

Hilal A. Ganie; Rezwan Ul Shaban; Bilal A. Rather; Shariefuddin Pirzada
On distance Laplacian energy in terms of graph invariants

Czechoslovak Mathematical Journal, Vol. 73 (2023), No. 2, 335–353

Persistent URL: <http://dml.cz/dmlcz/151661>

Terms of use:

© Institute of Mathematics AS CR, 2023

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

ON DISTANCE LAPLACIAN ENERGY IN TERMS
OF GRAPH INVARIANTSHILAL A. GANIE, REZWAN UL SHABAN, BILAL A. RATHER,
SHARIEFUDDIN PIRZADA, Srinagar

Received September 28, 2020. Published online March 20, 2023.

Abstract. For a simple connected graph G of order n having distance Laplacian eigenvalues $\varrho_1^L \geq \varrho_2^L \geq \dots \geq \varrho_n^L$, the distance Laplacian energy $DLE(G)$ is defined as $DLE(G) = \sum_{i=1}^n |\varrho_i^L - 2W(G)/n|$, where $W(G)$ is the Wiener index of G . We obtain a relationship between the Laplacian energy and the distance Laplacian energy for graphs with diameter 2. We obtain lower bounds for the distance Laplacian energy $DLE(G)$ in terms of the order n , the Wiener index $W(G)$, the independence number, the vertex connectivity number and other given parameters. We characterize the extremal graphs attaining these bounds. We show that the complete bipartite graph has the minimum distance Laplacian energy among all connected bipartite graphs and the complete split graph has the minimum distance Laplacian energy among all connected graphs with a given independence number. Further, we obtain the distance Laplacian spectrum of the join of a graph with the union of two other graphs. We show that the graph $K_k \nabla (K_t \cup K_{n-k-t})$, $1 \leq t \leq \lfloor \frac{n-k}{2} \rfloor$, has the minimum distance Laplacian energy among all connected graphs with vertex connectivity k . We conclude this paper with a discussion on the trace norm of a matrix and the importance of our results in the theory of the trace norm of the matrix $D^L(G) - (2W(G)/n)I_n$.

Keywords: distance matrix; energy; distance Laplacian matrix; distance Laplacian energy

MSC 2020: 05C50, 05C12, 15A18

1. INTRODUCTION

A graph is denoted by $G(V(G), E(G))$, where $V(G) = \{v_1, v_2, \dots, v_n\}$ is its vertex set and $E(G)$ is its edge set. Throughout, G is connected, simple and finite. $|V(G)| = n$ is the *order* and $|E(G)| = m$ is the *size* of G . The set of vertices ad-

The research of S. Pirzada is supported by SERB-DST, New Delhi under the research project number CRG/2020/000109.

adjacent to $v \in V(G)$, denoted by $N(v)$, refers to the *neighborhood* of v . The *degree* of v , denoted by $d_G(v)$ (we simply write d_v if it is clear from the context) means the cardinality of $N(v)$. A graph is *regular* if each of its vertices has the same degree. The adjacency matrix $A = (a_{ij})$ of G is a $(0, 1)$ -square matrix of order n whose (i, j) -entry is equal to 1 if v_i is adjacent to v_j , and equal to 0 otherwise. Let $\text{Deg}(G) = \text{diag}(d_1, d_2, \dots, d_n)$ be the diagonal matrix of vertex degrees $d_i = d_G(v_i)$, $i = 1, 2, \dots, n$ associated to G . Matrix $L(G) = \text{Deg}(G) - A(G)$ is the Laplacian matrix and its spectrum is the Laplacian spectrum of G . This matrix is real symmetric and positive semi-definite. We take $0 = \mu_n \leq \mu_{n-1} \leq \dots \leq \mu_1$ to be the Laplacian eigenvalues of G . The Laplacian energy of a graph (see [12]), denoted by $\text{LE}(G)$, is defined as $\text{LE}(G) = \sum_{i=1}^n |\mu_i - 2m/n|$. For some recent papers and related results on Laplacian energy, we refer to [10], [17] and the references therein. For other undefined notations and terminology, the readers are referred to [5], [14], [16].

The *distance* between two vertices $u, v \in V(G)$, denoted by d_{uv} , is defined as the length of a shortest path between u and v . The *diameter* of G is the maximum distance between any two vertices of G . The *distance matrix* of G , denoted by $D(G)$, is defined as $D(G) = (d_{uv})_{u,v \in V(G)}$. The *transmission* $\text{Tr}_G(v)$ of a vertex v is defined to be the sum of the distances from v to all other vertices in G , i.e., $\text{Tr}_G(v) = \sum_{u \in V(G)} d_{uv}$. A graph G is said to be *k-transmission regular* if $\text{Tr}_G(v) = k$ for each $v \in V(G)$. The *transmission number* or Wiener index of a graph G , denoted by $W(G)$, is the sum of distances between all unordered pairs of vertices in G . Clearly, $W(G) = \frac{1}{2} \sum_{v \in V(G)} \text{Tr}_G(v)$. For any vertex $v_i \in V(G)$, the transmission $\text{Tr}_G(v_i)$ is called the *transmission degree*, shortly denoted by Tr_i and the sequence $\{\text{Tr}_1, \text{Tr}_2, \dots, \text{Tr}_n\}$ is called the *transmission degree sequence* of the graph G .

Let $\text{Tr}(G) = \text{diag}(\text{Tr}_1, \text{Tr}_2, \dots, \text{Tr}_n)$ be the diagonal matrix of vertex transmissions of G . Aouchiche and Hansen in [1] defined the *distance Laplacian matrix* of G as $D^L(G) = \text{Tr}(G) - D(G)$.

Let $\varrho_1^D \geq \varrho_2^D \geq \dots \geq \varrho_n^D$ and $\varrho_1^L \geq \varrho_2^L \geq \dots \geq \varrho_n^L$ be respectively the distance and distance Laplacian eigenvalues of the graph G . The distance energy (see [13]) of G is the sum of the absolute values of the distance eigenvalues of G , that is, $\text{DE}(G) = \sum_{i=1}^n |\varrho_i^D|$. For some recent works on distance energy we refer to [2], [6], [7] and the references therein.

The distance Laplacian energy $\text{DLE}(G)$ (see [18]) of a connected graph G is defined as

$$\text{DLE}(G) = \sum_{i=1}^n \left| \varrho_i^L - \frac{2W(G)}{n} \right|.$$

Let σ be the largest positive integer such that $\varrho_\sigma^L \geq 2W(G)/n$ and let $U_k^L(G) = \sum_{i=1}^k \varrho_i^L$ be the sum of k largest distance Laplacian eigenvalues of G . Using $\sum_{i=1}^n \varrho_i^L = 2W(G)$, it can be seen that

$$\begin{aligned} \text{DLE}(G) &= 2 \left(U_\sigma^L(G) - \frac{2\sigma W(G)}{n} \right) = 2 \max_{1 \leq j \leq n} \left(\sum_{i=1}^j \varrho_i^L(G) - \frac{2jW(G)}{n} \right) \\ &= 2 \max_{1 \leq j \leq n} \left(U_j^L(G) - \frac{2jW(G)}{n} \right). \end{aligned}$$

For some recent works on $\text{DLE}(G)$, see [6], [7], [8], [9].

The rest of the paper is organized as follows. In Section 2, we obtain a relationship between the Laplacian energy and the distance Laplacian energy for graphs with diameter 2. We also obtain a lower bound for the distance Laplacian energy $\text{DLE}(G)$ in terms of the order n , the Wiener index $W(G)$, etc. and characterize the extremal graphs. In Section 3, we study the distance Laplacian energy of connected bipartite graphs and connected graphs with a given independence number. We show that the complete bipartite graph has the minimum distance Laplacian energy among all connected bipartite graphs and that the complete split graph has the minimum distance Laplacian energy among all connected graphs with a given independence number. In Section 4, we study the distance Laplacian spectrum of the join of a graph with the union of two other graphs. We show that the graph $K_k \nabla (K_t \cup K_{n-k-t})$, $1 \leq t \leq \lfloor \frac{n-k}{2} \rfloor$, has the minimum distance Laplacian energy among all connected graphs with vertex connectivity k . We conclude the paper with a conclusion highlighting the importance of our results.

