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S-SHAPED COMPONENT OF NODAL SOLUTIONS FOR PROBLEM INVOLVING ONE-DIMENSION MEAN CURVATURE OPERATOR

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Abstract. Let $E = \{u \in C^1[0,1]\colon u(0) = u(1) = 0\}$. Let S_k^ν with $\nu = \{+,-\}$ denote the set of functions $u \in E$ which have exactly $k-1$ interior nodal zeros in (0, 1) and νu be positive near 0. We show the existence of S-shaped connected component of S_k^ν-solutions of the problem

$$\begin{cases} \left(\dfrac{u'}{\sqrt{1-u'^2}}\right)' + \lambda a(x) f(u) = 0, \quad x \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where $\lambda > 0$ is a parameter, $a \in C([0,1],(0,\infty))$. We determine the intervals of parameter λ in which the above problem has one, two or three S_k^ν-solutions. The proofs of the main results are based upon the bifurcation technique.

Keywords: mean curvature operator; S_k^ν-solution; bifurcation; Sturm-type comparison theorem

MSC 2020: 34C23, 35J65, 35B40, 34C10

1. Introduction and main result

In this paper, we study the existence of S-shaped connected component of S_k^ν-solutions of the problem

$$\begin{cases} \left(\dfrac{u'}{\sqrt{1-u'^2}}\right)' + \lambda a(x) f(u) = 0, \quad x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1.1}$$

Here S_k^ν with $\nu = \{+,-\}$ and $k \in \mathbb{N}$ denotes the set of functions $y \in E = \{u \in C^1[0,1]\colon u(0) = u(1) = 0\}$ which have exactly $k-1$ interior nodal zeros

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in (0, 1), and νy is positive near 0. Moreover, $\lambda > 0$ is a parameter, a and f obey the conditions specified later.

The motion of the relativistic oscillator is described by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0\dot{x}}{\sqrt{1-\dot{x}^2/c^2}}\right)+\lambda u=0, \tag{1.2}$$

which has been investigated by several authors, see Hutten [13] and Mac-Coll [19]. Equation in (1.1) can be regarded as a more general form of (1.2).

Problem (1.1) is also the one-dimensional counterpart of the Dirichlet problem associated with the prescribed mean curvature equation in Minkowski space

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right)+\lambda f(x,u)=0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where λ is a positive parameter, Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 1$. Problem (1.3) plays an important role in certain fundamental issues in differential geometry and in the special theory of relativity, see for example [5], [6], [11]. We refer the readers, for motivations and results, to [1] and the references cited therein.

Existence, multiplicity and qualitative properties of positive solutions of (1.1) and (1.3) have been extensively studied by several authors via the method of critical point theory, fixed point theorem in cones as well as the bifurcation technique. We refer the reader to Coelho et al. [6], Bereanu et al. [2], [3], Corsato et al. [7], Dai and Wang [8] and Ma and Xu [17], [18] for references along this line. However, there are few papers dealing with the existence and multiplicity of sign-changing solutions of (1.3), see Boscaggin and Garrione [4], and Dai and Wang [9]. Boscaggin and Garrione in [4] showed that there are more and more sign-changing solutions on growing of the parameter $\lambda > 0$ for (1.3) with $\Omega = B_R(0)$ via the shooting method. Dai and Wang in [9] obtained the existence of $\subset$-type connected components for (1.3) with $\Omega = B_R(0)$ by bifurcation technique.

It is the purpose of this paper to give some conditions which will guarantee the existence of S-shaped connected component of S_k^ν-solutions of (1.1), and accordingly, we determine the intervals of parameter λ in which (1.1) has one, two and three S_k^ν-solutions.

As in [6], we understand that the solution of problem (1.1) is a function which belongs to $C^1[0,1]$ with $\|u'\|_\infty < 1$, such that $u'/\sqrt{1-u'^2}$ is differentiable and problem (1.1) is satisfied, where $\|\cdot\|_\infty$ denotes the usual sup-norm.

In order to study the global bifurcation phenomena of problem (1.1), we must consider the eigenvalue problem

$$\begin{cases} \varphi''(x) + \lambda a(x)\varphi(x) = 0, \quad x \in (0,1), \\ \varphi(0) = \varphi(1) = 0, \end{cases} \tag{1.4}$$

where a satisfies:

(F1) $a \in C[0,1]$ with $0 < a_* \leqslant a(x) \leqslant a^*$ on $[0, 1]$ for some $a_*, a^* \in (0,\infty)$.

