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EXPONENTIAL STABILITY CONDITIONS FOR NON-AUTONOMOUS DIFFERENTIAL EQUATIONS WITH UNBOUNDED COMMUTATORS IN A BANACH SPACE

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Abstract. We consider the equation $\mathrm{d}y(t)/\mathrm{d}t = (A+B(t))y(t)$ $(t \geqslant 0)$, where A is the generator of an analytic semigroup $(\mathrm{e}^{At})_{t\geqslant 0}$ on a Banach space $\mathcal{X}$, $B(t)$ is a variable bounded operator in $\mathcal{X}$. It is assumed that the commutator $K(t) = AB(t) - B(t)A$ has the following property: there is a linear operator S having a bounded left-inverse operator S_l^{-1} such that $\|S\mathrm{e}^{At}\|$ is integrable and the operator $K(t)S_l^{-1}$ is bounded. Under these conditions an exponential stability test is derived. As an example we consider a coupled system of parabolic equations.

Keywords: Banach space; differential equation; linear nonautonomous equation; exponential stability; commutator; parabolic equation

MSC 2020: 47D06, 35K51, 35B35, 34G10

1. Introduction and statement of the main result

Throughout this paper, $\mathcal{X}$ is a Banach space with a norm $\|\cdot\|$ and the identity operator I. By $\mathcal{B}(\mathcal{X})$ we denote the set of bounded linear operators in $\mathcal{X}$. For a linear operator C, $\mathrm{Dom}(C)$ is its domain, $\sigma(C)$ is its spectrum, and $\alpha(C) = \sup \mathrm{Re}\, \sigma(C)$. If $C \in \mathcal{B}(\mathcal{X})$, then $\|C\|$ is its operator norm.

Further, A denotes a generator of an analytic semigroup e^{At} on $\mathcal{X}$, and $B(t)$ $(t \geqslant 0)$ is a variable bounded piece-wise strongly continuous operator mapping $\mathrm{Dom}(A)$ into itself for each $t \geqslant 0$.

The paper deals with the exponential stability conditions for the equation

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = (A + B(t))y(t) \quad (t \geqslant 0). \tag{1.1}$$

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A solution to (1.1) for given $y_0 \in \operatorname{Dom}(A)$ is a function $y\colon [0,\infty) \to \operatorname{Dom}(A)$ having at each point $t > 0$ a strong derivative, at zero the right strong derivative, and satisfying (1.1) for all $t > 0$ and $y(0) = y_0$.

The existence, uniqueness and continuous dependence on initial vectors of solutions are due to Theorem II.3.4 from [14], since the operator $B(t)$ is bounded, and maps $\operatorname{Dom}(A)$ into itself, and the operator A generates an analytic semigroup.

We will say that (1.1) is an exponentially stable equation if there are positive constants m_1 and δ_1 such that $\|y(t)\| \leqslant m_1 \mathrm{e}^{-\delta_1 t}\|y(0)\|$ $(t \geqslant 0)$ for any solution $y(t)$ of (1.1).

Certainly, (1.1) can be rewritten as equation

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = C(t)y(t) \tag{1.2}$$

with the corresponding operator $C(t)$, but $C(t)$ in the present paper has a special form: it is the sum of A and $B(t)$. This allows us to use the information about A and $B(t)$ more completely than the theory of general equations (1.2).

The stability theory of abstract differential equations is well developed, cf. [1]–[9], [12], [15]–[18], etc. Mainly, equation (1.1) is considered as a perturbation of a stable semigroup generated by A. In paper [11], stability conditions for equation (1.1) have been established in terms of the commutator $K(t) = AB(t) - B(t)A$ $(t \geqslant 0)$. Besides, it was shown that stability conditions in terms of the commutator enable us to investigate equations with an unstable semigroup e^{At}. This fact gives us the conditions for the stabilization of systems with distributed parameters. Paper [11] deals with bounded commutators. *The aim of this paper is to generalize the main result from* [11] *to the case when $K(t)$ is unbounded.*

Denote by $U_B(t,s)$ $(t \geqslant s \geqslant 0)$ the evolution operator of the equation

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = B(t)u(t) \quad (t \geqslant 0) \tag{1.3}$$

and assume that there are real numbers b_0 and $c_0 = \text{const.} \geqslant 1$ such that

$$\|U_B(t,s)\| \leqslant c_0 \exp[b_0(t-s)] \quad (t \geqslant s \geqslant 0). \tag{1.4}$$