2. BOUNDS FOR THE DISTANCE LAPLACIAN ENERGY OF A GRAPH

We begin with the lemma, which gives the relation between the distance Laplacian spectrum of a graph and its connected spanning subgraph.

Lemma 2.1 ([1]). *Let G be a connected graph of order n and size m , where $m \geq n$ and let $G' = G - e$ be a connected graph obtained from G by deleting an edge. Let $\varrho_1^L(G) \geq \varrho_2^L(G) \geq \dots \geq \varrho_n^L(G)$ and $\varrho_1^L(G') \geq \varrho_2^L(G') \geq \dots \geq \varrho_n^L(G')$ be respectively the distance Laplacian eigenvalues of G and G' . Then $\varrho_i^L(G') \geq \varrho_i^L(G)$ holds for all $1 \leq i \leq n$.*

The following lemma shows that the distance Laplacian eigenvalues of a connected graph G of diameter 2 are completely determined by the Laplacian eigenvalues of the graph G .

Lemma 2.2 ([1]). *Let G be a connected graph of order $n \geq 2$ having diameter $d = 2$. Let $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{n-1} > \mu_n = 0$ be the Laplacian eigenvalues and $\varrho_1^L(G) \geq \varrho_2^L(G) \geq \dots \geq \varrho_{n-1}^L(G) > \varrho_n^L(G) = 0$ be the distance Laplacian eigenvalues of G . Then $\varrho_i^L(G) = 2n - \mu_{n-i}$ for $i = 1, 2, \dots, n-1$.*

As the complete graph K_n is the only graph of diameter 1, so $\text{DLE}(K_n) = 2(n-1)$. Therefore, in the rest of the paper we will deal with the graphs of diameter greater or equal to 2. The following result gives the distance Laplacian energy of a graph of diameter 2 in terms of sum of the Laplacian eigenvalues of G .

Theorem 2.1. *Let G be a connected graph of order $n \geq 3$ and size m having diameter 2. Then*

$$\text{DLE}(G) = 2 \left(\sigma \left(\frac{2m}{n} + 2 \right) - 2m + S_{n-\sigma-1}(G) \right),$$

where $S_{n-\sigma-1}(G) = \sum_{i=1}^{n-\sigma-1} \mu_i$ is the sum of the $n-\sigma-1$ largest Laplacian eigenvalues of G and σ , $1 \leq \sigma \leq n-2$, is the number of distance Laplacian eigenvalues of G which are greater than or equal to $2W(G)/n$.

Proof. Let G be a connected graph of order n having m edges. Since diameter of G is two, it follows that $\text{Tr}(v_i) = d_i + 2(n-1-d_i) = 2n-2-d_i$ for all $v_i \in V(G)$ and so $2W(G) = 2n(n-1) - 2m$. Let σ be the number of distance Laplacian eigenvalues of G which are greater than or equal to $2W(G)/n$. Using Lemma 2.2 and the definition of Laplacian energy, we have

$$\begin{aligned} \text{DLE}(G) &= 2 \left(\sum_{i=1}^{\sigma} \varrho_i^L(G) - \frac{2\sigma W(G)}{n} \right) = 2 \left(\sum_{i=1}^{\sigma} (2n - \mu_{n-i}) - \sigma \left(2n - 2 - \frac{2m}{n} \right) \right) \\ &= 2 \left(\sigma \left(\frac{2m}{n} + 2 \right) - \sum_{i=1}^{\sigma} \mu_{n-i} \right) = 2 \left(\sigma \left(\frac{2m}{n} + 2 \right) - 2m + S_{n-\sigma-1}(G) \right), \end{aligned}$$

where $S_{n-\sigma-1}(G) = \sum_{i=1}^{n-\sigma-1} \mu_i$ is the sum of the $n-\sigma-1$ largest Laplacian eigenvalues of G . □

From Theorem 2.1, it is clear that any lower bound or upper bound for the graph invariant $S_k(G)$, the sum of k largest Laplacian eigenvalues of G gives a lower bound or upper bound for $\text{DLE}(G)$. In fact, there is a conjecture by Brouwer (see [4], page 60) for the graph invariant $S_k(G)$, which is stated as follows.

Conjecture 2.1. *If G is any graph with order n and size m , then*

$$S_k(G) = \sum_{i=1}^k \mu_i \leq m + \binom{k+1}{2} \quad \text{for any } k \in \{1, 2, \dots, n\}.$$

This conjecture has been shown to be true for various families of graphs, but as a whole, this conjecture is still open. For some recent developments on Brouwer's conjecture, we refer to [11] and the references therein.

The following theorem gives a relation between the distance Laplacian energy and the Laplacian energy of a graph of diameter 2.

Theorem 2.2. *Let G be a connected graph of order $n \geq 3$ and size m having diameter 2. Let σ and t , $1 \leq \sigma, t \leq n-2$, be respectively the number of distance Laplacian eigenvalues and the number of Laplacian eigenvalues of G which are greater than or equal to $2W(G)/n$ and $2m/n$. Then*

$$\text{LE}(G) - 2\left(\frac{2m}{n} - 2(n-1) + 2t\right) \leq \text{DLE}(G) \leq \text{LE}(G) + 4\left(\sigma - \frac{m}{n}\right).$$

Proof. Let σ be the number of distance Laplacian eigenvalues of G which are greater than or equal to $2W(G)/n$. Then, by definition of distance Laplacian energy, we have

$$(2.1) \quad \text{DLE}(G) = 2\left(U_{\sigma}^L(G) - \frac{2\sigma W(G)}{n}\right) = 2 \max_{1 \leq j \leq n-1} \left(U_j^L(G) - \frac{2jW(G)}{n}\right).$$

Also, if t is the number of Laplacian eigenvalues of G which are greater than or equal to $2m/n$, then by definition of Laplacian energy, we have

$$(2.2) \quad \text{LE}(G) = 2\left(S_t(G) - \frac{2tm}{n}\right) = 2 \max_{1 \leq j \leq n-1} \left(S_j(G) - \frac{2jm}{n}\right).$$

Using Theorem 2.1 and the second equality of (2.2), we have

$$\begin{aligned} \text{DLE}(G) &= 2\left(\sigma\left(\frac{2m}{n} + 2\right) - 2m + S_{n-\sigma-1}(G)\right) \\ &= 2\left(2\sigma - \frac{2m}{n}\right) + 2\left(S_{n-\sigma-1}(G) - \frac{2m(n-\sigma-1)}{n}\right) \\ &\leq \text{LE}(G) + 4\left(\sigma - \frac{m}{n}\right), \end{aligned}$$

as $1 \leq n - \sigma - 1 \leq n - 1$. Using Lemma 2.2 and (2.1), we have

$$\begin{aligned}
\text{LE}(G) &= 2\left(S_t(G) - \frac{2tm}{n}\right) = 2\left(\left(2n - \frac{2m}{n}\right)t - \sum_{i=1}^t \varrho_{n-i}^L\right) \\
&= 2\left(\left(2n - \frac{2m}{n}\right)t - 2W(G) + \sum_{i=1}^{n-t-1} \varrho_i^L\right) \\
&= 2\left(\frac{2m}{n} - 2(n-1) + 2t\right) + 2\left(\sum_{i=1}^{n-t-1} \varrho_i^L - \frac{2(n-t-1)W(G)}{n}\right) \\
&\leq 2\left(\frac{2m}{n} - 2(n-1) + 2t\right) + \text{DLE}(G),
\end{aligned}$$

as $2W(G) = 2n(n-1) - 2m$ and $1 \leq n - t - 1 \leq n - 1$. This completes the proof. \square

From Theorem 2.2, the following observation is immediate.