It is well-known that (1.4) possesses infinitely many eigenvalues $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots \to \infty$, all of which are simple. The eigenfunction φ_k corresponding to λ_k has exactly $k-1$ simple zeros in $(0,1)$, see [22], page 269. Let

$$0 = t_0 < t_1 < \ldots < t_{k-1} < t_k = 1$$

be the successive zeros of φ_k.

In the special case of $a(x) \equiv 1$, we know that

$$\lambda_m = (m\pi)^2, \quad \varphi_m(x) = \sin m\pi x, \tag{1.5}$$

and accordingly,

$$t_{j+1} - t_j = \frac{1}{m}, \quad j \in \{0, 1, \ldots, m-1\}. \tag{1.6}$$

In the following, we assume that:

(F2) $f \in C(-\infty,\infty)$ with $f(0) = 0$, $sf(s) > 0$ for all $s \neq 0$.

(F3) There exists $\alpha > 1$, $f_0 > 0$ and $f_1 > 0$ such that

$$\lim_{|s|\to 0} \frac{f(s) - f_0 s}{s^{1+\alpha}} = -f_1.$$

(F4) There exists s_0: $s_0 \in [\frac{1}{24}c_*, \frac{1}{8}c_*)$ such that

$$\min_{|s|\in[s_0,4s_0]} \frac{f(s)}{s} > \frac{27 f_0}{5\sqrt{5}\lambda_k a_*}\Big(\frac{2\pi}{c_*}\Big)^2,$$

where $c_* := m_1^{-1}$ and m_1 be as in (1.9).

It is easy to find that if (F2) and (F3) hold, then

$$\lim_{|s|\to 0^+} \frac{f(s)}{s} = f_0. \tag{1.7}$$

Moreover, if (1.7) and (F2) hold, then there exists $f^* > 0$ such that

$$0 < \sup_{0\leqslant|s|\leqslant 1} \frac{f(s)}{s} \leqslant f^*. \tag{1.8}$$

Consider problem (1.1) and assume, in addition to (F1)–(F4), that f satisfies:

(F5) There exists a constant $m_1 \in \mathbb{N}$ such that

$$0 < \lambda_k a(x) \sup_{0<|s|<1} \frac{f(s)}{s} \leqslant (m_1 \pi)^2, \quad x \in [0,1], \tag{1.9}$$

for some $k \in \mathbb{N}$.

Indeed, under (F1), (F2) and (F3), we have an unbounded subcontinuum of S_k^ν-solutions of (1.1) which is bifurcating from $(\lambda_k/f_0, 0)$ and goes rightward. Conditions (F1), (F4) and (F5) lead the unbounded subcontinuum to the left at some point, and finally to the right near $\lambda = \infty$. Roughly speaking, we shall show that there exists an *S-shaped connected component* of S_k^ν-solutions.

Arguing the shape of bifurcation we have the following main result:

Theorem 1.1. *Assume that* (F1)–(F4) *hold. Let* (F5) *hold for some* $k \in \mathbb{N}$. *Then there exist* $\lambda_* \in (0, \lambda_k/f_0)$ *and* $\lambda^* > \lambda_k/f_0$ *such that*

(i) *problem* (1.1) *has at least one* S_k^ν*-solution if* $\lambda = \lambda_*$;
(ii) *problem* (1.1) *has at least two* S_k^ν*-solutions if* $\lambda_* < \lambda \leqslant \lambda_k/f_0$;
(iii) *problem* (1.1) *has at least three* S_k^ν*-solutions if* $\lambda_k/f_0 < \lambda < \lambda^*$;
(iv) *problem* (1.1) *has at least two* S_k^ν*-solutions if* $\lambda = \lambda^*$;
(v) *problem* (1.1) *has at least one* S_k^ν*-solution if* $\lambda > \lambda^*$.

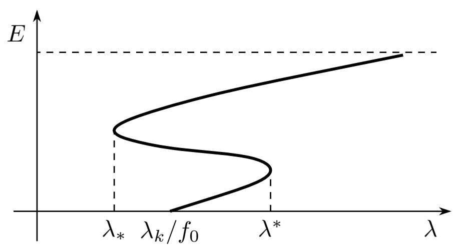

Figure 1. Bifurcation diagram of Theorem 1.1.

