Applications of Mathematics

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Applications of Mathematics, Vol. 68 (2023), No. 3, 357–376

Persistent URL: http://dml.cz/dmlcz/151658

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NON-LINEAR CHANDRASEKHAR-BÉNARD CONVECTION IN TEMPERATURE-DEPENDENT VARIABLE VISCOSITY BOUSSINESQ-STOKES SUSPENSION FLUID WITH VARIABLE HEAT SOURCE/SINK

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Received February 16, 2022. Published online July 1, 2022.

Abstract. The generalized Lorenz model for non-linear stability of Rayleigh-Bénard magneto-convection is derived in the present paper. The Boussinesq-Stokes suspension fluid in the presence of variable viscosity (temperature-dependent viscosity) and internal heat source/sink is considered in this study. The influence of various parameters like suspended particles, applied vertical magnetic field, and the temperature-dependent heat source/sink has been analyzed. It is found that the basic state of the temperature gradient, viscosity variation, and the magnetic field can be conveniently expressed using the half-range Fourier cosine series. This facilitates to determine the analytical expression of the eigenvalue (thermal Rayleigh number) of the problem. From the analytical expression of the thermal Rayleigh number, it is evident that the Chandrasekhar number, internal Rayleigh number, Boussinesq-Stokes suspension parameters, and the thermorheological parameter influence the onset of convection. The non-linear theory involves the derivation of the generalized Lorenz model which is essentially a coupled autonomous system and is solved numerically using the classical Runge-Kutta method of the fourth order. The quantification of heat transfer is possible due to the numerical solution of the Lorenz system. It has been shown that the effect of heat source and temperature-dependent viscosity advance the onset of convection and thereby give rise to enhancing the heat transport. The Chandrasekhar number and the couple-stress parameter have stabilizing effects and reduce heat transfer. This problem has possible applications in the context of the magnetic field which influences the stability of the fluid.

Keywords: Rayleigh-Bénard convection; heat source/sink; Boussinesq-Stokes suspension; Boussinesq approximation; Lorenz model

MSC 2020: 76E30, 76W05

DOI: 10.21136/AM.2022.0037-22

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Nomenclature

Latin symbols:

C	Couple stress parameter
d	Depth of the fluid layer (m)
g	Acceleration due to gravity ($g = 9.8\,\mathrm{ms}^{-2}$)
H_0	Magnetic field
Nu	Nusselt number
p	Pressure (Pa)
$\vec{q}$	Velocity components of u, v, w (ms^{-1})
Pr	Prandtl number
Pm	Magnetic Prandtl number
Q	Chandrasekhar number
R_E	External Rayleigh number
R_I	Internal Rayleigh number
T	Temperature (K)
t	Time (s)
ΔT	Temperature difference between the walls

Greek symbols:

μ	Dynamic viscosity (Pa s)
μ'	Couple stress viscosity
μ_m	Magnetic permeability
$\pi\alpha$	Wave number
β	Thermal expansion coefficient (K^{-1})
χ	Constant thermal diffusivity
ϱ	Density ($\mathrm{kg\,m}^{-3}$)
ψ	Stream function
Ψ	Perturbed stream function
φ	Magnetic potential
Φ	Perturbed magnetic potential
Θ	Perturbed temperature

Other symbols:

x, y, z	Cartesian coordinates (m)
$\hat{i}$	Unit vector normal in x-direction
$\hat{k}$	Unit vector normal in z-direction
∇^2	Laplace operator

1. Introduction

Thermal convection in a horizontal fluid layer in the presence of an internal heat source subject to constant but different temperatures at the boundaries has been extensively investigated by many researchers due to its practical importance in engineering and geophysical problems. The Rayleigh-Bénard convection in a Newtonian fluid with heat sources has been analyzed initially by Sparrow et al. [32]. Many theoretical studies on thermal convection have been explored by Roberts [21], McKenzie [12], Kulacki and Goldstein [8], and Thirlby et al. [34] and the experimental investigation on thermal convection by Palm [14], Tritton and Zarraga [36], Clever [5], and many others. The internal heat source $Q*$ is considered to be uniform in all of the above works. But in many practical problems and applications, Q^* is non-uniform in nature, which is due to many internal factors such as heat release of chemical reaction that takes place in the fluids, heat source produced by the radiation from the external medium, radioactive decay and so on. Riahi [19], [20] discussed the non-linear convection in the horizontal layer with the heat source. He showed that the effect of non-uniform internal heat source Q^* strongly affects the cell size, the stability of convective motion, and the internal motion of hexagonal cells. They have discussed the problem of non-linear thermal convection in a low Prandtl number fluid with internal heating. Bhattacharya and Jena [2] have analyzed the thermal instability of the horizontal layer of micropolar fluid with the heat source. They found that the heat source and heat sink both have the same destabilizing effect in micropolar fluids.

Siddheshwar and Titus [30] have made a detailed linear and non-linear stability analysis for the Newtonian fluid with heat source/sink analytically using Lorenz and Ginzburg-Landau models. They have used the minimal representation of the Fourier series in finite-amplitude analysis to find its mention in the study of chaotic thermal convection, which has also been implemented by Ramachandramurthy et al. [16], [18], [17], [1]. For the following few reasons, many researchers have found the truncated Fourier series expansion to be valuable:

(1) to explicit instability due to convection of many non-isothermal situations of practical interest,
(2) to undertake linear stability analysis and measure heat transport by generating an analytical expression for thermal Rayleigh number.

In most of the above works, researchers have considered the effect of heat sources on Newtonian fluids. However, in many real settings, the majority of fluids are not so pure, containing suspended particles which may be polymeric suspensions, liquid crystals, and so on. In the study of fluids, suspended particles have a significant influence. The presence of suspended particles does have a huge stabilizing/destabilizing influence on the fluid's thermal convection. Hence, the study related to non-Newtonian fluids with heat source/sink in our modern science technology is more desirable. Since the last few decades, the study of such fluids has been a brisk topic of research, particularly in many industrially essential fluids such as polymeric suspensions, chemicals, paints, liquid crystal solidifications, pharmaceuticals, food and beverages, refrigeration, and many others.