Corollary 2.1. *Let G be a connected graph of order $n \geq 3$ and size m having diameter 2. Let σ and t , $1 \leq \sigma, t \leq n - 2$, be respectively the number of distance Laplacian eigenvalues and the number of Laplacian eigenvalues of G which are greater than or equal to $2W(G)/n$ and $2m/n$. Then $\text{LE}(G) > \text{DLE}(G)$, provided that $m > \sigma n$ and $\text{LE}(G) < \text{DLE}(G)$, provided that $t < n - m/n - 2$.*

For the star graph $K_{1,n-1}$, the Laplacian spectrum is $\{n, 1^{[n-2]}, 0\}$ and $m = n - 1$. It is easy to see that $t = 1$ for $K_{1,n-1}$ and so the inequality $\text{LE}(K_{1,n-1}) < \text{DLE}(K_{1,n-1})$ is valid for $n \geq 4$.

The following observation also follows from Theorem 2.2.

Corollary 2.2. *Let G be a connected graph of order $n \geq 3$ and size m having diameter 2. Let σ and t , $1 \leq \sigma, t \leq n - 2$, be respectively the number of distance Laplacian eigenvalues and number of Laplacian eigenvalues of G which are greater than $2W(G)/n$ and $2m/n$. Then $\sigma \geq n - (t + 1)$.*

The following lemma (see [6]) gives an upper bound for the second smallest distance Laplacian eigenvalue $\varrho_{n-1}^L(G)$ in terms of order n and the minimum transmission degree Tr_{\min} of the graph G .

Lemma 2.3. *Let G be a connected graph of order $n \geq 3$ having minimum transmission Tr_{\min} and the second smallest distance Laplacian eigenvalue $\varrho_{n-1}^L(G)$. Then*

$$(2.3) \quad \varrho_{n-1}^L(G) \leq \frac{n}{n-1} \text{Tr}_{\min},$$

with equality if and only if G is a graph containing a vertex of transmission degree $n - 1$.

Next, we obtain a lower bound for the distance Laplacian energy in terms of the order n , the Wiener index $W(G)$ and the minimum transmission Tr_{\min} of the graph G .

Theorem 2.3. *Let G be a connected graph of order $n \geq 3$ having minimum transmission degree Tr_{\min} and Wiener index $W(G)$. Then*

$$(2.4) \quad \text{DLE}(G) \geq \frac{8W(G)}{n} - \frac{2n \text{Tr}_{\min}}{n-1},$$

equality occurs if and only if $\sigma(G) = n - 2$ and G contains a vertex of transmission degree $n - 1$.

Proof. Let G be a connected graph of order $n \geq 3$ having distance Laplacian eigenvalues $\varrho_1^L(G) \geq \varrho_2^L(G) \geq \dots \geq \varrho_{n-1}^L(G) \geq \varrho_n^L(G) = 0$. Let σ be the number of distance Laplacian eigenvalues of G which are greater than or equal to $2W(G)/n$. Using $\sum_{i=1}^n \varrho_i^L(G) = 2W(G)$ and the definition of Laplacian energy (see [6]), we have

$$\begin{aligned} \text{DLE}(G) &= 2 \max_{1 \leq j \leq n-1} \left(\sum_{i=1}^j \varrho_i^L(G) - \frac{2jW(G)}{n} \right) \geq 2 \left(\sum_{i=1}^{n-2} \varrho_i^L(G) - \frac{2(n-2)W(G)}{n} \right) \\ &= 2 \left(2W(G) - \varrho_{n-1}^L(G) - \frac{2(n-2)W(G)}{n} \right) = \frac{8W(G)}{n} - 2\varrho_{n-1}^L(G) \\ &\geq \frac{8W(G)}{n} - \frac{2n \text{Tr}_{\min}}{n-1}. \end{aligned}$$

Equality occurs in (2.4) if and only if equality occurs in

$$(2.5) \quad \max_{1 \leq j \leq n-1} \left(\sum_{i=1}^j \varrho_i^L(G) - \frac{2jW(G)}{n} \right) = \left(\sum_{i=1}^{n-2} \varrho_i^L(G) - \frac{2(n-2)W(G)}{n} \right)$$

and equality occurs in $\varrho_{n-1}^L(G) \leq (n/(n-1))\text{Tr}_{\min}$. Equality occurs in (2.5) if and only if $\sigma = n - 2$ and by Lemma 2.3 equality occurs in $\varrho_{n-1}^L(G) \leq (n/(n-1))\text{Tr}_{\min}$ if and only if G contains a vertex having transmission degree $n - 1$. This shows that equality occurs in (2.4) if and only if $\sigma = n - 2$ and G contains a vertex having transmission degree $n - 1$. This completes the proof. \square

The parameter t gives the number of Laplacian eigenvalues of a graph G which are in the interval $[2m/n, n]$. This parameter has been studied for various families of graphs and is presently an active topic of research in spectral study of graphs. Like the parameter t , the parameter σ is concerned with the distribution of distance Laplacian eigenvalues of a connected graph G . In fact, the value of σ gives the number of distance Laplacian eigenvalues which are in the interval $[2W(G)/n, \infty)$. It will be of interest to discuss the following problem for the parameter σ .

Problem 2.1. Characterize the connected graphs with $\sigma = 1$, $\sigma = n - 2$ and $\sigma = n - 1$. Establish relations between σ and the different parameters associated with the structure of G .

For the transmission regular graphs G , it can be seen that $\sigma = n - \gamma$, where γ is the positive inertia of the distance matrix of the graph G .

3. DISTANCE LAPLACIAN ENERGY OF BIPARTITE GRAPHS AND GRAPHS WITH GIVEN INDEPENDENCE NUMBER

In this section, we study the distance Laplacian energy of bipartite graphs and the distance Laplacian energy of graphs with a given independence number t . Among all connected bipartite graphs the complete bipartite graph has the minimum distance Laplacian energy. This can be seen as follows.

Theorem 3.1. Let G be a connected bipartite graph of order $n \geq 3$ with partite sets of cardinality a and b such that $a \leq b$ and $a + b = n$.

(1) If $a < b$, then

$$\text{DLE}(G) \geq \begin{cases} 4n^2 - 6n - 4ab - \frac{4(n-2)W(G)}{n} & \text{if } 2ab \geq n(b-2), \\ 2(b-1)\left(2n - a - \frac{2W(G)}{n}\right) & \text{if } 2ab < n(b-2), \end{cases}$$

with equality if and only if $G \cong K_{a,b}$.

(2) If $n = 2a$ and $n \geq 5$, then $\text{DLE}(G) \geq 12a(a-1) - 4(n-2)W(G)/n = 12b(b-1) - 4(n-2)W(G)/n$, with equality if and only if $G \cong K_{a,a}$.

Proof. Let G be a connected bipartite graph of order n with vertex set $V(G)$. Let $V(G) = V_1 \cup V_2$ with $|V_1| = a$, $|V_2| = b$ being a bipartition of the vertex set $V(G)$ of G . Since G is a connected bipartite graph with partite sets of cardinality a and b , it follows that G is a spanning subgraph of the complete bipartite graph $K_{a,b}$. Therefore, by Lemma 2.1, we have $\varrho_i^L(G) \geq \varrho_i^L(K_{a,b})$ for all $i = 1, 2, \dots, n$. With this and the definition of distance Laplacian energy $\text{DLE}(G)$, we have

$$\begin{aligned} (3.1) \quad \text{DLE}(G) &= 2 \left(\sum_{i=1}^{\sigma} \varrho_i^L(G) - \frac{2\sigma W(G)}{n} \right) = 2 \max_{1 \leq j \leq n} \left(\sum_{i=1}^j \varrho_i^L(G) - \frac{2jW(G)}{n} \right) \\ &\geq 2 \max_{1 \leq j \leq n} \left(\sum_{i=1}^j \varrho_i^L(K_{a,b}) - \frac{2jW(G)}{n} \right), \end{aligned}$$

where σ is the largest positive integer such that $\varrho_{\sigma}^L(G) \geq 2W(G)/n$.