For the recent solution bounds for the differential equations with bounded operators, see for instance [2]. It is also assumed that there is a linear operator S with $\operatorname{Dom}(S) \supseteq \operatorname{Dom}(A)$ having a bounded left-inverse one S_l^{-1} such that

$$J(S) := \int_0^\infty \|S\mathrm{e}^{(A+b_0 I)t}\|\,\mathrm{d}t < \infty \tag{1.5}$$

and

$$m(K(\cdot),S) := \sup_{t\geqslant 0} \|K(t)S_l^{-1}\| < \infty. \tag{1.6}$$

In addition, denote

$$J_0 := \int_0^\infty \|\mathrm{e}^{(A+b_0 I)t}\| \,\mathrm{d}t.$$

Due to (1.5) we have

$$J_0 = \int_0^\infty \|S_l^{-1} S \mathrm{e}^{(A+b_0 I)t}\| \,\mathrm{d}t \leqslant \|S_l^{-1}\| J(S) < \infty.$$

Now we are in a position to formulate the main result of the paper.

Theorem 1.1. *Let conditions* (1.4)–(1.6) *and*

$$c_0 m(K(\cdot), S) J(S) J_0 < 1 \tag{1.7}$$

hold. Then equation (1.1) *is exponentially stable.*

This theorem is proved in the next section.

For example, if $-A$ is sectorial and $\alpha(A) < \alpha_A < 0$, then as is well-known (see [13], Theorem 1.4.3, page 26) $\|\mathrm{e}^{At}\| \leqslant m_A \mathrm{e}^{\alpha_A t}$ ($m_A = \text{const.} \geqslant 1$; $t \geqslant 0$) and for any $\nu \in (0,1)$,

$$\|(-A)^\nu \mathrm{e}^{At}\| \leqslant m_\nu t^{-\nu} \exp(-\delta_\nu t) \quad (0 < \delta_\nu \leqslant |\alpha_A|;\ m_\nu = \text{const.} \geqslant 1;\ t \geqslant 0).$$

So if $\alpha_A + b_0 < 0$ and $-\delta_\nu + b_0 < 0$, in the considered case

$$J_0 \leqslant m_A \int_0^\infty \mathrm{e}^{(\alpha_A + b_0)t} \,\mathrm{d}t = \frac{m_A}{|\alpha_A + b_0|}$$

and

$$J(S) = J_\nu := \int_0^\infty \|(-A)^\nu \mathrm{e}^{(A+b_0)t}\| \,\mathrm{d}t \leqslant m_\nu \int_0^\infty t^{-\nu} \mathrm{e}^{(b_0 - \delta_\nu)t} \,\mathrm{d}t < \infty.$$

Note that

$$\int_0^\infty t^{-\nu} \mathrm{e}^{(b_0 - \delta_\nu)t} \,\mathrm{d}t = \frac{1}{(\delta_\nu - b_0)^{1-\nu}} \Gamma(1-\nu),$$

where

$$\Gamma(1+x) = \int_0^\infty s^x \mathrm{e}^{-s} \,\mathrm{d}s \quad (x \in (-1, \infty))$$

is the Euler Gamma function. Thus,

$$J_\nu \leqslant \frac{m_A \Gamma(1-\nu)}{|\delta_\nu - b_0|^{1-\nu}}.$$

Let us present an example of A satisfying (1.6). To this end recall that if A is a selfadjoint negative definite operator in a Hilbert space, then $\|f(A)\| = \sup_{s\leqslant\alpha(A)} |f(s)|$ for a function f bounded on $\sigma(A)$, cf. [14], and therefore, $\|\mathrm{e}^{At}\| = \mathrm{e}^{\alpha(A)t}$ $(t \geqslant 0)$, and

$$\|(-A)^{\nu}\mathrm{e}^{At}\| = \sup_{s\leqslant\alpha(A)} (-s)^{\nu}\mathrm{e}^{st} = \varphi_{\nu}(A,t) \quad (0 < \nu < 1),$$

where

$$\varphi_{\nu}(A,t) = \begin{cases} \left(\dfrac{\nu}{t}\right)^{\nu}\mathrm{e}^{-\nu} & \text{if } t \leqslant \dfrac{\nu}{|\alpha(A)|}, \\ |\alpha(A)|^{\nu}\mathrm{e}^{\alpha(A)t} & \text{if } t \geqslant \dfrac{\nu}{|\alpha(A)|}. \end{cases}$$