Convection in a fluid layer with varying viscosity has gotten a lot of interest in recent years because of its applications in the fields like terrestrial planets, heat transport, and so on. As a result, numerous academics have focused on the variable viscosity problem with Rayleigh-Bénard convection (Platten and Legros [15], Gebhart et al. [6], Siddheshwar and Pranesh [27], Siddheshwar [26] and the references therein). Gireesha et al. [7] discussed how the viscosity which depends on the temperature and also on the heat source will affect the kerosene-Alumina nano liquid. They made a non-linear convective analysis on the augmentation of heat transport rate in a liquid propellant rocket engine and found that the temperature of nano-liquid increases due to the effect of radiation and space-dependent heat sources. Yusuf and Ajibade [38] have done a detailed analysis on how the viscosity variation and thermal radiation are going to affect the convection which occurs naturally in the fluid. They found that the velocity of the fluid is directly proportional to variations in viscosity and the fluid temperature variation is also directly proportional to thermal radiation. This work is the extension work of Makinde et al. [9] in which the fluid viscosity is considered to have a constant status. Manjunatha et al. [10] also investigated the effect of variable viscosity, variable convection, and volume fraction for different types of fluid flow of nanofluids. The results show that the increase in the volume fraction will increase the thermal conductivity in the fluid and the Marangoni effect is very suitable for the cooling process. The viscosity of many engineering and geophysical problems is substantially influenced by temperature. Torrance and Turcotte [35], Busse and Frick [3], Stengel et al. [33], Siddheshwar [25], Severin and Herwig [22], and others have studied the thermorheological effect due to significant variation of viscosity with temperature, which is generally in the polynomial or exponential form through the truncated Taylor series approximation. Somerscales and Dougherty [31] investigated the experimental work on convection in fluids with the thermorheological effect. Siddheshwar [24] has also investigated the effect of temperature-dependent viscosity for a weak electrically conducting fluid under $1g$ and μg situations in the presence of a magnetic field. The results of the work mentioned above lead to many possible astrophysical as well as terrestrial applications in modern science.

Siddheshwar and Pranesh [28] studied the linear and non-linear convection for the fluid with Boussinesq-Stokes suspensions analytically. Siddheshwar et al. [29] analyzed the thermorheological effect of Rayleigh-Bénard magnetoconvection in Newtonian fluid numerically for various boundary combinations. Sharma and Sharma [23] have analyzed the effect of suspended particles on the couple-stress fluid in the presence of a magnetic field. Meenakshi and Siddheshwar [13] have made an analytical investigation on the Rayleigh-Bénard convection for the twenty nanoliquids in the presence of a volumetric heat source. Maruthamanikandan et al. [11] presented work on Marangoni convective instability in a ferromagnetic fluid layer in the presence of a spatial heat source and viscosity variation. In the presence of an internal heat source/sink, the influence of Boussinesq-Stokes suspension and variable viscosity on Rayleigh-Bénard magnetoconvection is investigated in this study. This research includes both linear and weak non-linear studies.

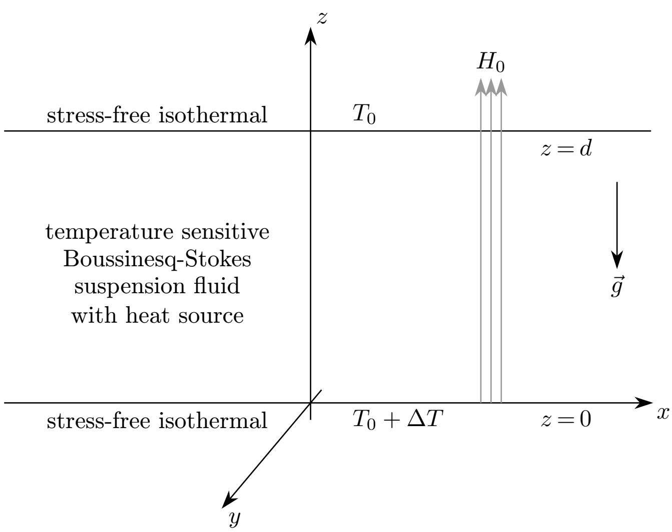

Figure 1. Physical model of the problem.

2. Schematic of flow configuration and mathematical modeling

Consider an electrically conducting Boussinesq-Stokes suspension fluid with heat source/sink between two infinitely extended parallel planes of depth d which support Rayleigh-Bénard situation as shown in Fig. 1. As known in the Rayleigh-Bénard convection problems, the lower and upper boundaries are maintained at different temperatures $T_0 + \Delta T$ and T_0, respectively, so that there is a temperature gradient $\Delta T > 0$ (see Fig. 1) across the fluid layer. In addition to temperature gradient, a vertically induced magnetic field $\vec{H} = H_x\hat{i} + H_z\hat{k}$ is also imposed. Here lower and upper boundaries are considered stress-free and isothermal (i.e., maintained at constant temperature). For the sake of non-linear stability analysis, all physical quantities of fluid are assumed to be independent of y and depend only on x and z.

The density, dynamic viscosity, and heat source/sink are assumed to be temperature-dependent. Subjected to the Oberbeck-Boussinesq approximations, the governing equations for the fluid motion in electrically conducting couple-stress fluid with a heat source in two-dimension are

$$\begin{aligned}(1)\qquad \frac{\partial \vec{q}}{\partial t}+(\vec{q}\cdot\nabla)\vec{q} &= -\frac{\nabla p}{\varrho_0}-\frac{\varrho(T)}{\varrho_0}g\hat{k}+\frac{1}{\varrho_0}\nabla\cdot(\mu_f(\nabla\vec{q}+\nabla\vec{q}^{\top}))\\ &\quad -\frac{\mu_m^2}{\varrho_0}(\vec{H}\cdot\nabla)\vec{H}-\frac{\mu'}{\varrho_0}\nabla^4\vec{q},\end{aligned}$$

$$(2)\qquad \frac{\partial T}{\partial t}+(\vec{q}\cdot\nabla)T-\chi\nabla^2T-Q^*(T-T_0)=0,$$

$$(3)\qquad \frac{\partial \vec{H}}{\partial t}+(\vec{q}.\nabla)\vec{H}-(\vec{H}.\nabla)\vec{q}-\nu_m\nabla^2\vec{H}=0,$$

$$(4)\qquad \varrho=\varrho_0(1-\beta(T-T_0)),$$

$$(5)\qquad \mu_f(T)=\mu_0e^{-\delta(T-T_0)},$$

$$(6)\qquad \nabla\cdot\vec{q}=0,$$

$$(7)\qquad \nabla\cdot\vec{H}=0,$$

where $\vec{q}=u\hat{\imath}+w\hat{k}$ is velocity vector, $\vec{H}=H_x\hat{\imath}+H_z\hat{k}$ is the magnetic intensity vector, $\nabla\cdot(\mu_f(\nabla\vec{q}+\nabla\vec{q}^{\top}))$ represents the variable viscosity, $Q^*(T-T_0)$ is the temperature-dependent heat source/sink, and $\mu_f(T)$ is the fluid viscosity, which is assumed to be an exponential model in-order to study the strong variation of viscosity with respect to variation of temperature.