In [3], it can be seen that the distance Laplacian spectrum of $K_{a,b}$ is $\{(2n-a)^{[b-1]}, (2n-b)^{[a-1]}, n, 0\}$ and $2W(K_{a,b}) = 2n^2 - 2n - 2ab$. Since a and b are positive integers, we assume that $a \leq b$. We first prove the result for $a < b$. If $a < b$, then $2n-a \geq 2n-b$. Also $2n-a > n$ holds as $n > a$. Likewise, if $n > b$, then $2n-b > n$. These observations imply that $2n-a$ is the distance Laplacian spectral radius of $K_{a,b}$ and so we always have $2n-a \geq 2W(K_{a,b})/n$. For the eigenvalue n , we have $n < 2W(K_{a,b})/n$ implying that $n^2 - 2n - 2ab > 0$, which further gives

$$(3.2) \quad 2a^2 - 2an + n^2 - 2n > 0 \quad \text{as } a+b=n.$$

Consider the polynomial $f(a) = 2a^2 - 2an + n^2 - 2n$ for $1 \leq a < n$. The discriminant of this polynomial is $d = 4n(4-n)$. Clearly, for $n > 4$ we have $d < 0$ and so for this n we always have $f(a) > 0$. For $n = 3, 4$ it can be seen by direct calculation that inequality (3.2) holds. This shows that $n < 2W(K_{a,b})/n$ holds for all $a < b$. For the eigenvalue $2n-b$ we have $2n-b \geq 2W(K_{a,b})/n$ implying that $2ab \geq n(b-2)$. This shows that if $2ab \geq n(b-2)$, then $2W(G)/n \leq 2n-b$ and if $2ab < n(b-2)$, we have $2W(G)/n > 2n-b$. In other words, if $2ab \geq n(b-2)$, then $\sigma = n-2$ while if $2ab < n(b-2)$, then $\sigma = b-1$. Therefore, we have the following cases to consider.

Case (i): If $2ab \geq n(b-2)$, then the number of eigenvalues which are greater or equal to $2W(G)/n$ are $n-2$, that is, $\sigma = n-2$. Since $1 \leq \sigma \leq n-1$, from inequality (3.1), it follows that

$$\begin{aligned} \text{DLE}(G) &\geq 2 \left(\sum_{i=1}^{n-2} \varrho_i^L(K_{a,b}) - \frac{2(n-2)W(G)}{n} \right) \\ &= 2 \left((b-1)(2n-a) + (a-1)(2n-b) - \frac{2(n-2)W(G)}{n} \right) \\ &= 4n^2 - 6n - 4ab - \frac{4(n-2)W(G)}{n}. \end{aligned}$$

Case (ii): If $2ab < n(b-2)$, then the number of eigenvalues which are greater or equal to $2W(G)/n$ is $b-1$, that is, $\sigma = b-1$. Since $1 \leq \sigma \leq n-1$, from inequality (3.1), it follows that

$$\text{DLE}(G) \geq 2 \left(\sum_{i=1}^{b-1} \varrho_i^L(K_{a,b}) - \frac{2(b-1)W(G)}{n} \right) = 2(b-1) \left(2n-a - \frac{2W(G)}{n} \right).$$

Equality occurs in each of the inequalities above if and only if equality occurs in (3.1). It is clear that equality occurs in (3.1) if and only if $G \cong K_{a,b}$. This implies that equality occurs if and only if $G \cong K_{a,b}$. This completes the proof in this case.

If $a = b$, then $n = 2a$ and so the distance Laplacian spectrum of $K_{a,a}$ is $\{(2n - a)^{\lfloor n-2 \rfloor}, n, 0\}$ and $2W(K_{a,a})/n = (2n^2 - an - 3n + 2a)/n$. Since $2n - a \geq 2W(K_{a,a})/n$ holds for all a and for $n \geq 5$, we have $2W(K_{a,a})/n > n$, from which it follows that $\sigma = n - 2$. Therefore, from (3.1) we have

$$\text{DLE}(G) \geq 2 \left(\sum_{i=1}^{n-2} \varrho_i^L(K_{a,b}) - \frac{2(n-2)W(G)}{2} \right) = 12a(a-1) - \frac{4(n-2)W(G)}{n}.$$

Equality case can be discussed similarly to the case $a < b$. This completes the proof. \square

For the complete bipartite graph $K_{a,b}$ with $a+b = n$ and $n \geq 3$, using Theorem 3.1, we have the following:

For $2ab \geq n(b-2)$ with $a < b$ we have

$$4n^2 - 6n - 4ab - \frac{4(n-2)W(K_{a,b})}{n} > 12a(a-1) - \frac{4(n-2)W(K_{a,b})}{n}$$

implying that $8a^2 + (4n-12)a - (4n^2-6n) < 0$. The zeros of the polynomial $f(a) = 8a^2 + (4n-12)a - (4n^2-6n)$ are $-n + \frac{3}{2}, \frac{1}{2}n$. This implies that $f(a) < 0$ for all $a \in (-n + \frac{3}{2}, \frac{1}{2}n)$. This shows that $4n^2 - 6n - 4ab - 4(n-2)W(K_{a,b})/n > 12a(a-1) - 4(n-2)W(G)/n$ holds for all $a < \frac{1}{2}n$. For $2ab < n(b-2)$ with $a < b$ we have

$$2(b-1) \left(2n - a - \frac{2W(K_{a,b})}{n} \right) > 12a(b-1) - \frac{4(2b-2)W(K_{a,b})}{n}$$

giving that $2n - a + 2W(K_{a,b})/n > 6a$. Using $2W(K_{a,b}) = 2n^2 - 2n - 2an + 2a^2 = (n-1)^2 + (n-a)^2 + a^2 - 1$, we get $2a^2 - 9an + 4n^2 - 2n > 0$. The zeros of the polynomial $g(a) = 2a^2 - 9an + 4n^2 - 2n$ are $y_1 = \frac{1}{4}(9n + \sqrt{49n^2 + 16n})$, $y_2 = \frac{1}{4}(9n - \sqrt{49n^2 + 16n})$. This shows that $g(a) > 0$ for all $a > y_1$ and for all $a < y_2$. Since $y_1 > n$, it follows that $g(a) > 0$ for all $a < y_2$. For $n \geq 3$ it is easy to see that $y_2 > \frac{1}{2}(n-1)$, implying that $g(a) > 0$ for all $a < \frac{1}{2}(n-1)$. This shows that

$$2(b-1) \left(2n - a - \frac{2W(K_{a,b})}{n} \right) > 12a(b-1) - \frac{4(2b-2)W(K_{a,b})}{n}$$

holds for all $a < \frac{1}{2}(n-1)$. Now, using Theorem 3.1 and the fact that a and $b = n - a$ are positive integers, we have the following observation.

Corollary 3.1. *Among all bipartite graphs of order $n \geq 3$, the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ has the minimum distance Laplacian energy.*

The next observation follows from Lemma 2.1 and Theorem 3.1.

Corollary 3.2. *Let G be a connected bipartite graph with partite sets of cardinality a and b , such that $a \leq b$ and $a + b = n$. Let $\varrho_1^L(G) \geq \varrho_2^L(G) \dots \geq \varrho_{n-1}^L(G) > \varrho_n^L(G) = 0$ be the distance Laplacian eigenvalues of G . Then $\varrho_i^L(G) \geq 2n - a$ for all $1 \leq i \leq b - 1$, $\varrho_i^L(G) \geq 2n - b$ for all $b \leq i \leq a + b - 2$, $\varrho_{n-1}^L(G) \geq n$. Equality occurs in each of these inequalities if and only if $G \cong K_{a,b}$.*

A complete split graph, denoted by $CS_{t,n-t}$, is the graph consisting of a clique on t vertices and an independent set (a subset of vertices of a graph is said to be an independent set if the subgraph induced by them is an empty graph) on the remaining $n - t$ vertices, such that each vertex of the clique is adjacent to every vertex of the independent set.