Figure 1 illustrates the global bifurcation graphs for Theorem 1.1. We point out that the solution branches shown in Figure 1 are qualitative sketches rather than computer calculated curves for representative f.

Remark 1.1. Let us consider that the nonlinear function

$$f(s) = s[(s-1)^2 + 1]e^{-s/m}, \quad m > 0,$$

satisfies (F3) with $m > 0$, $\alpha = 1$, $f_0 = 2$ and $f_1 = 2(m+1)/m$. Moreover, if $m > 0$ is sufficiently large, then this function satisfies (F4) with $s_0 = m+1$ since $g(s) := f(s)/s$ is increasing on $(m+1-\sqrt{m^2-1}, m+1+\sqrt{m^2-1})$ and is decreasing on $(m+1+\sqrt{m^2-1}, \infty)$ and hence, $\min\limits_{s\in[m+1,2(m+1)]} f(s)/s = \min\{g(m+1), g(2m+2)\} \to \infty$ as $m \to \infty$.

Remark 1.2. For other results concerning the existence of an S-*shaped connected component* in the set of solutions of semilinear problems, see [16], [20] for that of p-Laplacian problem, see [21].

Remark 1.3. Coelho et al. in [6] studied (1.1) when the nonlinearity f satisfies Carathéodory condition. Motivated by this paper, we may work in $L^\infty(0,1)$ and replace (F1) by a weaker condition

$$0 < a_* \leqslant \operatorname{essinf} a(\cdot) \leqslant \operatorname{esssup} a(\cdot) \leqslant a^*.$$

Remark 1.4. By the time-map method (see [12]), we may show that there exists $L_0 > 0$ such that

$$\begin{cases} \left(\dfrac{u'}{\sqrt{1-u'^2}}\right)' + \lambda L \arctan u = 0, \quad x \in (0,1), \\ u(0) = u(1) = 0 \end{cases}$$

has a unique positive solution for each $\lambda > \pi^2$ provided $L \in (0, L_0)$. So, (F4) is a crucial condition to guarantee the existence of S-shaped connected component of S_1^ν-solutions.

The rest of the paper is arranged as follows. In Section 2, we show a global bifurcation phenomena from the trivial branch with the rightward direction near the initial point. Section 3 is devoted to showing the change of direction of bifurcation, and completing the proof of Theorem 1.1.

2. Rightward bifurcation

Let $Y = C[0,1]$ with the norm

$$\|u\|_\infty = \max_{x\in[0,1]} |u(x)|.$$

Let $E = \{u \in C^1[0,1]\colon\ u(0) = u(1) = 0\}$ with the equivalent norm

$$\|u\| = \|u'\|_\infty.$$

Define $L\colon\ D(L) \to Y$ by setting

$$Lu := -u'', \quad u \in D(L),$$

where

$$D(L) = \{u \in C^2[0,1]\colon\ u(0) = u(1) = 0\}.$$

Then $L^{-1}\colon\ Y \to E$ is compact.

Let $\mathcal{S}$ be the closure of the set of all nontrivial solutions (λ, u) of (1.1) in $[0,\infty)\times E$. In the following, we shall show a global bifurcation phenomena from the trivial branch with the rightward direction of bifurcation. Let us rewrite (1.1) as

$$\begin{cases} u''(x) + \lambda a(x) f_0 u h(u') + \lambda a(x)\xi(u)h(u') = 0, & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{2.1}$$

where

$$h(s) = \begin{cases} (1-s^2)^{3/2}, & |s| \leqslant 1, \\ 0, & |s| > 1, \end{cases} \quad \text{and} \quad \xi(s) := f(s) - f_0 s, \quad s \in \mathbb{R}.$$

Since λ_k is simple and $\lim\limits_{|s|\to\infty} \xi(s)/s = 0$, it follows from (2.1) and a close analogue of Dancer unilateral global bifurcation theorem (see [10], Theorem 2) that there exist two continua $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$ of $\mathcal{S}$ bifurcating from $(\lambda_k/f_0, 0)$, such that either $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$ are both unbounded, or else $\mathcal{C}_k^+ \cap \mathcal{C}_k^- = \{(\lambda_k/f_0, 0)\}$. Notice that for any $\tau \in [0,1]$, the initial value problem

$$u''(x) + \lambda a(x) f_0 u h(u') + \lambda a(x)\xi(u)h(u') = 0, \quad u(\tau) = u'(\tau) = 0$$

has a unique solution. This implies the second case can never occur, see the proof of Theorem 1.1 in [9]. Thus, we get:

Lemma 2.1. *Assume that* (F1)–(F3) *hold. Let* (F5) *hold for some* $k \in \mathbb{N}$. *Then there exist two unbounded continua* $\mathcal{C}_k^+$ *and* $\mathcal{C}_k^-$ *of* $\mathcal{S}$ *bifurcating from* $(\lambda_k/f_0, 0)$ *such that for* $\nu \in \{+,-\}$ *and* $k \in \mathbb{N}$, *one has*

(1) $\mathcal{C}_k^\nu \subseteq (\mathbb{R}^+ \times S_k^\nu) \cup \{(\lambda_k/f_0, 0)\}$;

(2) u *has exactly* $k-1$ *simple zeros, and* νu *is positive near* 0 *for every* $(\lambda, u) \in \mathcal{C}_k^\nu \setminus (\lambda_k/f_0, 0)$;

(3) $\lim\limits_{\substack{(\lambda,u)\in\mathcal{C}_k^\nu,\\ \lambda\to\infty}} \|u\| = 1$.

By the same method as for proving Lemma 2.5 in [18] with obvious changes, we may get the following:

Lemma 2.2. *Assume that* (F1)–(F3) *hold. Let* (F5) *be satisfied for some* $k \in \mathbb{N}$. *Let* $\{(\beta_n, u_n)\}$ *be a sequence of* S_k^ν*-solutions of* (1.1) *which satisfies* $\|u_n\| \to 0$ *and* $\beta_n \to \lambda_k/f_0$. *Let* $\varphi_k(x)$ *be the* k*-eigenfunction of* (1.4) *which satisfies* $\|\varphi_k\| = 1$. *Then there exists a subsequence of* $\{u_n\}$, *again denoted by* $\{u_n\}$ *such that* $u_n/\|u_n\|$ *converges uniformly to* φ_k *on* $[0,1]$.

Lemma 2.3. *Assume that* (F1)–(F3) *hold. Let* (F5) *be satisfied for some* $k \in \mathbb{N}$. *Then there exists* $\delta > 0$ *such that* $(\lambda, u) \in \mathcal{C}_k^\nu$ *and* $|\lambda - \lambda_k/f_0| + \|u\| \leqslant \delta$ *imply* $\lambda > \lambda_k/f_0$.

Proof. We only deal with the case when $\nu = +$. The other case can be treated by a similar method.

Assume on the contrary that there exists a sequence $\{(\beta_n, u_n)\}$ such that $(\beta_n, u_n) \in \mathcal{C}_k^+$, $\beta_n \to \lambda_k/f_0$, $\|u_n\| \to 0$ and $\beta_n \leqslant \lambda_k/f_0$. By Lemma 2.2, there exists a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that $u_n/\|u_n\|$ converges uniformly to φ_k on $[0,1]$, where $\varphi_k \in S_k^+$ is the kth eigenfunction of (1.4) which satisfies $\|\varphi_k\| = 1$. Multiplying the equation of (1.1) with $(\lambda, u) = (\beta_n, u_n)$ by u_n and integrating it over $[0,1]$, we obtain

$$\beta_n \int_0^1 a(x)h(u_n'(x))f(u_n(x))u_n(x)\,\mathrm{d}x = \int_0^1 (u_n'(x))^2\,\mathrm{d}x, \tag{2.2}$$

and accordingly,

$$\beta_n \int_0^1 a(x)h(u_n'(x))\frac{f(u_n(x))}{\|u_n\|}\frac{u_n(x)}{\|u_n\|}\,\mathrm{d}x = \int_0^1 \frac{(u_n'(x))^2}{\|u_n\|^2}\,\mathrm{d}x. \tag{2.3}$$

On the other hand, we have

$$\int_0^1 (\varphi_k'(x))^2 = \lambda_k \int_0^1 a(x)\varphi_k^2(x)\,\mathrm{d}x. \tag{2.4}$$