When the fluid is at the basic state, velocity $\vec{q}_b$, magnetic field $H_b(z)$, density $\varrho_b(z)$, temperature $T_b(z)$ and viscosity μ_{f_b} have their solutions in the following forms:

$$(8)\qquad \begin{cases}\vec{q}=(0,0),\ H_b=H_b(z),\ \varrho_b\Big(\dfrac{z}{d}\Big)=\varrho_0\Big(1-\beta\Delta T\dfrac{\sin\sqrt{R_I}(1-z/d)}{\sin\sqrt{R_I}}\Big),\\ T_b=T_0+\Delta T\dfrac{\sin\sqrt{R_I}(1-z/d)}{\sin\sqrt{R_I}},\ \mu_{f_b}\Big(\dfrac{z}{d}\Big)=\mu_0\mathrm{e}^{-V\sin\sqrt{R_I}(1-z/d)/\sin\sqrt{R_I}},\\ p_b\Big(\dfrac{z}{d}\Big)=-\int\varrho_b\Big(\dfrac{z}{d}\Big)gd\Big(\dfrac{z}{d}\Big)+k_1,\end{cases}$$

where $R_I=Q^*d^2/\chi$ represents the internal Rayleigh number, $V=\delta\Delta T$ is thermorheological parameter and k_1 is the constant of integration. The finite amplitude perturbations are imposed for the basic state in the form:

$$(9)\qquad \begin{cases}\vec{q}=\vec{q}_b(z)+\vec{q}'(x,z,t),\ \vec{H}=\vec{H}_b(z)+\vec{H}'(x,z,t),\ \varrho=\varrho_b(z)+\varrho'(x,z,t),\\ T=T_b(z)+T'(x,z,t),\ \mu=\mu_{f_b}(z)+\mu_{f'}(x,z,t),\ p=p_b(z)+p'(x,z,t).\end{cases}$$

In the above equation, the prime denotes the perturbed quantity. We shall put (9) into the governing equations, the following component equations are obtained:

$$\frac{\partial \vec{q}'}{\partial t}+(\vec{q}'\cdot\nabla)\vec{q}'=-\frac{\nabla p'}{\varrho_0}-\frac{\varrho'(T)}{\varrho_0}g\hat{k}+\frac{1}{\varrho_0}\nabla\cdot(\mu_f(\nabla\vec{q}'+\nabla\vec{q}'^T))$$
$$-\frac{\mu_m^2}{\varrho_0}(\vec{H}'.\nabla)\vec{H}'+\mu_m H_b\frac{\partial\vec{H}'}{\partial z}-\frac{\mu'}{\varrho_0}\nabla^4\vec{q}', \tag{10}$$

$$\frac{\partial T'}{\partial t}+(\vec{q}'\cdot\nabla)T'+w'\frac{\partial T_b}{\partial z}-\chi\nabla^2 T'=0, \tag{11}$$

$$\frac{\partial\vec{H}'}{\partial t}+(\vec{q}'.\nabla)\vec{H}'-(\vec{H}'\cdot\nabla)\vec{q}'-H_b\frac{\partial w'}{\partial z}-\nu_m\nabla^2\vec{H}'=0, \tag{12}$$

$$\nabla\cdot\vec{q}'=0, \tag{13}$$

$$\nabla\cdot\vec{H}'=0, \tag{14}$$

$$\varrho'=-\varrho_0\beta T'. \tag{15}$$

For the sake of analyzing the fluid, we will assume two-dimensional disturbances, hence we introduce magnetic potential φ' and stream function ψ' as

$$\begin{cases} u'=-\dfrac{\partial\psi'}{\partial z},\ w'=\dfrac{\partial\psi'}{\partial x},\\ {H_x}'=-\dfrac{\partial\varphi'}{\partial z},\ {H_z}'=\dfrac{\partial\varphi'}{\partial x}. \end{cases} \tag{16}$$

The classical method of operating curl for (10) aids in the elimination of the pressure term. As a result, using the scaling described below, we may convert the given system of equations to dimensionless equations:

$$x=Xd,\quad z=Zd,\quad \psi'=\Psi\chi,\quad T'=\Theta\Delta T,\quad \varphi'=\Phi H_0 d. \tag{17}$$

The non-dimensional governing equations are obtained as follows:

$$\frac{1}{\Pr}\Big(\frac{\partial}{\partial t}(\nabla^2\Psi)+\frac{\partial(\Psi,\nabla^2\Psi)}{\partial(X,Z)}\Big)=R_E\frac{\partial\Theta}{\partial X}-C\nabla^6\Psi$$
$$+Q\,\mathrm{Pm}\Big(\frac{\partial(\nabla^2\Phi)}{\partial Z}+\frac{\partial(\Phi,\nabla^2\Phi)}{\partial(X,Z)}\Big)+\mu_{f_b}\nabla^4\Psi+\frac{\partial\mu_{f_b}}{\partial Z}\frac{\partial}{\partial Z}(\nabla^2\Psi), \tag{18}$$

$$\frac{\partial\Theta}{\partial t}=\frac{\partial\Psi}{\partial X}\Big(1+2\sum_{n=1}^{\infty}\frac{R_I}{R_I-n^2\pi^2}\cos n\pi Z\Big)+\nabla^2\Theta+R_I\Theta-\frac{\partial(\Psi,\Theta)}{\partial(X,Z)}, \tag{19}$$

$$\Big(\frac{\partial}{\partial t}-\mathrm{Pm}\nabla^2\Big)\Phi=\frac{\partial\Psi}{\partial Z}+\frac{\partial(\Psi,\Phi)}{\partial(X,Z)}, \tag{20}$$

where

$$\frac{\partial(\Psi,\nabla^2\Psi)}{\partial(X,Z)},\ \frac{\partial(\Phi,\nabla^2\Phi)}{\partial(X,Z)},\ \frac{\partial(\Psi,\Theta)}{\partial(X,Z)},\ \frac{\partial(\Psi,\Phi)}{\partial(X,Z)}$$

in the above equations represents the Jacobians, i.e.