The following theorem shows that among all connected graphs with given independence number $n - t$, $1 \leq t \leq n - 1$, the complete split graph $CS_{t,n-t}$ has the minimum distance Laplacian energy.

Theorem 3.2. *Let G be a connected graph of order $n \geq 3$ having independence number $n - t$, $1 \leq t \leq n - 1$. Then*

$$\text{DLE}(G) \geq \begin{cases} 2(n-t-1)\left(2n-t-\frac{2W(G)}{n}\right) & \text{if } t < n - \frac{1}{2} - \sqrt{n + \frac{1}{4}}, \\ 4n^2 - 2nt - 4n + 2t(t+1) - \frac{4(n-1)W(G)}{n} & \text{if } t \geq n - \frac{1}{2} - \sqrt{n + \frac{1}{4}}, \end{cases}$$

equality occurs in each case if and only if $G \cong CS_{t,n-t}$, $1 \leq t \leq n - 1$.

Proof. Let G be a connected graph of order $n \geq 3$ having the independence number $n - t$. Let $CS_{t,n-t}$ be the complete split graph having the independence number $n - t$. It is clear that G is a spanning subgraph of $CS_{t,n-t}$. Therefore, by Lemma 2.1, we have $\varrho_i^L(G) \geq \varrho_i^L(CS_{t,n-t})$. Let σ be the largest positive integer such that $\varrho_\sigma^L(G) \geq 2W(G)/n$. With this information, it follows that

$$\begin{aligned} (3.3) \quad \text{DLE}(G) &= 2\left(\sum_{i=1}^{\sigma} \varrho_i^L(G) - \frac{2\sigma W(G)}{n}\right) = 2 \max_{1 \leq j \leq n} \left(\sum_{i=1}^j \varrho_i^L(G) - \frac{2jW(G)}{n}\right) \\ &\geq 2 \max_{1 \leq j \leq n} \left(\sum_{i=1}^j \varrho_i^L(CS_{t,n-t}) - \frac{2jW(G)}{n}\right). \end{aligned}$$

The distance Laplacian spectrum (see [3]) of the complete split graph $CS_{t,n-t}$ is $\{(2n-t)^{[n-t-1]}, n^{[t]}, 0\}$ with $2W(CS_{t,n-t})/n = (2n(n-t-1) + t(t+1))/n$. Since $n-t \geq 1$, it follows that $2n-t$ is the distance Laplacian spectral radius of $CS_{t,n-t}$.

For the eigenvalue n , we have $n < 2W(CS_{t,n-t})/n = (2n(n-t-1) + t(t+1))/n$, which after simplification gives

$$(3.4) \quad t^2 - (2n-1)t + n^2 - 2n > 0.$$

Consider the polynomial $f(t) = t^2 - (2n-1)t + n^2 - 2n$ for $1 \leq t \leq n-1$. The roots of this polynomial are $x_1 = n - \frac{1}{2} - \sqrt{n + \frac{1}{4}}$ and $x_2 = n - \frac{1}{2} + \sqrt{n + \frac{1}{4}}$. This implies that $f(t) > 0$ for all $t < x_1$ and $f(t) > 0$ for all $t > x_2$. Since $x_2 > n$ and $t \leq n-1$, it follows that inequality (3.4) holds for all $t < n - \frac{1}{2} - \sqrt{n + \frac{1}{4}}$. From this, it follows that for $t < n - \frac{1}{2} - \sqrt{n + \frac{1}{4}}$ we have $\sigma = n-t-1$ and for $t \geq n - \frac{1}{2} - \sqrt{n + \frac{1}{4}}$ we have $\sigma = n-1$. The rest of the proof is omitted and follows on similar lines as done in Theorem 3.1. \square

We characterized the extremal graphs which attain the minimum value for the distance Laplacian energy among all connected bipartite graphs and among all connected graphs with a given independence number. The following problems will be of interest for the future research.

Problem 3.1. Characterize the extremal graphs which attain the maximum value for the distance Laplacian energy among all connected bipartite graphs of order n .

Problem 3.2. Characterize the extremal graphs which attain the maximum value for the distance Laplacian energy among all connected graphs of order n with independence number α .

4. DISTANCE LAPLACIAN ENERGY OF GRAPHS WITH GIVEN CONNECTIVITY

In this section, we obtain the distance Laplacian spectrum of the join of a connected graph G_0 with the union of two connected graphs G_1 and G_2 . We show the existence of some new families of graphs having all the distance Laplacian eigenvalues as integers. We also determine the graph with the minimum distance Laplacian energy among all the connected graphs with given vertex connectivity.

The vertex connectivity of a graph G , denoted by $\kappa(G)$, is the minimum number of vertices of G whose deletion disconnects G . Let \mathcal{F}_n be the family of simple connected graphs on n vertices and let $\mathcal{V}_n^k = \{G \in \mathcal{F}_n : \kappa(G) \leq k\}$.

Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs on disjoint vertex sets V_1 and V_2 of order n_1 and n_2 , respectively. The *union* of graphs G_1 and G_2 , is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The *join* of graphs G_1 and G_2 , denoted by $G_1 \nabla G_2$, is the graph consisting of $G_1 \cup G_2$ and all edges joining the vertices in V_1 and the vertices in V_2 . In other words, the join of two graphs G_1 and G_2 , denoted by $G_1 \nabla G_2$, is the graph obtained from G_1 and G_2 by joining each vertex of G_1 to every vertex of G_2 .

In the following theorem, we find the distance Laplacian spectrum of the join of a connected graph G_0 with the union of two connected graphs G_1 and G_2 in terms of the distance Laplacian spectrum of the graphs G_0, G_1 and G_2 .

Theorem 4.1. *Let G_0, G_1 and G_2 be the connected graphs of order n_0, n_1 and n_2 , respectively. The distance Laplacian spectrum of $G_0 \nabla (G_1 \cup G_2)$ consists of eigenvalues $(\lambda + n_1 + n_2), (\mu + n_0 + 2n_2), (\zeta + n_0 + 2n_1), 2n - n_0, n, 0$, where $n = n_0 + n_1 + n_2$ and λ, μ, ζ varies over the nonzero distance Laplacian eigenvalues of G_0, G_1, G_2 , respectively.*

Proof. Let $G_i, i = 0, 1, 2$, be the connected graphs of order n_i . Let $G = G_0 \nabla (G_1 \cup G_2)$ be the join of graphs G_0 and $G_1 \cup G_2$. Clearly, G is a graph of diameter 2. Let $D^L(G_0), D^L(G_1)$ and $D^L(G_2)$ be respectively the distance Laplacian matrices of the graphs G_0, G_1 and G_2 . By suitably labelling the vertices of G , it can be seen that the distance Laplacian matrix of G is

$$D^L(G) = \begin{pmatrix} b_0 & -J_{n_0 \times n_1} & -J_{n_0 \times n_2} \\ -J_{n_1 \times n_0} & b_1 & -2J_{n_1 \times n_2} \\ -J_{n_2 \times n_0} & -2J_{n_2 \times n_1} & b_2 \end{pmatrix},$$