So if A is negative definite and $S = (-A)^{\nu}$, then

$$(1.8)\quad J_0 = \int_0^{\infty} \mathrm{e}^{(b_0+\alpha(A))t}\,\mathrm{d}t = \frac{1}{|b_0+\alpha(A)|} \quad \text{and} \quad J(S) \leqslant J_{\nu} = \int_0^{\infty} \mathrm{e}^{b_0 t}\varphi_{\nu}(A,t)\,\mathrm{d}t,$$

provided that $\alpha(A) + b_0 < 0$.

2. Proof of Theorem 1.1

Lemma 2.1. *Let A generate an analytic semigroup $(\mathrm{e}^{At})_{t\geqslant 0}$ and $B(r)$ map* $\mathrm{Dom}(A)$ *into itself for all $r \geqslant 0$. In addition, let there be a linear operator S with* $\mathrm{Dom}(S) \supseteq \mathrm{Dom}(A)$ *having a bounded left-inverse one S_l^{-1} such that $S\mathrm{e}^{At}$ be integrable on each finite interval, and the conditions* (1.6) *and*

$$\int_0^t \|\mathrm{e}^{sA}\|\|S\mathrm{e}^{sA}\|\,\mathrm{d}s < \infty \quad (0 < t < \infty)$$

hold. Then with the notation

$$[\mathrm{e}^{At}, B(r)] := \mathrm{e}^{tA}B(r) - B(r)\mathrm{e}^{At} \quad (t, r \geqslant 0),$$

one has

$$[\mathrm{e}^{At}, B(r)] = \int_0^t \mathrm{e}^{sA}K(r)\mathrm{e}^{(t-s)A}\,\mathrm{d}s \quad (0 \leqslant t, r < \infty).$$

In addition,

$$\|[\mathrm{e}^{At}, B(r)]\| \leqslant m(K(\cdot), S)\int_0^t \|\mathrm{e}^{sA}\|\|S\mathrm{e}^{(t-s)A}\|\,\mathrm{d}s \quad (0 \leqslant t, r < \infty)$$

and $[\mathrm{e}^{At}, B(r)]$ maps $\mathrm{Dom}(A)$ *into itself.*

P r o o f. In this proof for a fixed $r > 0$, for the brevity we put $B(r) = B$ and $K(r) = K$. Condition (1.6) implies

$$\begin{aligned}\Big\|\int_0^t \mathrm{e}^{sA} K \mathrm{e}^{(t-s)A}\,\mathrm{d}s\Big\| &= \Big\|\int_0^t \mathrm{e}^{sA} K S_l^{-1} S \mathrm{e}^{(t-s)A}\,\mathrm{d}s\Big\| \\ &\leqslant m(K(\cdot), S)\int_0^t \|\mathrm{e}^{sA}\|\|S\mathrm{e}^{(t-s)A}\|\,\mathrm{d}s \\ &\leqslant m(K(\cdot), S)\max_{s\leqslant t}\|\mathrm{e}^{sA}\|\int_0^t \|S\mathrm{e}^{(t-s)A}\|\,\mathrm{d}s < \infty.\end{aligned}$$

So the operator $\int_0^t \mathrm{e}^{sA} K \mathrm{e}^{(t-s)A}\,\mathrm{d}s$ is bounded for all finite t. On $\mathrm{Dom}(A)$ we have

$$\begin{aligned}\int_0^t \mathrm{e}^{sA} K \mathrm{e}^{(t-s)A}\,\mathrm{d}s &= \int_0^t \mathrm{e}^{sA}(AB - BA)\mathrm{e}^{(t-s)A}\,\mathrm{d}s \\ &= \int_0^t (\mathrm{e}^{sA} AB \mathrm{e}^{(t-s)A} - \mathrm{e}^{sA} BA \mathrm{e}^{(t-s)A})\,\mathrm{d}s \\ &= \int_0^t \Big(\frac{\partial}{\partial s}\mathrm{e}^{sA} B \mathrm{e}^{(t-s)A} + \mathrm{e}^{sA} B \frac{\partial}{\partial s}\mathrm{e}^{(t-s)A}\Big)\,\mathrm{d}s \\ &= \int_0^t \frac{\partial}{\partial s}(\mathrm{e}^{sA} B \mathrm{e}^{(t-s)A})\,\mathrm{d}s = \mathrm{e}^{At}B - B\mathrm{e}^{At},\end{aligned}$$

