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UPPER AND LOWER CONVERGENCE RATES FOR STRONG SOLUTIONS OF THE 3D NON-NEWTONIAN FLOWS ASSOCIATED WITH MAXWELL EQUATIONS UNDER LARGE INITIAL PERTURBATION

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Abstract. We show the upper and lower bounds of convergence rates for strong solutions of the 3D non-Newtonian flows associated with Maxwell equations under a large initial perturbation.

Keywords: non-Newtonian fluid; MHD equation; decay estimate; large initial perturbation

MSC 2020: 35Q30, 76A05, 35B35

1. Introduction

In this paper, we study the non-Newtonian fluids associated with Maxwell equations:

$$(1.1)\quad \begin{cases} u_t - \nabla\cdot S(Du) + (u\cdot\nabla)u + \nabla\pi - (b\cdot\nabla)b = f,\\ b_t - \triangle b + (u\cdot\nabla)b - (b\cdot\nabla)u = g,\\ \operatorname{div}\ u = 0 \quad\text{and}\quad \operatorname{div} b = 0,\\ u(x,0) = u_0(x) \quad\text{and}\quad b(x,0) = b_0(x). \end{cases} \qquad \text{in } Q_T := \mathbb{R}^3\times(0,T),$$

Here $u\colon \mathbb{R}^3\times(0,T)\to\mathbb{R}^3$ is the flow velocity vector, $b\colon \mathbb{R}^3\times(0,T)\to\mathbb{R}^3$ is the magnetic vector, $\pi\colon \mathbb{R}^3\times(0,T)\to\mathbb{R}$ is the total pressure, f and g are the external forces, and Du is the symmetric part of the velocity gradient, i.e.,

$$Du = D_{ij}u := \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad i,j = 1,2,3.$$

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We assume that $u_0, b_0 \in L^2(\mathbb{R}^3)$ and $\operatorname{div} u_0 = 0 = \operatorname{div} b_0$. To motivate the conditions on the stress tensor S, we recall the following examples of constitutive laws:

$$S(Du) = (\mu_0 + \mu_1|Du|^{p-2})Du \quad \text{or} \quad S(Du) = (\mu_0 + \mu_1|Du|^2)^{(p-2)/2}Du, \quad 1 < p < \infty,$$

where $\mu_0 \geqslant 0$ and $\mu_1 > 0$ are constants, see [1], [19].

For notational convenience, we denote by $\mathbb{M}^3_{\text{sym}}$ a vector space of all symmetric 3×3 matrices $\zeta = (\zeta_{ij})_{1 \leqslant i,j \leqslant 3}$. We note that the deviatoric stress tensor $S = (S_{ij})$, $i, j = 1, 2, 3$ satisfies the following assumptions: For $(x, t) \in Q_T$,

(i) $S\colon Q_T \times \mathbb{M}^3_{\text{sym}} \to \mathbb{M}^3_{\text{sym}}$ is a Carathéodory function,

(ii) (symmetry) $S_{ij} = S_{ji}$,

(iii) (polynomial growth) $|S_{ij}(\xi)| \leqslant (\mu_0 + \mu_1|\xi|^{p-2})|\xi|$,

(iv) (coercivity condition) there exists $c_1 > 1$ such that

$$(\mu_0 + \mu_1|\xi|^{p-2})|\eta|^2 \leqslant \frac{\partial S_{ij}}{\partial \xi_{kl}} \eta_{kl}\eta_{ij} \leqslant c_1(\mu_0 + \mu_1|\xi|^{p-2})|\eta|^2,$$

(v) (strict monotonicity) for all $\zeta, \eta \in \mathbb{M}^3_{\text{sym}}$ $(\zeta \neq \eta)$, $S(\zeta) - S(\eta) : (\zeta - \eta) > 0$.