where $b_0 = (2n - n_1 - n_2)I_{n_0} - 2J_{n_0} - D^L(G_0)$, $b_i = (2n - n_0)I_{n_i} - 2J_{n_i} - D^L(G_i)$, for $i = 1, 2$, $J_{n_i \times n_j}$ is the all one matrix of order $n_i \times n_j$, I_{n_k} is the identity matrix of order n_k . Let $e_{n_i} = (1, 1, \dots, 1)$ be the all 1-vector of order n_i , $i = 0, 1, 2$. It is well known that e_{n_i} is an eigenvector of G_i for the distance Laplacian eigenvalue 0 and any other eigenvector x of G_i is orthogonal to e_{n_i} . Let x be any eigenvector of G_0 for the nonzero distance Laplacian eigenvalue λ . Then $x \perp e_{n_0}$. Consider the column vector $X = (x^\top \ 0_{n_1 \times 1}^\top \ 0_{n_2 \times 1}^\top)^\top$. We have

$$D^L(G)X = \begin{pmatrix} (2n - n_1 - n_2)I_{n_0} - 2J_{n_0} - D^L(G_0)x \\ 0 \\ 0 \end{pmatrix} = (2n - n_1 - n_2 - \lambda)X,$$

which implies that $2n - n_1 - n_2 - \lambda$ is an eigenvalue of $D^L(G)$ for each nonzero distance Laplacian eigenvalue λ of G_0 . In this way, we get $n_0 - 1$ distance Laplacian eigenvalues of G . Similarly, if $0 \neq \mu$ is a distance Laplacian eigenvalue of G_1 with eigenvector y , $y \perp e_{n_1}$, then it can be seen that the column vector $Y = (0_{n_0 \times 1}^\top \ y^\top \ 0_{n_2 \times 1}^\top)^\top$ is an eigenvector of $D^L(G)$ for the eigenvalue $2n - n_0 - \mu$. This implies that $2n - n_0 - \mu$ is an eigenvalue of $D^L(G)$ for each nonzero distance Laplacian eigenvalue μ of G_1 . In this way, we get other $n_1 - 1$ distance Laplacian eigenvalues of G . Lastly, if $0 \neq \zeta$ is a distance Laplacian eigenvalue of G_2 with eigenvector z , $z \perp e_{n_2}$, then it can be seen that the column vector $Z = (0_{n_0 \times 1}^\top \ 0_{n_1 \times 1}^\top \ z^\top)^\top$ is an eigenvector of $D^L(G)$ for

the eigenvalue $2n - n_0 - \zeta$. This implies that $2n - n_0 - \zeta$ is an eigenvalue of $D^L(G)$ for each nonzero distance Laplacian eigenvalue ζ of G_2 . In this way, we get another $n_2 - 1$ distance Laplacian eigenvalues of G . Thus, we get a total of $n_0 - 1 + n_1 - 1 + n_2 - 1 = n - 3$ distance Laplacian eigenvalues of G . The remaining three distance Laplacian eigenvalues of G are given by the quotient matrix of $D^L(G)$, which is

$$M = \begin{pmatrix} n_1 + n_2 & -n_1 & -n_2 \\ -n_0 & n_0 + 2n_2 & -2n_2 \\ -n_0 & -2n_1 & n_0 + 2n_1 \end{pmatrix}.$$

Since each row sum of matrix M is zero, it follows that one of the eigenvalues of this matrix is 0. The other two eigenvalues of M are given by the roots of $x^2 - (3n - n_0)x + n(2n - n_0) = 0$. This completes the proof. \square

A graph G is said to be an adjacency integral (Laplacian integral, distance integral) graph if all its adjacency eigenvalues (Laplacian eigenvalues, distance eigenvalues) are integers. Likewise, a graph G is called a *distance Laplacian integral graph* if all its distance Laplacian eigenvalues are integers. From Theorem 4.1 it is clear that if all the graphs G_0 , G_1 and G_2 are distance Laplacian integrals, then the graph $G = G_0 \nabla (G_1 \cup G_2)$ is also a distance Laplacian integral graph. It is well known that the complete graph K_n , the complete bipartite graph $K_{a,n-a}$, the complete split graph $CS_{a,n-a}$, the pineapple graph $PA_{n,p}$ (the graph obtained from a clique K_{n-p} by adding p pendent edges to a vertex of K_{n-p}) and the graph S^+ (the graph obtained from the star graph $K_{1,n-1}$ by adding an edge between two pendent vertices) are all Laplacian integral graphs. Therefore, using Theorem 4.1, we have the following observation.

Theorem 4.2. *Let $n = n_0 + n_1 + n_2$. Then each of the graphs $K_{n_0} \nabla (K_{n_1} \cup K_{n_2})$, $K_{n_0} \nabla (K_{n_1-a,a} \cup K_{n_2})$, $K_{n_0-a,a} \nabla (K_{n_1} \cup K_{n_2})$, $K_{n_0-a,a} \nabla (K_{n_1-b,b} \cup K_{n_2})$, $K_{n_0-a,a} \nabla (K_{n_1-b,b} \cup K_{n_2-c,c})$, $CS_{a,n_0-a} \nabla (K_{n_1} \cup K_{n_2})$, $CS_{a,n_0-a} \nabla (CS_{b,n_1-b} \cup K_{n_2})$, $CS_{a,n_0-a} \nabla (CS_{b,n_1-b} \cup CS_{c,n_2-c})$, $K_{n_0-a,a} \nabla (CS_{b,n_1-b} \cup K_{n_2})$, $PA_{n_0,p} \nabla (K_{n_1} \cup K_{n_2})$, $CS_{a,n_0-a} \nabla (PA_{n_1,p} \cup K_{n_2})$, $CS_{a,n_0-a} \nabla (PA_{n_1,p} \cup K_{n_2-c,c})$, etc. are distance Laplacian integral graphs.*

The following theorem shows that among all connected graphs with a given vertex connectivity k the graph $K_k \nabla (K_t \cup K_{n-k-t})$ has the minimum distance Laplacian energy.

Theorem 4.3. *Let $G \in \mathcal{V}_n^k$ be a connected graph of order $n \geq 4$ having vertex connectivity number k . Then $\text{DLE}(G) \geq t(2n - k - t + 1) - 2tW(G)/n$ for $1 \leq k < \frac{1}{2}(n - 2t) - n/(2t)$; $\text{DLE}(G) \geq t(3n - 2k - 2t) + n(n - t - k - 1) - 2(n - k - 1)W(G)/n$*

for $\frac{1}{2}(n-2t) - n/(2t) \leq k < n-t - n/(2t)$ and $\text{DLE}(G) \geq n(n-1) + 2t(n-k-t) - 2(n-1)W(G)/n$ for $n-t - n/(2t) \leq k \leq n-2$; if $k = n-1$, then $G \cong K_n$ and so $\text{DLE}(G) = 2n-2$. Equality occurs in each of these inequalities if and only if $G \cong K_k \nabla (K_t \cup K_{n-k-t})$, $1 \leq t \leq \lfloor \frac{n-k}{2} \rfloor$.

Proof. Let G be a connected graph of order n with vertex connectivity k , $1 \leq k \leq n-1$. We first show that $U_i^L(G) \geq U_i^L(K_k \nabla (K_t \cup K_{n-k-t}))$ for all $i = 1, 2, \dots, n$. If $k = n-1$, then $G \cong K_n$ and $K_k \nabla (K_t \cup K_{n-k-t}) = K_n$ and so equality holds in this case. Assume that $1 \leq k \leq n-2$, that is, G is not a complete graph. Suppose that G is the connected graph of order n with vertex connectivity k for which the spectral parameter $U_i^L(G)$ has the minimum possible value. Then it is clear that $G \in \mathcal{V}_n^k$ and $U_i^L(G)$ attains the minimum value for G . Let S be a vertex cut set of G with $|S| = k$. Let G_1, G_2, \dots, G_r be the connected components of the graph $G - S$. We will show the number of components of graph $G - S$ is two, that is, $r = 2$. For if $r > 2$, then adding an edge between any two components, say G_1 and G_2 of $G - S$, gives the graph $G' = G + e$, which is such that the vertex connectivity of G' is k . Clearly, $G' \in \mathcal{V}_n^k$, also by Lemma 2.1, we have $U_i^L(G) > U_i^L(G')$. This is a contradiction to the fact $U_i^L(G)$ attains the minimum possible value for G . Therefore, we must have $r = 2$. Further, we claim that each of the components G_1, G_2 and the vertex induced subgraph $\langle S \rangle$ are cliques. For if one among them, say G_1 , is not a clique, then adding an edge between the two non adjacent vertices of G_1 gives a graph H having vertex connectivity the same as the vertex connectivity of G . Clearly, $H \in \mathcal{V}_n^k$ and by Lemma 2.1, we have $U_i^L(G) > U_i^L(H)$, which is a contradiction as $U_i^L(G)$ attains the minimum possible value for G . Thus, G must be of the form $G = K_k \nabla (K_t \cup K_{n-k-t})$, $1 \leq t \leq \lfloor \frac{n-k}{2} \rfloor$. This shows that for all $G \in \mathcal{V}_n^k$, the spectral parameter $U_i^L(G)$ has the minimum possible value for the graph $K_k \nabla (K_t \cup K_{n-k-t})$. That is, for all $G \in \mathcal{V}_n^k$ we have $U_i^L(G) \geq U_i^L(K_k \nabla (K_t \cup K_{n-k-t}))$. With this, from the definition of distance Laplacian energy, it follows that