as claimed. □

For an operator function $Z(t,s)$ defined and uniformly bounded on $0 \leqslant s \leqslant t \leqslant \infty$ set $\|Z\|_C := \sup_{t\geqslant s\geqslant 0}\|Z(t,s)\|$.

Lemma 2.2. *Let $X(t,s)$ be the evolution operator of* (1.1), *and with the notations $W(t,s) = \exp[A(t-s)]U_B(t,s)$ and*

$$H(t,s) := [\mathrm{e}^{A(t-s)}, B(t)]U_B(t,s) \quad (t \geqslant s \geqslant 0),$$

let $\|W\|_C < \infty$ and

$$\zeta(H) := \sup_s \int_s^\infty \|H(t,s)\|\,\mathrm{d}t < 1. \tag{2.1}$$

Then the inequalities

$$\|X\|_C \leqslant \frac{\|W\|_C}{1-\zeta(H)} \tag{2.2}$$

and

$$\|X - W\|_C \leqslant \frac{\zeta(H)\|W\|_C}{1-\zeta(H)} \tag{2.3}$$

are valid.

P r o o f. Note that for all $h \in \operatorname{Dom}(A)$ we have

$$\frac{\mathrm{d}X(t,s)h}{\mathrm{d}t} = (A+B(t))X(t,s)h \tag{2.4}$$

and

$$\begin{aligned}\frac{\mathrm{d}W(t,s)h}{\mathrm{d}t} &= (A\mathrm{e}^{A(t-s)}U_B(t,s) + \mathrm{e}^{A(t-s)}B(t)U_B(t,s))h \\ &= ((A+B(t))\mathrm{e}^{A(t-s)}U_B(t,s) + \mathrm{e}^{A(t-s)}B(t)U_B(t,s) \\ &\quad - B(t)\mathrm{e}^{A(t-s)}U_B(t,s))h \\ &= (A+B(t))W(t,s)h + H(t,s)h.\end{aligned} \tag{2.5}$$

Due to Lemma 2.1, operator $H(t,s)$ is bounded for all finite t, s and maps $\operatorname{Dom}(A)$ into itself. Subtracting (2.4) from (2.5), on $\operatorname{Dom}(A)$ we get

$$\frac{\mathrm{d}(W(t)-X(t))}{\mathrm{d}t} = (A+B(t))(W(t,s)-X(t,s)) + H(t,s).$$

Making use of the variation of constants formula, (see [14], Theorem II.3.1) we can write

$$(W(t,s)-X(t,s))h = \int_s^t X(t,s_1)H(s_1,s)h\,\mathrm{d}s_1 \quad \forall\, h \in \operatorname{Dom}(A).$$

Since $\operatorname{Dom}(A)$ is dense in $\mathcal{X}$, and $W(t,s)$, $X(t,s)$ and $H(t,s)$ are bounded, we can write

$$W(t,s)-X(t,s) = \int_s^t X(t,s_1)H(s_1,s)\,\mathrm{d}s_1.$$

Consequently,

$$\|W(t,s)-X(t,s)\| \leqslant \int_s^t \|X(t,s_1)\|\|H(s_1,s)\|\,\mathrm{d}s_1, \tag{2.6}$$

and therefore,

$$\|X(t,s)\| \leqslant \|W(t,s)\| + \int_s^t \|X(t,s_1)\|\|H(s_1,s)\|\,\mathrm{d}s_1.$$

Hence, for any finite $t > s$ we obtain

$$\sup_{0\leqslant s\leqslant v\leqslant t} \|X(v,s)\| \leqslant \|W\|_C + \sup_{0\leqslant s\leqslant v\leqslant t} \|X(v,s)\|\zeta(H).$$

Now (2.1) implies (2.2). From (2.6) and (2.2), inequality (2.3) follows. This proves the lemma. □