Gunzburger et al. in [4] considered (1.1) for the case of bounded or periodic domains, and they established unique solvability of the initial-boundary value problem. More specifically, assuming that $u_0 \in H^2(\Omega)$ and $b_0 \in H^1(\Omega)$ with some boundary conditions for a bounded domain, it was shown in [4] that if $\frac{5}{2} < q \leqslant 6$, a generalized solution exists, and moreover, it satisfies

$$\begin{gathered} u \in L^\infty(0,T; L^2(\Omega) \cap H^1(\Omega)), \quad \nabla u \in L^\infty(0,T; L^q(\Omega)), \\ b \in L^\infty(0,T; L^2(\Omega) \cap H^1(\Omega)), \quad \nabla b \in L^\infty(0,T; L^2(\Omega)), \quad b \in L^2(0,T; H^2(\Omega)), \\ u_t \in L^2(\Omega \times (0,T)), \quad b_t \in L^2(\Omega \times (0,T)). \end{gathered} \tag{1.2}$$

Furthermore, they have shown the uniqueness of solutions. Here strong solutions means that the solutions satisfy (1.1) pointwise a.e. and the energy equality holds. Recently, the authors in [6] established global unique solvability to (1.1) for $u_0 \in (W^{1,2} \cap W^{1,p})$ and $b_0 \in W^{1,2}$, $\frac{5}{2} \leqslant p$ in the same class above, see [15] for weak solutions. For a half space, the proof in [4] also holds.

Let us take a look at the main results on the Navier-Stokes equation. Secchi in [17] considered the L^2-asymptotic stability for weak solutions of 3D Navier-Stokes equations under the large initial data perturbation. He proved that if one has a smooth solution (for small data), then every weak solution $v(x,t)$ to the perturbed Navier-Stokes equations converges asymptotically to $u(x,t)$, that is,

$$\|v(t) - u(t)\|_{L^2} \to 0 \quad \text{as } t \to \infty. \tag{1.3}$$

After his work, Kozono in [11] considered the asymptotic stability of large solutions with large perturbation to the Navier-Stokes equations in $\Omega \subset \mathbb{R}^3$:

$$u_t - \Delta u + u \cdot \nabla u + \nabla p = \overline{f} + f, \quad \operatorname{div}(u) = 0,$$

$u|_{\partial\Omega} = 0$, $u(0) = w_0 + a$. Here Ω is an open subset of $\mathbb{R}^3$ which may be unbounded as well as its boundary $\partial\Omega \in C^3$. Assume further that $\overline{f}, f \in L^1_{\text{loc}}([0,\infty), L^2(\Omega))$ and $w_0, a \in L^2(\Omega)$ with zero divergence and zero normal component on the boundary in the weak sense. Let w be a weak solution of (1) with $a = 0$, $f = 0$, which also belongs to Serrin's class $L^\alpha((0,\infty), L^q(\Omega))$ with $2/\alpha + 3/q = 1$ and $q > 3$. In addition, he proved that every perturbed flow v with the energy inequality converges asymptotically to w in Serrin's class, that is, $\|v(t) - w(t)\|_{L_2} \to 0$ as $t \to \infty$, or $\|\nabla v(t) - \nabla w(t)\|_{L_2} = O(t^{-1/2})$. In this direction, we refer to [21] and [22]. We also refer to [8] for L^2-asymptotic stability of singular solutions corresponding to singular external force to the Navier-Stokes system of equations in $\mathbb{R}^3$, see also [7].

On the other hand, for the upper and lower bounds for weak solutions to 3D Navier-Stokes equations, a lot of important work has been done see e.g. [2], [3], [12], [14], [16] and [18]. The L^2-decay rates of weak solutions with both upper and lower bounds reads

$$C_1(1+t)^{-3/4} \leqslant \|u(t)\|_{L^2(\mathbb{R}^3)} \leqslant C_2(1+t)^{-3/4}.$$

Motivated by Secchi's result (see [17]), very recently, the authors in [20] have obtained the optimal upper and lower bounds of convergence rates for the 3D Navier-Stokes equations under large initial perturbation. For w_0 with

$$\int |\widehat{w}_0(\lambda w)|^2 \,\mathrm{d}w = C\lambda^{2\gamma-3} + o(\lambda^{2\gamma-3}) \quad \text{as } \lambda \to 0, \tag{1.4}$$

they have shown that for the global Leray weak solution u(x, t) of the 3D Navier-Stokes equations then even for the large initial perturbation, every weak solution $v(x,t)$ corresponding to $u_0 + w_0$ of the perturbed Navier-Stokes equations converges algebraically to $u(x,t)$ corresponding to u_0 with the optimal upper and lower bounds

$$C_1(1+t)^{-\gamma/2} \leqslant \|v(t) - u(t)\|_{L^2(\mathbb{R}^3)} \leqslant C_2(1+t)^{-\gamma/2} \text{ for large } t > 1,\ 2 < \gamma < \frac{5}{2}. \tag{1.5}$$

For the asymptotic behavior of strong solutions to (1.1), the author in [9] or [10] recently have examined the L^2-algebraic decay, that is $\|(u,b)(t)\|_{L^2} \leqslant C(1+t)^{-3/4}$ in the whole space $\mathbb{R}^3$ with respect to the monopolar shear thickening fluids using Fourier splitting method in [16]. We also refer to [5] and [13] for Navier–Stokes equations of non-Newtonian type.