$$\frac{\partial(M,N)}{\partial(X,Z)} = \frac{\partial M}{\partial X}\frac{\partial N}{\partial Z} - \frac{\partial M}{\partial Z}\frac{\partial N}{\partial X}, \quad \nabla^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2}$$

is the Laplace operator, $\mathrm{Pr} = \mu/(\varrho_0\chi)$ is the Prandtl number, $\mathrm{Pm} = \nu_m/\chi$ is the magnetic Prandtl number, $R_E = \beta\varrho_0 g d^3\Delta T/(\mu_0\chi)$ is the thermal Rayleigh number, $C = \mu'/(\mu d^2)$ is the couple stress parameter, $Q = \mu_m^2\sigma H_0^2 d^2/\mu$ is the Chandrasekhar number, $R_I = Q^* d^2/\chi$ is the internal Rayleigh number. Furthermore, $\mu_{f_b} = \mu_0\mathrm{e}^{-V(1-Z)}$ and $\partial\mu_{f_b}/\partial Z = \mu_0 V\mathrm{e}^{-V(1-Z)}$ both will be stated as half-range Fourier cosine series in the interval $[0,1]$ and this representation helps in obtaining the analytical expression for the thermal Rayleigh number. Moreover, the truncated Fourier cosine series is good enough to represent the basic viscosity. It is even clear that the basic states of the temperature-gradient and viscosity are linear. The boundary conditions for the present problem on velocity, temperature are

$$\begin{cases} \dfrac{\partial\Psi}{\partial X} = \dfrac{\partial^2\Psi}{\partial Z^2} = \Theta = \dfrac{\partial\Phi}{\partial Z} = 0 \quad \text{at} \quad Z = 0, \\ \dfrac{\partial\Psi}{\partial X} = \dfrac{\partial^2\Psi}{\partial Z^2} = \Theta = \dfrac{\partial\Phi}{\partial Z} = 0 \quad \text{at} \quad Z = 1. \end{cases} \tag{21}$$

In the case of free-free surface, the boundary conditions on fluid velocity depend on the existence of surface tension. If the free surface does not deform in the direction normal to itself, we must require that $w = 0$ at the boundaries. In case the surface tension is absent, the condition on velocity at the free surface is $w = \mathrm{d}^2w/\mathrm{d}z^2 = 0$; this condition is called stress-free condition. If the fluid layer's bounding wall has a high heat conductivity and a large heat capacity, the temperature will be consistent and stable throughout time, i.e., the boundary temperature would be unperturbed by any flow or temperature perturbations in the fluid. Thus, $T = 0$ at the boundaries. This boundary condition is known as isothermal or boundary condition of the first kind. This condition is also known as the Dirichlet condition.

2.1. Linear stability using truncated Fourier series expansion. The onset of convection is due to the buoyancy-driven condition with the vertical temperature gradient by a non-dimensional number called thermal Raleigh number. The thermal Raleigh number is a non-dimensional form of the temperature gradient across the fluid layer. In the absence of an oscillatory mode of convection, one can derive the analytical expression of the thermal Raleigh number by considering the linearized version of equations (18), (19), and (20). Here we neglected the terms

$$\frac{\partial(\Psi,\nabla^2\Psi)}{\partial(X,Z)}, \quad \frac{\partial(\Phi,\nabla^2\Phi)}{\partial(X,Z)}, \quad \frac{\partial(\Psi,\Theta)}{\partial(X,Z)} \quad \text{and} \quad \frac{\partial(\Psi,\Phi)}{\partial(X,Z)},$$

since these terms give the product of amplitudes which contributed less towards linear theory. However, these terms are retained while performing the non-linear stability analysis. As Chandrasekhar's linear theory states, the solutions of (18), (19), and (20) are truncated Fourier series which are periodic in the form given by Chandrasekhar [4]

$$
\begin{cases}
\Psi(X,Z) = A_0 \sin(\pi\alpha X)\sin(\pi Z),\\
\Theta(X,Z) = B_0 \cos(\pi\alpha X)\sin(\pi Z),\\
\Phi(X,Z) = C_0 \sin(\pi\alpha X)\cos(\pi Z).
\end{cases}
\tag{22}
$$

Here, A_0, B_0, and C_0 denote the amplitudes of stream function, temperature field, and magnetic potential, respectively. Substituting these in the linearized version of the dimensionless (18), (19), and (20) and incorporating the standard Galerkin procedure, we obtain the following system of linear equations of order 3×3

$$
\begin{cases}
\Big(\pi^2\eta^2 a_2 + \dfrac{\eta^4}{2}(a_0 - a_2) + C\eta^6\Big)\Psi_0 - R_E\pi\alpha\Theta_0 + Q\,\mathrm{Pm}\eta^2\pi\Phi_0 = 0,\\
\Big(1 - \dfrac{R_I}{R_I - 4\pi^2}\Big)\pi\alpha\Psi_0 + (R_I - \eta^2)\Theta_0 = 0,\\
\pi\Psi_0 - \mathrm{Pm}\eta^2\Phi_0 = 0.
\end{cases}
\tag{23}
$$

Clearly, the above equations are the homogeneous linear system of equations that has a trivial solution when the determinant of the coefficient matrix vanishes. Therefore, one can derive the following analytical expression for the thermal Rayleigh number:

$$
R_E = \frac{(\eta_1^2 - R_I)(4\pi^2 - R_I)(Q\pi^2 + C\eta_1^6 - \frac{\eta_1^4}{2}((1-2\pi^2)a_2 - a_0))}{4\pi^4\alpha^2},
\tag{24}
$$

where $\eta_1^2 = \pi^2(1+\alpha^2)$, $a_0 = 2\int_0^1 \mu_{f_b}\,\mathrm{d}Z$ and $a_2 = 2\int_0^1 \mu_{f_b}\cos(2\pi Z)\,\mathrm{d}Z$ represents the half range Fourier series expansion for the function $\mu_{f_b} = \mu_0 \mathrm{e}^{-V\sin\sqrt{R_I}(1-z/d)/\sin\sqrt{R_I}}$ and $\pi\alpha$ is the horizontal wave number. It is clear from the analytical expansion of $\mu_{f_b}(Z)$ that it is more conveniently represented by employing the half range Fourier cosine series. This representation facilitates the analytical solution of the problem. While deriving the above expression of R_E, we substitute (22) into the linearized version of (18), (19), and (20). It is then integrated with respect to X between the limits $[0, 2/\alpha]$ and with respect to Z between the limits $[0,1]$ (orthogonal procedure gives the set of linear equations). Clearly, the analytical expression obtained in (24) is a function of Chandrasekhar number Q, thermorheological parameter V, internal Rayleigh number R_I and couple stress parameter C. It is evident that these parameters influence the onset of convection. The influence of these parameters on stability has been well discussed in the result and discussion. It is also evident that when we put $Q = V = C = 0$, the analytical expression of the thermal Raleigh number reduces to its standard form as

$$R_E = \frac{\eta_1^2(\eta_1^2 - R_I)(4\pi^2 - R_I)}{4\pi^4\alpha^2}. \tag{25}$$

2.2. Non-linear stability analysis using truncated yet representative using truncated Fourier modes. The linear stability analysis discussed above shows only conductive to convective mode transition, i.e., stationary instability, but it fails to explain heat transfer within the fluid layer. We now embark upon a weak non-linear stability analysis employing minimal representation of Fourier series for the temperature, magnetic field and velocity.