By (2.3) and (2.4), it follows that

$$\beta_n \int_0^1 a(x)h(u_n'(x))f(u_n(x))u_n(x)\,\mathrm{d}x = \lambda_k \int_0^1 a(x)(u_n(x))^2\,\mathrm{d}x - \zeta(n)\|u_n\|^2, \tag{2.5}$$

with a function $\zeta\colon \mathbb{N} \to \mathbb{R}$ satisfying

$$\lim_{n\to\infty} \zeta(n) = 0.$$

Therefore,

$$\int_0^1 a(x)\frac{h(u_n')f(u_n(x)) - f_0u_n(x)}{u_n^{1+\alpha}(x)}\Big|\frac{u_n(x)}{\|u_n\|}\Big|^{2+\alpha}\,\mathrm{d}x = \frac{1}{\|u_n\|^\alpha}\Big[\frac{\lambda_k - f_0\beta_n}{\beta_n}\int_0^1 a(x)\Big|\frac{u_n(x)}{\|u_n\|}\Big|^2\,\mathrm{d}x - \zeta(n)\Big].$$

Lebesgue's dominated convergence theorem and condition (F3) imply that

$$\int_0^1 a(x)\frac{h(u_n')f(u_n(x))-f_0u_n(x)}{u_n^{1+\alpha}(x)}\Big|\frac{u_n(x)}{\|u_n\|}\Big|^{2+\alpha}\,\mathrm{d}x \to -f_1\int_0^1 a(x)|\varphi_k|^{2+\alpha}\,\mathrm{d}x<0,$$

and

$$\int_0^1 a(x)\Big|\frac{u_n(x)}{\|u_n\|}\Big|^2\,\mathrm{d}x \to \int_0^1 a(x)|\varphi_k|^2\,\mathrm{d}x>0.$$

This contradicts $\beta_n \leqslant \lambda_k/f_0$. □

3. Direction turn of bifurcation

Lemma 3.1. *Let $v \in C[0,1]$ with $v(t) \geqslant 0$ for $t \in [0,1]$. If $v'(t)$ is nonincreasing on $[0,1]$, then*

$$v(t) \geqslant \min\{t,1-t\}\|v\|_\infty, \quad t\in[0,1].$$

In particular, for any $\alpha,\beta\in(0,1)$ we have

$$\min_{\alpha\leqslant t\leqslant\beta} v(t) \geqslant \min\{\alpha,1-\beta\}\|v\|_\infty.$$

P r o o f. It is an immediate consequence of the fact that v is concave down in $[0,1]$ and hence we omit it. □

In the sequel, we will need the following lemma.

Lemma 3.2. *Let $\kappa\in(0,1)$, $0\leqslant t_j<t_{j+1}\leqslant 1$ such that $u(t_j)=u(t_{j+1})=0$ and $\beta_0\in(0,(\frac18(1-\kappa))(t_{j+1}-t_j))$ be given. Let $I_{\kappa,\beta_0} := [t_j+4\beta_0/(1-\kappa), t_{j+1}-4\beta_0/(1-\kappa)]$. Then*

$$\frac{t_{j+1}+t_j}{2}\in I_{\kappa,\beta_0}, \quad \text{and} \quad |u'(s)|\leqslant 1-\kappa \quad \forall\, u\in\mathcal{A},\ s\in I_{\kappa,\beta_0},$$

where

$$\mathcal{A} := \{u\in E\colon\ u \text{ is concave in } [t_j,t_{j+1}],\ 0<u'(t_j)<1,\ 0>u'(t_{j+1})>-1,\ \|u\|_\infty\leqslant 4\beta_0\}.$$

P r o o f. Set $1-\kappa=\alpha$ and $\xi=4\beta_0/(1-\kappa)$, then the conditions can be rewritten as

$$0<\alpha<1, \quad \xi\in\Big(0,\frac{t_{j+1}-t_j}{2}\Big), \quad \text{and} \quad I := I_{\kappa,\beta_0}=[t_j+\xi,t_{j+1}-\xi].$$

Assume on the contrary that there exists $s\in I$ such that $|u'(s)|>1-\kappa=\alpha$, then $u'(s)>\alpha$ or $u'(s)<-\alpha$.