In light of the results of a series for the viscous fluid flows, recently, Xie, Guo and Dong in [20] have studied the upper and lower convergence rates for weak solutions to 3D shear thickening non-Newtonian fluid equation of bipolar type:

$$(1.6)\qquad \begin{cases} u_t - \mu_0\Delta u + \mu_2\Delta^2 u + (u\cdot\nabla)u + \nabla p = f + \mu_1\nabla\cdot(|\nabla u|^{r-2}\nabla u), \\ \operatorname{div} u = 0, \\ u(x,0) = u_0(x). \end{cases}$$

For w_0 with (1.4), they obtain the asymptotic stability of large solutions with large perturbation for the difference between a weak solution u corresponding to the initial data u_0 and the weak solution $\widetilde{u}$ corresponding to the perturbed initial data u_0+w_0:

$$C_1(1+t)^{-(5-\gamma)/4} \leqslant \|\widetilde{u}(t) - u(t)\|_{L^2(\mathbb{R}^3)} \leqslant C_2(1+t)^{-(5-\gamma)/4}$$

for a sufficiently small $\gamma > 0$.

The purpose of this paper is to investigate asymptotic stability of strong solutions inspired by [20]. If w and h are perturbed initially, then the perturbed flow $\widetilde{u}$ and $\tilde{b}$ satisfy

$$(1.7)\qquad \begin{cases} \widetilde{u}_t - \nabla\cdot S(D\widetilde{u}) + (\widetilde{u}\cdot\nabla)\widetilde{u} + \nabla\widetilde{\pi} - (\tilde{b}\cdot\nabla)\tilde{b} = f, \\ \tilde{b}_t - \triangle\tilde{b} + (\widetilde{u}\cdot\nabla)\tilde{b} - (\tilde{b}\cdot\nabla)\widetilde{u} = g, \\ \operatorname{div}\widetilde{u} = 0 \quad\text{and}\quad \operatorname{div}\tilde{b} = 0, \\ \widetilde{u}(x,0) = u_0(x) + w_0(x) \quad\text{and}\quad \tilde{b}(x,0) = b_0(x) + h_0(x). \end{cases}$$

Theorem 1.1. *Suppose (u,b) is a strong solution of the 3D non-Newtonian flows associated with Maxwell equations (1.1) with the initial data $u_0, b_0 \in L^2(\mathbb{R}^3)$ and external force $f, g \in L^2(Q_T)$. If the initial perturbation $w_0, h_0 \in L^2(\mathbb{R}^3)$ satisfies (1.4) with $2 < \gamma < \frac{5}{2}$, then there exists a strong solution $\widetilde{u}(x,t)$ and $\tilde{b}(x,t)$ of the perturbed equations (1.7) and satisfies the optimal convergence rates*

$$(1.8)\qquad C_1(1+t)^{-\gamma/2} \leqslant \|\widetilde{u}(t) - u(t)\|_{L^2(\mathbb{R}^3)} + \|\tilde{b}(t) - b(t)\|_{L^2(\mathbb{R}^3)} \leqslant C_2(1+t)^{-\gamma/2}$$

for large $t > 1$.

Remark 1.2. Comparing to [20], we focus on the monopolar fluid for the stress tensor, which means only the first derivative of the velocity is involved in the stress tensor. For this case of the monopolar fluid since there is not enough regularity of weak solutions, we need a solution with slightly higher regularity, which is called *strong solutions*.

Remark 1.3. For the existence of (weak or strong) solutions to the perturbed system, it is checked according to the standard arguments, see [6] and [15].