The truncated yet representative Fourier modes which describe the non-linear state of the system are given by

$$\begin{cases} \Psi(X,Z,\tau) = A_1(\tau)\sin(\pi\alpha X)\sin(\pi Z), \\ \Theta(X,Z,\tau) = B_1(\tau)\cos(\pi\alpha X)\sin(\pi Z) - D_1(\tau)\sin(2\pi Z), \\ \Phi(X,Z,\tau) = E_1(\tau)\sin(\pi\alpha X)\cos(\pi Z) - F_1(\tau)\sin(2\pi\alpha X). \end{cases} \tag{26}$$

We now substitute (26) and perform standard orthogonalization process to obtain the following Lorenz model:

$$\frac{\mathrm{d}A_1}{\mathrm{d}\tau} = \Big[(C\eta_1^2 + C_1)A_1 + C_2B_1 - \frac{(\mathrm{Pm}\,Q\pi)}{\eta_1^2}E_1 - \mathrm{Pm}\,Q\Big(\frac{4\pi^4\alpha^3 - 2\pi^2\alpha\eta_1^2}{\eta_1^4}\Big)E_1F_1\Big], \tag{27}$$

$$\frac{\mathrm{d}B_1}{\mathrm{d}\tau} = \frac{1}{\eta_1^2}[\pi\alpha A_1 - \eta_1^2 B_1 - \pi^2\alpha A_1 D_1], \tag{28}$$

$$\frac{\mathrm{d}D_1}{\mathrm{d}\tau} = \frac{1}{\eta_1^2}\Big[\frac{\pi^2\alpha}{2}A_1B_1 - 4\pi^2 D_1\Big], \tag{29}$$

$$\frac{\mathrm{d}E_1}{\mathrm{d}\tau} = \frac{1}{\eta_1^2}[\pi A_1 - \mathrm{Pm}\,\eta_1^2 E_1 + \pi^2\alpha A_1 F_1], \tag{30}$$

$$\frac{\mathrm{d}F_1}{\mathrm{d}\tau} = \frac{1}{\eta_1^2}[-\pi^2\alpha A_1 E_1 - 4\pi^2\alpha^2\mathrm{Pm}\,F_1], \tag{31}$$

where

$$C_1 = \mathrm{Pr}\Big[\Big(\frac{\eta_1^2 - 2\pi^2}{2\eta_1^2}\Big)a_2 - \frac{a_0}{2}\Big], \quad C_2 = \frac{\mathrm{Pr}\,\pi\alpha R_E}{\eta_1^4} \quad \text{and} \quad \tau = \eta_1^2\tau.$$

It is evident that the Lorenz model in (27) to (31) is a generalized one but its coefficient involves all the parameters which influence the onset of convection. The analytical solution of such a system is not possible but one can solve these equations numerically using the classical Runge-Kutta method. The amplitudes $A_1(\tau)$, $B_1(\tau)$, $D_1(\tau)$, $E_1(\tau)$ and $F_1(\tau)$ represent the dynamics of the system which varies wrt the variation of the parameter. As we know the time-dependent amplitude $A_1(\tau)$ represents the intensity of convection while the remaining represent the different behavior of the system.

3. Heat transport using generalized Lorenz model

The quantification of heat transport is possible due to the numerical solution of the generalized Lorenz model. The horizontally averaged Nusselt number Nu at the lower boundary, i.e., $Z = 0$ for the stationary mode of convection (preferred mode in the problem), is given by the following expression:

$$\mathrm{Nu}(\tau) = \frac{\text{Heat transfer due to conduction+convection}}{\text{Heat transfer due to conduction}}, \tag{32}$$

where "Heat transfer due to conduction+convection" is

$$\left[\frac{\alpha_C}{2}\int_{X=0}^{2/\alpha}\Big(\frac{\sin(\sqrt{R_I(1-Z)})}{\sin\sqrt{R_I}}+\Theta\Big)_{,Z}\,\mathrm{d}X\right]_{Z=0}$$

and "Heat transfer due to conduction" is

$$\left[\frac{\alpha_C}{2}\int_{X=0}^{2/\alpha}\Big(\frac{\sin(\sqrt{R_I(1-Z)})}{\sin\sqrt{R_I}}\Big)_{,Z}\,\mathrm{d}X\right]_{Z=0}.$$

Substituting (26b) into (33), the Nusselt number $\mathrm{Nu}(\tau)$ expression is obtained as:

$$\mathrm{Nu}(\tau) = 1 + \frac{2\tan\sqrt{R_I}}{\sqrt{R_I}}D_1(\tau). \tag{33}$$

4. Results and discussion

In this paper, we focus on analyzing the effect of heat source/sink and temperature-dependent viscosity on the onset of magnetoconvection for the Boussinesq-Stokes suspension fluid. The effect of suspended particles, heat source/sink, temperature-dependent viscosity and applied magnetic field for the Rayleigh-Bénard convection are respectively represented by the couple stress parameter C, internal Rayleigh number R_I, variable viscosity parameter V and the Chandrasekhar number Q. The problem with these limitations is subjected to a linear and non-linear stability analysis. The parameters Pm and Q control the effects of electrical conductivity and magnetic field. The impact of these variables on the onset of stability and heat transmission is thoroughly examined.

The linear stability analysis clearly illustrates the parameter influence on the eigenvalue problem, whereas the non-linear theory is used in examining the quantification of heat transport in the system.

4.1. Linear stability analysis. The following are some of the key features of linear stability analysis:

(1) Deriving the expression for the half range Fourier series for basic non-uniform temperature gradient and the basic viscosity over the interval $[0, 1]$.
(2) Obtaining an expression for the eigenvalue, critical Rayleigh number R_{E_C} for the stationary convection using Galerkin technique.
(3) Plotting Neutral stability curve (Rayleigh—wave number graphs) in order to analyze the convective instability.

Figs. 3–8 are the plots of external Rayleigh number R_E versus wave number α, for the different combination values of Q, V, C and R_I. The effect of couple stress parameter C can be observed from each of the linear plots. The stabilizing effect of the system can also be noticed, that is, the increasing values of C ($C = 0$, $C = 0.02$, $C = 0.04$) increase R_E but decrease the wave number α. The effect of magnetic field Q is similar to the effect of C, which can be observed by comparing Fig. 3 and Fig. 6, that is, the increase of Q corresponds to the increase in both R_E and α.