We only deal with the case when $u'(s) > \alpha$, the other case can be treated by a similar method. By the fact that $u \in C^1[0,1]$ and u is concave in $[t_j, t_{j+1}]$, u' is decreasing. If $u'(s) > \alpha$, then $u(s) - u(t_j) = u'(t)(s - t_j)$ for some $t \in (t_j, s)$. Hence, $u(s)/(s - t_j) \geqslant u'(s) > \alpha$. Therefore, $u(s) > \alpha(s - t_j) \geqslant \xi\alpha = 4\beta_0 \geqslant \|u\|_\infty$. This contradicts $\|u\|_\infty \leqslant 4\beta_0$. □

Let $\kappa = \frac{1}{3}$ and $\frac{1}{24}(t_{j+1} - t_j) = \beta_0 \in (0, \frac{1}{12}(t_{j+1} - t_j))$. Then we have the following:

Corollary 3.1. *For any concave function $u \in E$ with*

$$u(0) = u(1) = 0, \quad 0 < u'(t_j) < 1, \quad -1 < u'(t_{j+1}) < 0, \quad \|u\|_\infty \leqslant \frac{t_{j+1} - t_j}{6},$$

we have

$$|u'(x)| \leqslant \frac{2}{3}, \quad x \in \Big[\frac{3t_j + t_{j+1}}{4}, \frac{t_j + 3t_{j+1}}{4}\Big].$$

Lemma 3.3. *Assume that* (F1) *and* (F2) *hold. Let* (F5) *be satisfied for some $k \in \mathbb{N}$. Let u be a S_k^ν solutions of* (1.1). *Then there exists $I_u := (\alpha_u, \beta_u)$ such that*

$$u(\alpha_u) = u(\beta_u) = 0, \quad \beta_u - \alpha_u \geqslant c_*; \quad |u| > 0 \text{ in } I_u, \quad \|u\|_\infty = u(t_0),\ t_0 \in I_u. \tag{3.1}$$

Moreover,

$$\frac{1}{4}\|u\|_\infty \leqslant u(x) \leqslant \|u\|_\infty, \quad x \in \Big[\frac{3\alpha_u + \beta_u}{4}, \frac{\alpha_u + 3\beta_u}{4}\Big] =: J_u, \tag{3.2}$$

or

$$\frac{1}{4}\|u\|_\infty \leqslant -u(x) \leqslant \|u\|_\infty, \quad x \in J_u. \tag{3.3}$$

Proof. By condition (F5) and the Sturm-type comparison theorem, see [14], [15], it is easy to see that there exist α_u, β_u such that $\beta_u - \alpha_u \geqslant c_*$, which follows that (3.1) is valid.

Since $-u'' = \lambda a(x)h(u')f(u)$, conditions (F1) and (F2) combined with the fact that $u(\alpha_u) = u(\beta_u) = 0$ imply that u' is decreasing on I_u and $u(x) > 0,\ x \in (\alpha_u, \beta_u)$ or u' is increasing on I_u and $u(x) < 0,\ x \in (\alpha_u, \beta_u)$. Suppose the former case occurs (in the latter one the argument would be similar).

By Lemma 3.1, it is easy to check that

$$\frac{1}{4}\|u\|_\infty \leqslant u(x) \leqslant \|u\|_\infty, \quad x \in J_u.$$

So, (3.2) is also valid. □

Lemma 3.4. *Assume that* (F1)–(F4) *hold. Let* (F5) *hold for some* $k \in \mathbb{N}$. *Let* $\mathcal{C}_k^\nu$ *be as in Lemma* 2.1. *If* $(\lambda, u) \in \mathcal{C}_k^\nu$ *such that* $\|u\|_\infty = 4s_0$, *we have* $\lambda < \lambda_k/f_0$.

P r o o f. Using the same notations used in the proof of Lemma 3.3, let $(\lambda, u) \in \mathcal{C}_k^\nu$, then by Lemma 3.3, we obtain

$$s_0 \leqslant u(x) \leqslant 4s_0, \quad x \in J_u, \tag{3.4}$$

or

$$s_0 \leqslant -u(x) \leqslant 4s_0, \quad x \in J_u. \tag{3.5}$$

We only deal with case (3.4) since case (3.5) can be treated in a similar way.