Corollary 1.4. *For a sufficiently smooth solution* (u_0, b_0), *if* (u, b) *of 3D MHD equations, that is,* $p = 2$ *in* (1.1) *satisfies one of the following conditions:*
(a) $u \in L^{2r/(r-3)}(0, T; L^r(\mathbb{R}^3))$ *with* $3 < r \leqslant \infty$,
(b) $\nabla u \in L^{2r/(2r-3)}(0, T; L^r(\mathbb{R}^3))$ *with* $\frac{3}{2} < r \leqslant \infty$,
then (1.8) *holds.*

2. Preliminary

We first introduce some notations. Let $(X, \|\cdot\|)$ be a normed space. By $L^q(0, T; X)$, we denote the space of all Bochner measurable functions $\varphi\colon (0, T) \to X$ such that

$$\begin{cases} \|\varphi\|_{L^q(0,T;X)} := \left(\int_0^T \|\varphi(t)\|^q \,\mathrm{d}t\right)^{1/q} < \infty & \text{if } 1 \leqslant q < \infty, \\ \|\varphi\|_{L^\infty(0,T;X)} := \sup\limits_{t\in(0,T)} \|\varphi(t)\| < \infty & \text{if } q = \infty. \end{cases}$$

For $1 \leqslant q \leqslant \infty$, we mean by $W^{k,q}(\mathbb{R}^3)$ the usual Sobolev space. Let $A = (a_{ij})_{i,j=1}^3$ and $B = (b_{ij})_{i,j=1}^3$ be matrix valued maps and we then denote $A : B = \sum\limits_{i,j=1}^{3} a_{ij} b_{ij}$. For vector fields u, v we write $(u_i v_j)_{i,j=1,2,3}$ as $u \otimes v$. Unless specifically mentioned, the letter C is used to represent a generic constant, which may change from line to line. And also, we denote by $A \lesssim B$ an estimate of the form $A \leqslant CB$ with an absolute constant C.

Next, we recall a notion of weak and strong solutions to (1.1) satisfying the following definition:

Definition 2.1 (Weak solutions [6])**.** Let $p > \frac{6}{5}$. Suppose that $u_0 \in L^2(\mathbb{R}^3)$ and $b_0 \in L^2(\mathbb{R}^3)$. We say that (u, b) is a weak solution of the magnetohydrodynamics with power-law type nonlinear viscous fluid (1.1) if u and b satisfy the following:
(i) $u \in L^\infty(I; L^2(\mathbb{R}^3)) \cap L^q(I; W^{1,q}(\mathbb{R}^3))$, $b \in L^\infty(I; L^2(\mathbb{R}^3)) \cap L^2(I; H^1(\mathbb{R}^3))$,
(ii) (u, b) satisfies (1.1) in the sense of distribution, that is,

$$\int_0^T \int_{\mathbb{R}^3} \Big(\frac{\partial \phi}{\partial t} + (u \cdot \nabla)\phi\Big) u \,\mathrm{d}x \,\mathrm{d}t - \int_0^T \int_{\mathbb{R}^3} S : \nabla\phi \,\mathrm{d}x \,\mathrm{d}t = \int_0^T \int_{\mathbb{R}^3} (b \cdot \nabla\phi) b \,\mathrm{d}x \,\mathrm{d}t$$

and

$$\int_0^T \int_{\mathbb{R}^3} \Big(\frac{\partial \phi}{\partial t} + \Delta\phi\Big) b \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} (u \cdot \nabla\phi) b \,\mathrm{d}x \,\mathrm{d}t = \int_0^T \int_{\mathbb{R}^3} (b \cdot \nabla\phi) u \,\mathrm{d}x \,\mathrm{d}t$$

for all $\phi \in C_0^\infty(\mathbb{R}^3 \times [0, T))$ with $\nabla \cdot \phi = 0$ and

$$\int_{\mathbb{R}^3} u \cdot \nabla\psi \,\mathrm{d}x = 0, \quad \int_{\mathbb{R}^3} b \cdot \nabla\psi \,\mathrm{d}x = 0$$

for every $\psi \in C_0^\infty(\mathbb{R}^3)$,

(iii) (u,b) satisfies the energy inequality

$$\sup_{0\leqslant t\leqslant T}(\|u(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}+\|b(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}) \\ +2\int_0^T(\mu_0\|\nabla u\|^2_{L^2(\mathbb{R}^3)}+\mu_1\|\nabla u\|^q_{L^q(\mathbb{R}^3)}+\|\nabla b\|^2_{L^2(\mathbb{R}^3)})\,\mathrm{d}t \\ \leqslant \|u_0\|^2_{L^2(\mathbb{R}^3)}+\|b_0\|^2_{L^2(\mathbb{R}^3)}.$$