P r o o f of Theorem 1.1. By (1.4),

$$\int_s^\infty \|H(t,s)\|\,\mathrm{d}t \leqslant c_0 \int_s^\infty \mathrm{e}^{b_0(t-s)} \|[\mathrm{e}^{A(t-s)}, B(t)]\|\,\mathrm{d}t \leqslant c_0 \int_0^\infty \mathrm{e}^{b_0 v} \|[\mathrm{e}^{Av}, B(v+s)]\|\,\mathrm{d}v.$$

Inequality (2.2) means that (1.1) is Lyapunov stable, i.e., there is a constant $m_1 \geqslant 1$, independent of the initial vector, such that $\|y(t)\| \leqslant m_1\|y(0)\|$ $(t \geqslant 0)$ for any solution $y(t)$ of (1.1), see [6].

Furthermore, substitute

$$y(t) = u_\varepsilon(t)\mathrm{e}^{-\varepsilon t} \quad (\varepsilon > 0) \tag{2.7}$$

into (1.1). Then

$$\frac{\mathrm{d}u_\varepsilon(t)}{\mathrm{d}t} = (A + B(t) + \varepsilon I)u_\varepsilon(t). \tag{2.8}$$

If ε is small enough, then conditions (1.4), (1.5) and (1.6) hold with $B(t)+\varepsilon I$ instead of $B(t)$.

Applying our above arguments to equation (2.8) we can assert that it is Lyapunov stable: $\|u_\varepsilon(t)\| \leqslant m_1\|u_\varepsilon(0)\|$ $(t \geqslant 0)$. So due to (2.7), equation (1.1) is exponentially stable. This proves the theorem. □

3. Equations with the Lipschitz property

In this section we illustrate Theorem 1.1 in the case when

$$\|B(t) - B(t_1)\| \leqslant q_0|t - t_1| \quad (t, t_1 \geqslant 0;\ q_0 = \text{const.} > 0), \tag{3.1}$$

and

$$\|\exp[B(\tau)t]\| \leqslant p(t) \quad (t, \tau \geqslant 0), \tag{3.2}$$

where $p(t)$ is a piecewise-continuous function independent of τ uniformly bounded on $[0, \infty)$.

Lemma 3.1. *Let conditions* (3.1), (3.2) *and*

$$\theta_0 := q_0 \int_0^\infty tp(t)\,\mathrm{d}t < 1 \tag{3.3}$$

hold. Then the evolution operator $U_B(t,s)$ *of* (1.3) *satisfies the inequality*

$$\sup_{t\geqslant s} \|U_B(t,s)\| \leqslant \frac{\chi}{1-\theta_0} \quad (t \geqslant s \geqslant 0),$$

where $\chi := \sup_{t\geqslant 0} p(t)$.

P r o o f. Equation (1.3) can be rewritten in the form

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} - B(\tau)u(t) + [B(t) - B(\tau)]u(t)$$

with an arbitrary fixed $\tau \geqslant 0$. This equation is equivalent to the following one:

$$u(t) = \exp[B(\tau)(t-s)]u(s) + \int_s^t \exp[B(\tau)(t-t_1)][B(t_1) - B(\tau)]u(t_1)\,\mathrm{d}t_1.$$

So

$$\|u(t)\| \leqslant \|\exp[B(\tau)(t-s)]\|\|u(s)\| + \int_s^t \|\exp[B(\tau)(t-t_1)]\|\|B(t_1) - B(\tau)\|\|u(t_1)\|\,\mathrm{d}t_1.$$

According to (3.1) and (3.2),

$$\|u(t)\| \leqslant p(t-s)\|u(s)\| + q_0 \int_s^t p(t-t_1)|t_1 - \tau|\|u(t_1)\|\,\mathrm{d}t_1.$$

With $\tau = t$, this relation gives us

$$\|u(t)\| \leqslant p(t-s)\|u(s)\| + q_0 \int_s^t p(t-t_1)(t-t_1)\|u(t_1)\|\,\mathrm{d}t_1.$$

Hence,

$$\sup_{s \leqslant t \leqslant T} \|u(t)\| \leqslant \chi\|u(s)\| + \sup_{s \leqslant t \leqslant T} \|u(t)\|\theta_0$$

for any positive finite T. By (3.3) we arrive at the inequality

$$\sup_{0 \leqslant t \leqslant T} \|u(t)\| \leqslant \frac{\chi\|u(s)\|}{1-\theta_0}.$$