Fix $s_0 = \frac{1}{24}(\beta_u - \alpha_u)$, then from Corollary 3.1 for any $(\lambda, u) \in \mathcal{C}_k^\nu$ with $\|u\|_\infty = 4s_0$ we have

$$0 \leqslant |u'(x)| \leqslant \frac{2}{3}, \quad x \in J_u.$$

Assume on the contrary that $\lambda \geqslant \lambda_k/f_0$, then for $x \in J_u$, by (F5) and Corollary 3.1, we have

$$\lambda a(x)\frac{f(u)}{u}(1 - u'^2)^{3/2} \geqslant \frac{\lambda_k}{f_0} a_* \frac{27 f_0}{5\sqrt{5}\lambda_k a_*}\Big(\frac{2\pi}{c_*}\Big)^2 \frac{5\sqrt{5}}{27} \geqslant \Big(\frac{2\pi}{\beta_u - \alpha_u}\Big)^2.$$

Let

$$v(x) = \sin\Big(\frac{2\pi}{\beta_u - \alpha_u}\Big(x - \frac{3\alpha_u + \beta_u}{4}\Big)\Big),$$

then v is a positive solution of

$$\begin{cases} v''(x) + \Big(\dfrac{2\pi}{\beta_u - \alpha_u}\Big)^2 v(x) = 0, & x \in J_u, \\ v\Big(\dfrac{3\alpha_u + \beta_u}{4}\Big) = v\Big(\dfrac{\alpha_u + 3\beta_u}{4}\Big) = 0. \end{cases}$$

We note that $u(x)$ for all $x \in (\alpha_u, \beta_u)$ is a positive solution of

$$u''(x) + \lambda a(x)\frac{f(u)}{u}(1 - u'^2)^{3/2} u(x) = 0.$$

Sturm comparison theorem (see [14], [15]) implies that u has at least one zero on J_u. This contradicts the fact that $u(x) > 0$ on J_u. $\square$

Lemma 3.5. *Assume that* (F1) *holds. Let* (F5) *be satisfied for some* $k \in \mathbb{N}$. *Let* $\mathcal{C}_k^\nu$ *be as in Lemma* 2.1. *Then*

$$\lim_{\lambda \to \infty} \|u\|_\infty \in \Big[\frac{c_*}{2}, \frac{1}{2}\Big].$$

P r o o f. In the case of $k = 1$, Ma and Xu in [18] proved that

$$\lim_{\lambda\to\infty} \|u\| = 1, \quad \lim_{\lambda\to\infty} \|u\|_\infty = \frac{1}{2}.$$

Next, we shall consider the case when $k \geqslant 2$. Let u be a S_k^ν-solution of (1.1) and

$$0 = z_0 < z_1 < \ldots < z_{k-1} < z_k = 1$$

be the successive zeros of u. By the same method as for proving Lemma 3.3, we know that there exists z_j for some $j \in \{0, 1, \ldots, k-1\}$ such that $z_{j+1} - z_j \geqslant c_*$ and $\|u\|_\infty = u(t_0)$ for some $t_0 \in (z_j, z_{j+1})$.

Next, we deal with the case when $u(x) > 0$, $x \in (z_j, z_{j+1})$, the other case can be treated by a similar method. Using the same method as for proving (see [9], Theorem 1.1) we know that $u' \to 1$ in $C[\varepsilon + z_j, t_0 - \varepsilon_1]$ as $\lambda \to \infty$. Here ε and ε_1 are some arbitrary sufficiently small positive constants.

Since $f(s)s > 0$ for $s \neq 0$, it follows from (F5) that u is concave down in $[z_j, z_{j+1}]$, then

$$u'(x) \geqslant 0, \quad x \in [z_j, t_0], \quad -u'(x) < 0, \quad x \in (t_0, z_{j+1}].$$

Then for $(\lambda, u) \in \mathcal{C}_k^\nu$ we have

$$\lim_{\lambda\to\infty} \|u\|_\infty = \lim_{\lambda\to\infty} u(t_0) = \lim_{\lambda\to\infty} \int_{z_j}^{t_0} u'(t)\,\mathrm{d}t \geqslant \lim_{\lambda\to\infty} \int_{z_j-\varepsilon}^{t_0-\varepsilon_1} u'(t)\,\mathrm{d}t = t_0 - \varepsilon_1 - z_j + \varepsilon.$$