From Fig. 3 (both) we can observe that the increase of V from $V = 0$ to $V = 0.5$ for $Q = 2$ and $R_I = -1$ corresponds to the decrease of R_E and the increase of α; this shows the destabilizing effect of variable viscosity V. Similar effect can be observed for the other values of Q and R_I (i.e., $Q = 4$, $R_I = -1, 0, 1$). The effect of variable heat source R_I is also similar to the effect of V. The influence of heat source R_I advance the onset of convection, whereas the heat sink delays the onset of convection.

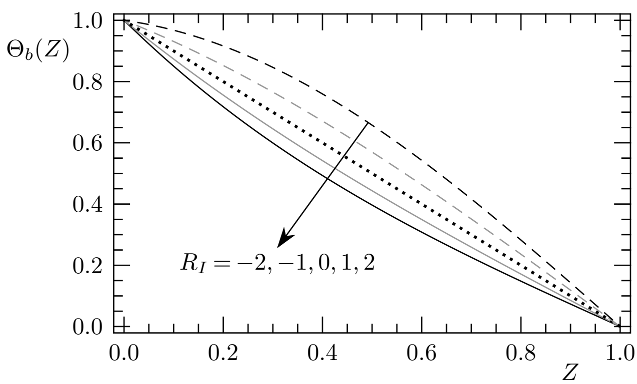

Figure 2. Temperature profile of quiescent basic state for different values of R_I.

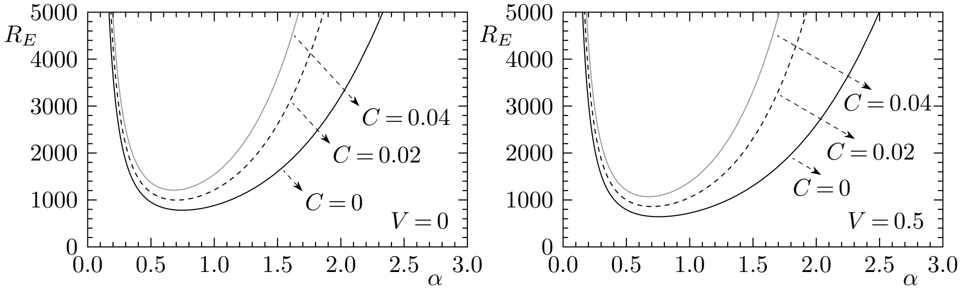

Figure 3. Plot of R_{E_C} vs. α_C for $Q = 2, R_I = -1$.

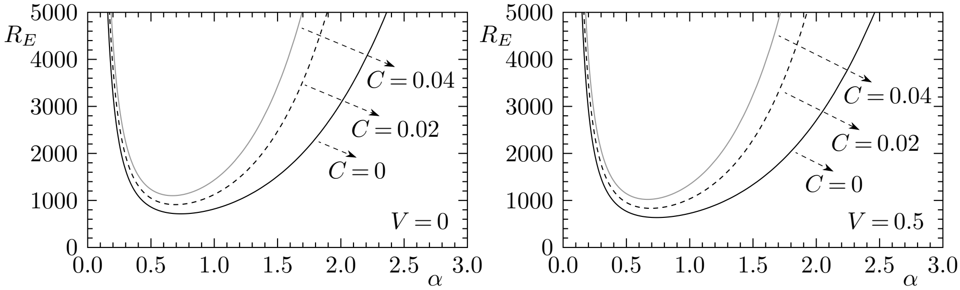

Figure 4. Plot of R_{E_C} vs. α_C for $Q = 2, R_I = 0$.

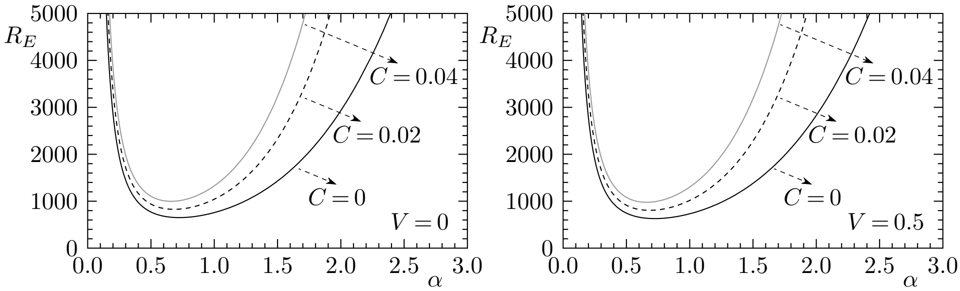

Figure 5. Plot of R_{E_C} vs. α_C for $Q = 2$, $R_I = 1$.

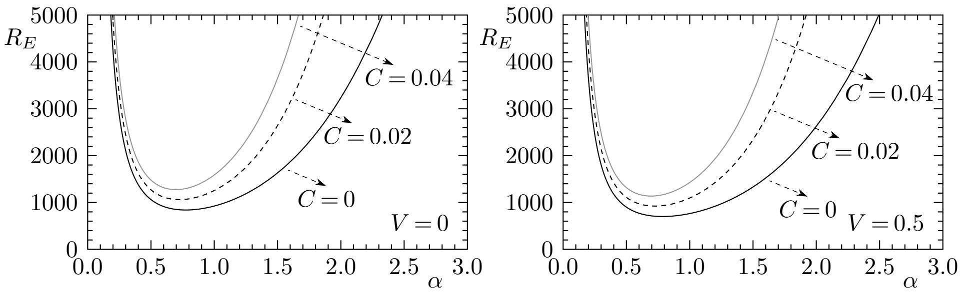

Figure 6. Plot of R_{E_C} vs. α_C for $Q = 4$, $R_I = -1$.

In the view of analyzing the results obtained in the present problem, we examine the non-linear temperature distribution in the basic state, which sheds light on the observed effect of the heat source (sink) on stability. The following model is known for a scaled, dimensionless distribution of temperature:

$$\theta_b(Z) = \frac{T_b(Z) - T_0}{\Delta T} = \frac{\sin(\sqrt{R_I}(1-Z))}{\sin(\sqrt{R_I})}. \tag{34}$$

Fig. 2 represents the non-uniform basic temperature gradient of $\Theta_b(Z)$ versus Z for the different values of R_I. It can be seen that the plots are not symmetric about the line $\Theta_b(Z) = 1 - Z$, which is the basic temperature distribution when there is no heat source/sink. Such asymmetry due to temperature-dependent heat source/sink helps to conclude that the findings on the issue with a heat sink cannot be achieved from that of a heat source through an acceptable transformation, as can be done with a constant heat source (Watson [37]).

