), where we set $\varphi(x) = \frac{1}{60}(x+0.1) \in C(J, \mathbb{R}^n)$.

By elementary calculation, one has $\widehat{M} = \frac{1}{60} \sup_{x \in [0,2.5]} (x+0.1) = 0.044$, $\|A\| = 0.9$, $\|B\| = 1.5$, $M_1 = 21.75$, $M_2 = 3.737$, and $\|\hat{f}\| = \sup_{x \in J} \|f(s, \mathbf{0})\| = 0$. Therefore, one can get $M\widehat{M}M_2^2\|B\| + \widehat{M}M_2 = 0.91 < 1$, which guarantees that (4.5) holds. By Theorem 4.1, the system (5.4) is completely controllable on J.

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