Since the right-hand side of the latter inequality does not depend on T, we get the required inequality. □

Under the hypothesis of Lemma 3.1, condition (1.4) holds with $b_0 = 0$ and $c_0 = \chi/(1-\theta_0)$, hence Theorem 1.1 implies:

Corollary 3.2. *Let conditions* (1.6), (3.1) *and* (3.2) *hold. Let*

$$\hat{J}(S) := \frac{\chi}{1-\theta_0} \int_0^\infty \|S\mathrm{e}^{At}\|\,\mathrm{d}t < \infty \quad \text{and} \quad \hat{J}_0 := \int_0^\infty \|\mathrm{e}^{At}\|\,\mathrm{d}t,$$

and

$$\frac{\chi m(K(\cdot), S)\hat{J}(S)\hat{J}_0}{1-\theta_0} < 1.$$

Then equation (1.1) *is exponentially stable.*

For estimates for the exponential function of various finite and infinite dimensional operators, see for example [10].

4. Example

Consider the problem

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u}{\partial x^2}(t,x) + M(t,x)u(t,x) \quad (0 < x < 1), \tag{4.1}$$

$$u(t,0) = u(t,1) = 0 \quad (t > 0), \tag{4.2}$$

where $M(t,x) = (m_{jk}(t,x))$ is a variable real $n \times n$-matrix function defined and uniformly bounded on $[0,\infty) \times [0,1]$, twice continuously differentiable in x and continuous in t.

Take $\mathcal{X} = L^2([0,1];\mathbb{C}^n)$ – the Hilbert space of n-vector valued functions defined on $[0,1]$ with the scalar product

$$(v,w) = \int_0^1 (v(x),w(x))_n \,\mathrm{d}x \quad (v,w \in L^2([0,1];\mathbb{C}^n)),$$

where $(\cdot,\cdot)_n$ is the scalar product in $\mathbb{C}^n$. For the brevity put $L^2_n = L^2([0,1];\mathbb{C}^n)$ and take

$$(Af)(x) = f''(x) \quad \text{and} \quad (B(t)f)(x) = M(t,x)f(x) \quad (f \in \mathrm{Dom}(A),\ 0 \leqslant x \leqslant 1)$$

and $S = (-A)^{1/2}$ with

$$\mathrm{Dom}(A) = H^2(0,1)^n \cap H^1_0(0,1)^n = \{h \in L^2_n\colon\ h'' \in L^2_n,\ h(0) = h(1) = 0\}.$$

Then $(K(t)f)(x) = M''_{xx}(t,x)f(x) + 2M'_x(t,x)f'(x)$. Obviously,

$$e_{k,j}(x) = \sqrt{2}\sin(\pi k x)e_j \quad \forall\, j = 1,\ldots,n,$$

where $\{e_j\}_{j=1}^n$ is the standard basis in $\mathbb{C}^n$, are the eigenfunctions of A of the algebraic multiplicity n, $P_{kj} = (\cdot,e_{k,j})e_{k,j}$ are the eigen-projections of A and $-\pi^2k^2$ $(k = 1,2,\ldots)$ are its eigenvalues of multiplicity n. We have

$$A = -\pi^2\sum_{j=1}^n\sum_{j=1}^\infty k^2P_{kj}, \quad (-A)^{1/2} = \pi\sum_{j=1}^n\sum_{j=1}^\infty kP_{kj} \quad \text{and} \quad \mathrm{e}^{At} = \sum_{j=1}^n\sum_{j=1}^\infty \mathrm{e}^{-\pi^2k^2t}P_{kj},$$

and by (1.8)

$$\|(-A)^{1/2}\mathrm{e}^{At}\| = \varphi_{1/2}(A,t), \tag{4.3}$$

where

$$\varphi_{1/2}(A,t) = \begin{cases} \dfrac{1}{\sqrt{2t}}\mathrm{e}^{-1/2} & \text{if } t \leqslant \dfrac{1}{2\pi^2}, \\ \pi\mathrm{e}^{-\pi^2 t} & \text{if } t \geqslant \dfrac{1}{2\pi^2}. \end{cases}$$