By the arbitrariness of ε and ε_1 we have

$$\lim_{\lambda\to\infty} \|u\|_\infty \geqslant t_0 - z_j.$$

Similarly, using the fact that $u' \to -1$ in $C[t_0 + \varepsilon_2, z_{j+1} - \varepsilon_3]$ as $\lambda \to \infty$ for arbitrary positive constants ε_2 and ε_3, we may deduce

$$\lim_{\lambda\to\infty} \|u\|_\infty \geqslant z_{j+1} - t_0.$$

Therefore, it yields that

$$\lim_{\lambda\to\infty} \|u\|_\infty \geqslant \frac{z_{j+1} - z_j}{2}. \tag{3.6}$$

On the other hand,

$$\lim_{\lambda\to\infty} \|u\|_\infty = \lim_{\lambda\to\infty} u(t_0) = \lim_{\lambda\to\infty} \int_{z_j}^{t_0} u'(t)\,\mathrm{d}t \leqslant t_0 - z_j, \quad t_0 \in (z_j, z_{j+1})$$

and

$$\lim_{\lambda\to\infty} \|u\|_\infty = \lim_{\lambda\to\infty} u(t_0) = \lim_{\lambda\to\infty} \int_{t_0}^{z_{j+1}} -u'(t)\,\mathrm{d}t \leqslant z_{j+1} - t_0, \quad t_0 \in (z_j, z_{j+1})$$

and accordingly,

$$\lim_{\lambda\to\infty}\|u\|_\infty \leqslant \frac{z_{j+1}-z_j}{2}. \tag{3.7}$$

Therefore, from (3.6) and (3.7) we have

$$\lim_{\lambda\to\infty}\|u\|_\infty = \frac{z_{j+1}-z_j}{2} \geqslant \frac{c_*}{2}.$$

□

P r o o f of Theorem 1.1. Let $\mathcal{C}_k^\nu$ be as in Lemma 2.1. We only deal with $\mathcal{C}_k^+$ since the case $\mathcal{C}_k^-$ can be treated similarly. By Lemma 2.3, $\mathcal{C}_k^+$ is bifurcating from $(\lambda_k/f_0, 0)$ and goes rightward.

By Lemmas 2.1, 3.5, we have $\lim\limits_{\lambda\to\infty}\|u\| = 1$ and $\lim\limits_{\lambda\to\infty}\|u\|_\infty \in [\frac{1}{2}c_*, \frac{1}{2}]$. Then there exists $(\lambda_0, u_0) \in \mathcal{C}_k^+$ such that $\|u_0\|_\infty = 4s_0$. Lemma 3.4 implies that $\lambda_0 < \lambda_k/f_0$.

By Lemmas 2.3, 3.4, $\mathcal{C}_k^+$ passes through some points $(\lambda_k/f_0, v_1)$ and $(\lambda_k/f_0, v_2)$ with $\|v_1\|_\infty < 4s_0 < \|v_2\|_\infty$, and there exist $\underline{\lambda}$ and $\overline{\lambda}$ which satisfy $0 < \underline{\lambda} < \lambda_k/f_0 < \overline{\lambda}$ and both (i) and (ii):

(i) If $\lambda \in (\lambda_k/f_0, \overline{\lambda}]$, then there exist u and v such that $(\lambda, u), (\lambda, v) \in \mathcal{C}_k^+$ and $\|u\|_\infty < \|v\|_\infty < 4s_0$.

(ii) If $\lambda \in (\underline{\lambda}, \lambda_k/f_0]$, then there exist u and v such that $(\lambda, u), (\lambda, v) \in \mathcal{C}_k^+$ and $\|u\|_\infty < 4s_0 < \|v\|_\infty$.

Define $\lambda^* = \sup\{\overline{\lambda}\colon \overline{\lambda}$ satisfies (i)$\}$ and $\lambda_* = \inf\{\underline{\lambda}\colon \underline{\lambda}$ satisfies (ii)$\}$. Then by the standard arguments, (1.1) has a S_k^+ solution at $\lambda = \lambda_*$ and $\lambda = \lambda^*$, respectively.

Clearly, $\mathcal{C}_k^+$ turns to the left at $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and to the right at $(\lambda_*, \|u_{\lambda_*}\|_\infty)$, finally to the right near $\lambda = \infty$. This completes the proof of Theorem 1.1.

□

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