$$(4.1) \quad \begin{aligned} \text{DLE}(G) &= 2 \left(U_\sigma^L(G) - \frac{2\sigma W(G)}{n} \right) = 2 \max_{1 \leq j \leq n} \left(\sum_{i=1}^j \varrho_i^L(G) - \frac{2jW(G)}{n} \right) \\ &\geq 2 \max_{1 \leq j \leq n} \left(\sum_{i=1}^j \varrho_i^L(K_k \nabla (K_t \cup K_{n-k-t})) - \frac{2jW(G)}{n} \right). \end{aligned}$$

Taking $G_{n_0} = K_k$, $G_{n_1} = K_t$, $G_{n_2} = K_{n-k-t}$, $n_0 = k$, $n_1 = t$ and $n_2 = n-t-k$ in Theorem 4.1, we find that the distance Laplacian spectrum of the graph $K_k \nabla (K_t \cup K_{n-k-t})$ is $\{2n-k, (2n-t-k)^{[t-1]}, (n+t)^{[n-t-k-1]}, n^{[k]}, 0\}$. Let σ be the number of distance Laplacian eigenvalues of $K_k \nabla (K_t \cup K_{n-k-t})$ which are greater than or equal to that $2W(K_k \nabla (K_t \cup K_{n-k-t}))/n = (n^2 - n + 2nt - 2t^2 - 2kt)/n$.

It is easy to see that $2n - k$ is the distance Laplacian spectral radius of the graph $K_k \nabla (K_t \cup K_{n-t-k})$ and for this eigenvalue we always have $2n - k \geq 2W(K_k \nabla (K_t \cup K_{n-t-k}))/n$. For the eigenvalue $2n - k - t$ we have $2n - k - t \geq 2W(K_k \nabla (K_t \cup K_{n-t-k}))/n = (n^2 - n + 2nt - 2t^2 - 2kt)/n$, which gives

$$(4.2) \quad 2t^2 - (3n - 2k)t + (n^2 + n - kn) \geq 0.$$

The zeros of the polynomial $g_1(t) = 2t^2 - (3n - 2k)t + (n^2 + n - kn)$ are $y_1 = \frac{1}{2}(3n - 2k + \sqrt{(n - 2k)^2 - 8n})$ and $y_2 = \frac{1}{2}(3n - 2k - \sqrt{(n - 2k)^2 - 8n})$. This shows that $g_1(t) \geq 0$ for all $t \leq y_2$ and $t \geq y_1$. Since

$$\frac{n - k}{2} < \frac{3n - 2k - \sqrt{(n - 2k)^2 - 8n}}{2} = y_2$$

always holds, it follows that $g_1(t) \geq 0$ for all $t \leq \frac{1}{2}(n - k)$. For the eigenvalue $n + t$ we have $n + t \geq 2W(K_k \nabla (K_t \cup K_{n-t-k}))/n = (n^2 - n + 2nt - 2t^2 - 2kt)/n$ giving that $2t^2 - (n - 2k)t + n \geq 0$, which in turn gives $k \geq \frac{1}{2}(n - 2t) - n/(2t)$. This shows that $n + t \geq 2W(K_k \nabla (K_t \cup K_{n-t-k}))/n$ for all $k \geq \frac{1}{2}(n - 2t) - n/(2t)$ and $n + t < 2W(K_k \nabla (K_t \cup K_{n-t-k}))/n$ for all $k < \frac{1}{2}(n - 2t) - n/(2t)$. Lastly, for the eigenvalue n we have $n \geq 2W(K_k \nabla (K_t \cup K_{n-t-k}))/n = (n^2 - n + 2nt - 2t^2 - 2kt)/n$ giving that $2t^2 - (2n - 2k)t + n \geq 0$, which in turn gives that $k \geq n - t - n/(2t)$. This shows that $n \geq 2W(K_k \nabla (K_t \cup K_{n-t-k}))/n$ for all $k \geq n - t - n/(2t)$ and $n < 2W(K_k \nabla (K_t \cup K_{n-t-k}))/n$ for all $k < n - t - n/(2t)$. From this discussion it follows that if $1 \leq k < \frac{1}{2}(n - 2t) - n/(2t)$, then $\sigma = t$, if $\frac{1}{2}(n - 2t) - n/(2t) \leq k < n - t - n/(2t)$, then $\sigma = n - k - 1$ and if $k \geq n - t - n/(2t)$, then $\sigma = n - 1$. If $1 \leq k < \frac{1}{2}(n - 2t) - n/(2t)$, then from (4.1) it follows that

$$\begin{aligned} \text{DLE}(G) &\geq 2 \max_{1 \leq j \leq n} \left(\sum_{i=1}^j \varrho_i^L(K_k \nabla (K_t \cup K_{n-t-k})) - \frac{2jW(G)}{n} \right) \\ &\geq 2 \left(\sum_{i=1}^t \varrho_i^L(K_k \nabla (K_t \cup K_{n-t-k})) - \frac{2tW(G)}{n} \right) \\ &= t(2n - k - t + 1) - \frac{2tW(G)}{n}. \end{aligned}$$

If $\frac{1}{2}(n - 2t) - n/(2t) \leq k < n - t - n/(2t)$, then from (4.1) it follows that

$$\begin{aligned} \text{DLE}(G) &\geq 2 \max_{1 \leq j \leq n} \left(\sum_{i=1}^j \varrho_i^L(K_k \nabla (K_t \cup K_{n-t-k})) - \frac{2jW(G)}{n} \right) \\ &\geq 2 \left(\sum_{i=1}^{n-k-1} \varrho_i^L(K_k \nabla (K_t \cup K_{n-t-k})) - \frac{2(n - k - 1)W(G)}{n} \right) \\ &= t(3n - 2k - 2t) + n(n - t - k - 1) - \frac{2(n - k - 1)W(G)}{n}. \end{aligned}$$

If $n - t - n/(2t) \leq k \leq n - 1$, then from (4.1) it follows that

$$\begin{aligned} \text{DLE}(G) &\geq 2 \max_{1 \leq j \leq n} \left(\sum_{i=1}^j \varrho_i^L(K_k \nabla (K_t \cup K_{n-t-k})) - \frac{2jW(G)}{n} \right) \\ &\geq 2 \left(\sum_{i=1}^{n-1} \varrho_i^L(K_k \nabla (K_t \cup K_{n-t-k})) - \frac{2(n-1)W(G)}{n} \right) \\ &= n(n-1) + 2t(n-k-t) - \frac{2(n-1)W(G)}{n}. \end{aligned}$$

This completes the proof. \square

The next observation follows from Lemma 2.1 and the proof of Theorem 4.3.