4.2. Non-linear stability analysis. The following are some of the key features of non-linear stability analysis:

(1) A double Fourier series representation is used to perform finite amplitude analysis for the magnetic potential Φ, temperature Θ, and stream function Ψ.
(2) The non-linear system of equations is constructed in the form of generalized Lorenz model.
(3) The Nusselt number is considered to be an important factor to analyze the quantification of heat transport for the Boussinesq-Stokes suspension liquid in the existence of a variable heat source for the stationary mode of magnetoconvection.
(4) To analyze the individual effects of the parameters, the variation of the Nusselt number with respect to time is plotted.

Before we start the discussion of the results, we empathize that the truncated Fourier series models are adequate enough to depict Rayleigh-Bénard convection (Siddeheshwar (31)). In addition to this, Lorenz model of the problem is quantitatively evaluated using the Runge-Kutta method with the appropriate step size. We have used the initial conditions $A_1(0) = B_1(0) = D_1(0) = E_1(0) = F_1(0) = 5$ in order to carry out numerical integration of the coupled system. The Prandtl number of the couple-stress liquid is higher than that of the Newtonian carrier liquid because the carrier liquid contains suspended particles. As a result, we have set $\Pr = 10$ as the reference for our discussion.

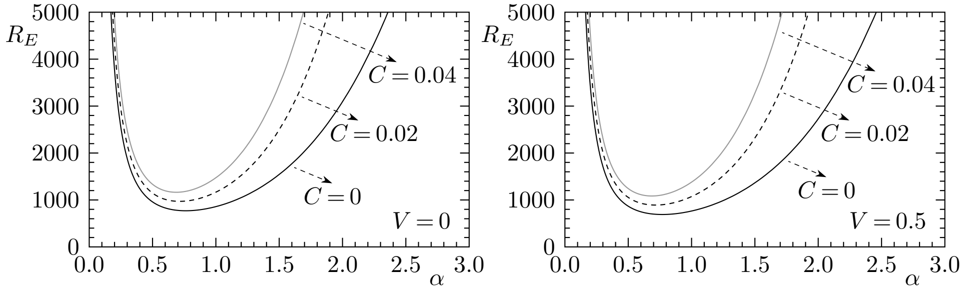

Figure 7. Plot of R_{E_C} vs. α_C for $Q = 4$, $R_I = 0$.

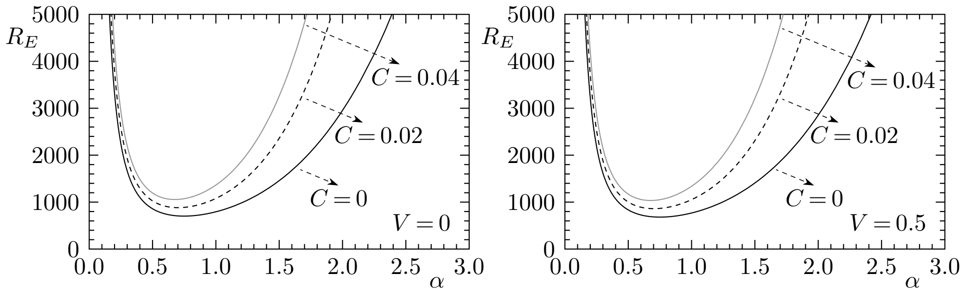

Figure 8. Plot of R_{E_C} vs. α_C for $Q = 4$, $R_I = 1$.

The plots of Nusselt number Nu variation vs time τ for different values of Q, C, V, R_I and for the fixed value of $\mathrm{Pr} = 10$ are shown in Figs. 9–14. Each of the Nusselt number plots shows the distinct or the combined effects of the magnetic field, suspended particles, temperature-dependent viscosity, and the variable heat source. On comparing the Nu plots from Figs. 9, 10 and 11, we observe that Nu increases with the increasing values of R_I (i.e., $R_I = -1, 0, 1$) for the fixed value of $Q = 2$. Here the negative value of R_I represents the heat sink, whereas positive value of R_I represents heat source. The impact of heat source is dominant when compared to the effect of heat sink in the quantification of heat transport. Hence, internal heating in the liquid results in the increase of heat transport in the system. Thus, the heat source in the fluid advances the onset of convection. However, the presence of a heat sink leads to a delay in the onset of convection. By observing Fig. 8 we can notice a similar effect of variable viscosity V for the heat transport. The Nusselt number Nu rises as V rises, indicating the convective contribution to heat transport. When Figs. 8 and 11 are compared, the influence of the magnetic field Q can be seen. The reducing effect of heat transfer owing to the increase in the magnetic field may be seen in these figures. Hence, the increase of Q corresponds to a decrease in Nusselt number Nu. From each of these plots we have analyzed the variation of Nusselt number Nu in the inhabitance of Boussinesq-Stokes suspension parameter C and noticed how the increase of couple stress parameter declines the heat transport both in the presence and absence of magnetic field Q, variable viscosity V and heat source R_I. As a result, it is evident that Nu drops as Q and C increase. Because of the increased viscosity caused by the presence of suspended particles in the fluid, more heating is required to make the system unstable, resulting in the stabilizing effect of C.

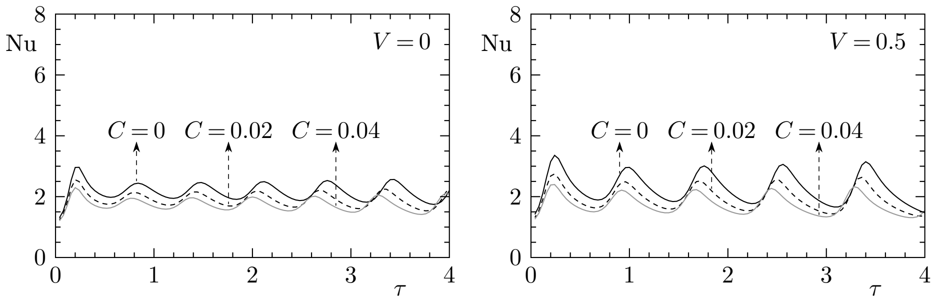

Figure 9. Variation of Nusselt number plot with τ for $Q = 2$, $R_I = -1$ and $\Pr = 10$, $\mathrm{Pm} = 10$.