In addition, $\|(-A)^{-1/2}\| = \|S^{-1}\| = 1/\pi$, and by Green's formula we have

$$\Big(\frac{\mathrm{d}}{\mathrm{dx}}S^{-1}f, \frac{\mathrm{d}}{\mathrm{dx}}S^{-1}f\Big) = -\Big(\frac{\mathrm{d}^2}{\mathrm{d}x^2}S^{-1}f, S^{-1}f\Big) = -(AS^{-1}f, S^{-1}f).$$

As S^{-1} is selfadjoint and commutes with A, this yields

$$\Big(\frac{\mathrm{d}}{\mathrm{dx}}S^{-1}f, \frac{\mathrm{d}}{\mathrm{dx}}S^{-1}f\Big) = -(AS^{-2}f, f) = (f, f),$$

and therefore, for any $f \in \mathrm{Dom}(A)$ with $\|f\| = 1$ we obtain

$$\begin{aligned}\|K(t)S^{-1}f\| &= \Big\|M''_{xx}(t,\cdot)S^{-1}f + 2M'_x(t,\cdot)\frac{\mathrm{d}}{\mathrm{dx}}S^{-1}f\Big\| \\ &\leqslant \sup_x(\|M''_{xx}(t,x)\|_n\|S^{-1}\| + 2\|M'_x(t,x)\|_n)\end{aligned}$$

Here $\|\cdot\| = \sqrt{(\cdot,\cdot)}$ is the norm in L^2_n and $\|\cdot\|_n$ is the norm in $\mathbb{C}^n$. Suppose that

$$\widehat{m}(K(\cdot), M) := \sup_{x,t}\Big(\frac{1}{\pi}\|M''_{xx}(t,x)\|_n + 2\|M'_x(t,x)\|_n\Big) < \infty. \tag{4.4}$$

Then $m(K(\cdot), S) = \sup_t \|K(t,\cdot)S^{-1}\| \leqslant \widehat{m}(K(\cdot), M)$. Consider the vector equation

$$\frac{\partial v}{\partial t} = M(t,x)v \quad (v = v(t,x),\ 0 < x < 1). \tag{4.5}$$

Assume that there are constant q_M independent of x and a piecewise-continuous function $p_M(t)$ independent of s and x, and uniformly bounded on $[0,\infty)$ such that

$$\|M(t,x) - M(t_1,x)\|_n \leqslant q_M|t - t_1| \quad (t, t_1 \geqslant 0;\ q_M = \text{const.} > 0), \tag{4.6}$$

$$\|\exp[M(\tau,x)t]\| \leqslant p_M(t) \quad (t, \tau \geqslant 0;\ 0 \leqslant x \leqslant 1) \tag{4.7}$$

and

$$\theta_M := q_M \int_0^\infty t p_M(t)\,\mathrm{d}t < 1 \tag{4.8}$$

hold. Then due to Lemma 3.1 the evolution operator $U_M(t,s)$ of (4.5) satisfies the inequality

$$\|U_M(t,s)\|_n \leqslant \frac{\chi_M}{1-\theta_M} \quad (t \geqslant s \geqslant 0),$$

where $\chi_M := \sup_{t\geqslant 0} p_m(t)$. Hence, condition (1.4) holds with $b_0 = 0$ and $c_0 = \chi_M/(1-\theta_M)$. Thus,

$$\hat{J}_0 = \int_0^\infty \|\mathrm{e}^{At}\|\,\mathrm{d}t \leqslant \int_0^\infty \mathrm{e}^{-\pi^2 t}\,\mathrm{d}t = \frac{1}{\pi^2}.$$

In addition, due to (4.3)

$$\hat{J}(S) = \int_0^\infty \|S\mathrm{e}^{At}\|\,\mathrm{d}t \leqslant \hat{J}_{1/2}, \quad \text{where } \hat{J}_{1/2} := \int_0^\infty \varphi_{1/2}(A,t)\,\mathrm{d}t.$$

This integral is simply calculated. Now Corollary 3.2 yields:

Corollary 4.1. *Let conditions* (4.6)–(4.8) *and*

$$\frac{\chi_M m(K(\cdot),S)J_{1/2}}{(1-\theta_M)\pi^2} < 1$$

hold. Then equation (4.1), (4.2) *is exponentially stable.*

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