Corollary 4.1. *Let G be a connected graph of order $n \geq 4$ having vertex connectivity $\kappa \leq k$. Let $\varrho_1^L(G) \geq \varrho_2^L(G) \geq \dots \geq \varrho_{n-1}^L(G) > \varrho_n^L(G) = 0$ be the distance Laplacian eigenvalues of G . Then $\varrho_1^L(G) \geq 2n - k$, $\varrho_i^L(G) \geq 2n - t - k$, for $2 \leq i \leq t$, $\varrho_i^L(G) \geq n + t$, for $t + 1 \leq i \leq n - k - 1$ and $\varrho_i^L(G) \geq n$ for $n - k \leq i \leq n - 1$. Equality occurs in each of these inequalities if and only if $G \cong K_k \nabla (K_t \cup K_{n-t-k})$.*

In the latter part of this section, we characterized the extremal graphs which attain the minimum value for the distance Laplacian energy among all connected graphs with given vertex connectivity. The following problem will be of interest for the future research.

Problem 4.1. Characterize the extremal graphs which attain the maximum value for the distance Laplacian energy among all connected graphs of order n with given vertex connectivity.

5. CONCLUSION

Let $\mathbb{M}_n(\mathbb{C})$ be the set of all square matrices of order n with complex entries. The trace norm of a matrix $M \in \mathbb{M}_n(\mathbb{C})$ is defined as $\|M\|_* = \sum_{i=1}^n \sigma_i(M)$, where $\sigma_1(M) \geq \sigma_2(M) \geq \dots \geq \sigma_n(M)$ are the singular values of M . It is well known that for a symmetric matrix M , if $\sigma_i(M)$ is the i th singular value and $\lambda_i(M)$ is the i th eigenvalue, then $\sigma_i(M) = |\lambda_i(M)|$. In the light of this definition, it follows that the distance Laplacian energy $\text{DLE}(G)$ of a connected graph G is the trace norm of the matrix $D^L(G) - (2W(G)/n)I_n$, where I_n is the identity matrix of order n . It is an interesting problem in matrix theory to determine among a given class of matrices the matrix (or the matrices) which attains the maximum value and the minimum

value for the trace norm. The trace norm of matrices associated with the graphs and digraphs are extensively studied. For some recent papers in this direction, see [15] and the references therein.

Therefore, in this language, Theorem 2.2 gives a relation between the trace norm of the matrices $D^L(G) - (2W(G)/n)I_n$ and $L(G) - (2m/n)I_n$ when G is a connected graph of diameter two; Theorem 2.3 gives a lower bound for the trace norm of $D^L(G) - (2W(G)/n)I_n$ in terms of the order and the trace of the matrix $D^L(G)$; Theorem 3.1 gives that among all connected bipartite graphs G , the complete bipartite $K_{a,b}$ attains the minimum trace norm for the matrix $D^L(G) - (2W(G)/n)I_n$; Theorem 3.2 gives that among all connected graphs G with given independence number $n - t$, $1 \leq t \leq n - 1$, the complete split graph $CS_{t,n-t}$ attains the minimum trace norm for the matrix $D^L(G) - (2W(G)/n)I_n$ and Theorem 4.3 gives that among all connected graphs G with given vertex connectivity k , the graph $K_k \nabla (K_t \cup K_{n-k-t})$ attains the minimum trace norm for the matrix $D^L(G) - (2W(G)/n)I_n$.

Acknowledgements. We are highly thankful to the anonymous referee for his comments and suggestions to improve the presentation of the paper.

References

- [1] *M. Aouchiche, P. Hansen*: Two Laplacians for the distance matrix of a graph. *Linear Algebra Appl.* **439** (2013), 21–33. [zbl](#) [MR](#) [doi](#)
- [2] *M. Aouchiche, P. Hansen*: Distance spectra of graphs: A survey. *Linear Algebra Appl.* **458** (2014), 301–386. [zbl](#) [MR](#) [doi](#)
- [3] *M. Aouchiche, P. Hansen*: Some properties of the distance Laplacian eigenvalues of a graph. *Czech. Math. J.* **64** (2014), 751–761. [zbl](#) [MR](#) [doi](#)
- [4] *A. E. Brouwer, W. H. Haemers*: Spectra of graphs. Universitext. Berlin: Springer, 2012. [zbl](#) [MR](#) [doi](#)
- [5] *D. M. Cvetković, M. Doob, H. Sachs*: Spectra of Graphs: Theory and Application. Pure and Applied Mathematics 87. Academic Press, New York, 1980. [zbl](#) [MR](#)
- [6] *K. C. Das, M. Aouchiche, P. Hansen*: On (distance) Laplacian energy and (distance) signless Laplacian energy of graphs. *Discrete Appl. Math.* **243** (2018), 172–185. [zbl](#) [MR](#) [doi](#)
- [7] *R. C. Díaz, O. Rojo*: Sharp upper bounds on the distance energies of a graph. *Linear Algebra Appl.* **545** (2018), 55–75. [zbl](#) [MR](#) [doi](#)
- [8] *H. A. Ganie*: On distance Laplacian spectrum (energy) of graphs. *Discrete Math. Algorithms Appl.* **12** (2020), Article ID 2050061, 16 pages. [zbl](#) [MR](#) [doi](#)
- [9] *H. A. Ganie*: On the distance Laplacian energy ordering of tree. *Appl. Math. Comput.* **394** (2021), Article ID 125762, 10 pages. [zbl](#) [MR](#) [doi](#)
- [10] *H. A. Ganie, B. A. Chat, S. Pirzada*: Signless Laplacian energy of a graph and energy of line graph. *Linear Algebra Appl.* **544** (2018), 306–324. [zbl](#) [MR](#) [doi](#)
- [11] *H. A. Ganie, S. Pirzada, B. A. Rather, V. Trevisan*: Further developments on Brouwer’s conjecture for the sum of Laplacian eigenvalues of graphs. *Linear Algebra Appl.* **588** (2020), 1–18. [zbl](#) [MR](#) [doi](#)
- [12] *I. Gutman, B. Zhou*: Laplacian energy of a graph. *Linear Algebra Appl.* **414** (2006), 29–37. [zbl](#) [MR](#) [doi](#)
- [13] *G. Indulal, I. Gutman, A. Vijayakumar*: On distance energy of graphs. *MATCH Commun. Math. Comput. Chem.* **60** (2008), 461–472. [zbl](#) [MR](#)

- [14] *X. Li, Y. Shi, I. Gutman*: Graph Energy. Springer, New York, 2012. [zbl](#) [MR](#) [doi](#)
- [15] *J. Monsalve, J. Rada*: Oriented bipartite graphs with minimal trace norm. *Linear Multilinear Algebra* *67* (2019), 1121–1131. [zbl](#) [MR](#) [doi](#)
- [16] *S. Pirzada*: An Introduction to Graph Theory. Orient Blackswan, Hyderabad, 2012.
- [17] *S. Pirzada, H. A. Ganie*: On the Laplacian eigenvalues of a graph and Laplacian energy. *Linear Algebra Appl.* *486* (2015), 454–468. [zbl](#) [MR](#) [doi](#)
- [18] *J. Yang, L. You, I. Gutman*: Bounds on the distance Laplacian energy of graphs. *Kragujevac J. Math.* *37* (2013), 245–255. [zbl](#) [MR](#)

Authors' addresses: Hilal A. Ganie, Department of School Education, MA Road, Kothi Bagh, Srinagar, Jammu and Kashmir 190001, India, e-mail: hilahmad1119kt@gmail.com; Rezwan Ul Shaban, Bilal A. Rather, Shariefuddin Pirzada (corresponding author), Department of Mathematics, School of Physical and Mathematical Sciences, University of Kashmir, Hazaratbal, Srinagar, Jammu and Kashmir 190006, India, e-mail: rezwanbhat21@gmail.com, bilalahmadrr@gmail.com, pirzadasd@kashmiruniversity.ac.in.