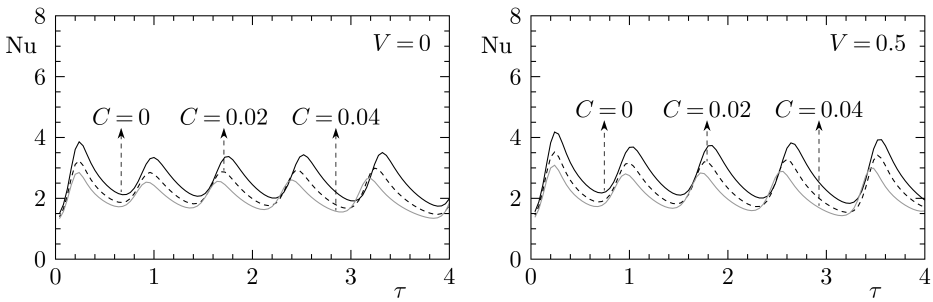

Figure 10. Variation of Nusselt number plot with τ for $Q = 2$, $R_I = 0$ and $\Pr = 10$, $\mathrm{Pm} = 10$.

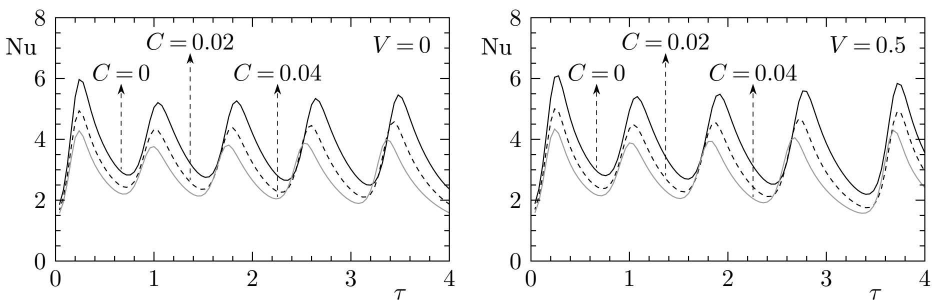

Figure 11. Variation of Nusselt number plot with τ for $Q = 2$, $R_I = 1$ and $\Pr = 10$, $\mathrm{Pm} = 10$.

5. Conclusion

(1) The effect of Chandrasekhar number and couple-stress parameter is that the system is stabilized.

(2) The effect of variable viscosity and internal Rayleigh number destabilizes the system and convection sets in soon.

(3) Both R_{E_c} and $\pi\alpha_C$ decrease as V and R_I increase in the absence of C.

(4) The Nusselt number decreases with an increase in Q and C.

(5) The internal Rayleigh number on thermal instability leads to destabilizing effect as Nu increases on increasing R_I, which results in the increase of heat transport of the fluid.

(6) The effect of increases in Chandrasekhar number Q and couple stress parameter C leads to a decrease in the Nusselt number, which indicates the stabilizing effect of the system.

(7) The effect of the increase in internal Rayleigh number R_I and variable viscosity V is that the heat transfer increases, which leads to the increase in the Nusselt number value.

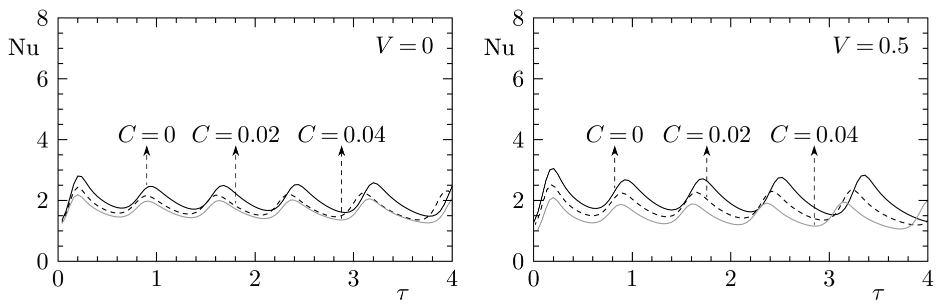

Figure 12. Variation of Nusselt number plot with τ for $Q = 4$, $R_I = -1$ and $\mathrm{Pr} = 10$, $\mathrm{Pm} = 10$.

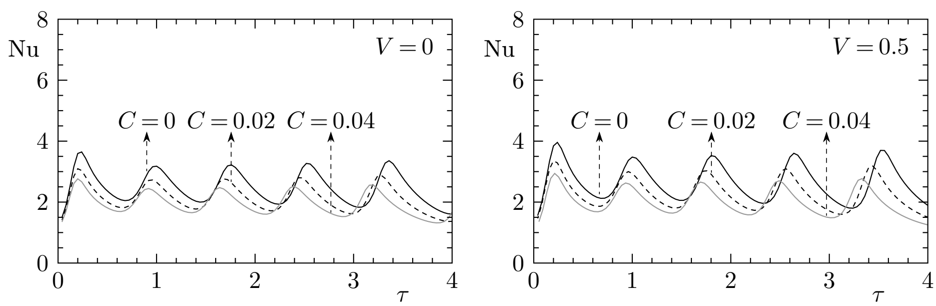

Figure 13. Variation of Nusselt number plot with τ for $Q = 4$, $R_I = 0$ and $\mathrm{Pr} = 10$, $\mathrm{Pm} = 10$.

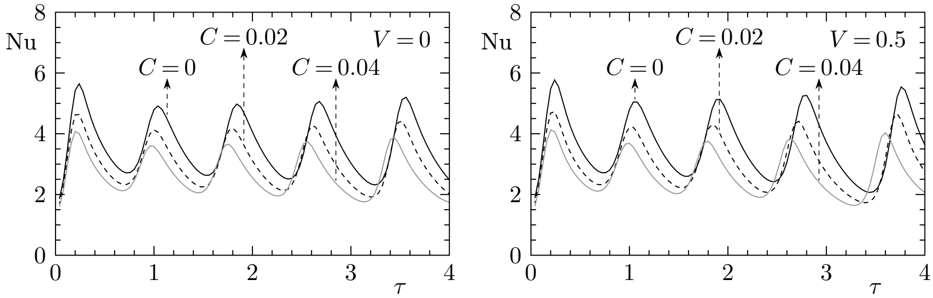

Figure 14. Variation of Nusselt number plot with τ for $Q = 4$, $R_I = 1$ and $\Pr = 10$, $\mathrm{Pm} = 10$.

A c k n o w l e d g e m e n t s. The first three authors are grateful to the principal and management of Ramaiah Institute of Technology, Bengaluru, whereas the fourth author is grateful to the management of R. R. Institute of Technology for the encouragement of research activities.

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