Definition 2.2 (Strong solutions [6]). Let $p\geqslant \frac{5}{2}$. Suppose that

$$u_0\in (W^{1,2}\cap W^{1,q})(\mathbb{R}^3) \quad \text{and} \quad b_0\in W^{1,2}(\mathbb{R}^3), \quad q\in(1,\infty)$$

and q' be the Hölder conjugate of q. We say that (u,b) is a strong solution of (1.1) if (u,b) is a weak solution in Definition 2.1 and (u,b) satisfies

$$\nabla u\in L^3(\mathbb{R}^3\times(0,T))\cap L^\infty(0,T;L^q\cap L^2(\mathbb{R}^3)),$$
$$u_t\in L^2(\mathbb{R}^3\times(0,T)),\ S(Du)\in L^{q'}(0,T;W^{1,q'}_{\rm loc}(\mathbb{R}^3)),\ \nabla|Du|^{q/2}\in L^2((0,T);L^2(\mathbb{R}^3)),$$
$$b_t\in L^2(\mathbb{R}^3\times(0,T)),\ \nabla b\in L^\infty(0,T;L^2(\mathbb{R}^3)),\ b\in L^2(0,T;W^{2,2}(\mathbb{R}^3)).$$

We denote the pressure difference $p(x,t)=\pi_2(x,t)-\pi_1(x,t)$ and the difference $w(x,t)=\widetilde{u}(x,t)-u(x,t)$ and $h(x,t)=\widetilde{b}(x,t)-b(x,t)$. Hence, $w(x,t)$ and $h(x,t)$ satisfy the following equations in a weak sense:

$$\begin{cases}\partial_t w-\mu_0\Delta w+\widetilde{u}\cdot\nabla w+w\cdot\nabla u-\widetilde{b}\cdot\nabla h-h\cdot\nabla b+\nabla p\\ \qquad\qquad =\mu_1\nabla\cdot(|\nabla\widetilde{u}|^{r-2}\nabla\widetilde{u}-|\nabla u|^{r-2}\nabla u),\\ \partial_t h-\Delta h+\widetilde{u}\cdot\nabla h+w\cdot\nabla b-\widetilde{b}\cdot\nabla w-h\cdot\nabla u=0,\\ \nabla\cdot w=0 \quad \text{and} \quad \nabla\cdot h=0,\\ w(x,0)=w_0(x) \quad \text{and} \quad h(x,0)=h_0(x).\end{cases} \tag{2.1}$$

We also recall the optimal upper and lower bounds of the linear heat equations, see Olive and Titi [14].

Lemma 2.3. *Suppose the initial data $z_0\in L^2(\mathbb{R}^3)$ and satisfies*

$$\int_{\mathbb{S}^2}|\hat{z}_0(rw)|^2\,\mathrm{d}w=Cr^{2\gamma-3}+o(2\gamma-3) \quad \textit{for as } r\to 0.$$

Then there exist two positive constants C_1 and C_2 such that the solution of the heat equation

$$\begin{cases}\partial_t z-\Delta z=0,\\ z(x,0)=z_0(x),\end{cases} \tag{2.2}$$

has the following upper and lower bounds:

$$C_1(1+t)^{-\gamma/2}\leqslant \|\mathrm{e}^{\Delta t}z_0\|_{L^2}\leqslant C_2(1+t)^{-\gamma/2} \quad \textit{for large } t>1.$$

Lemma 2.4. *Let (u,b) be a strong solution to the initial value problem of* (2.1) *with the initial data $u_0, b_0 \in H^1(\mathbb{R}^3)$. Then we have for $\frac{11}{5} \leqslant p < 3$*

$$
\begin{aligned}
|\widehat{w}(\xi,t)| + |\hat{h}(\xi,t)| \leqslant C\Big(&\mathrm{e}^{-|\xi|^2 t}|\widehat{w}_0(\xi)| + |\mathrm{e}^{-|\xi|^2 t}\hat{h}_0(\xi)| \\
&+ |\xi|\int_0^t (\|w(s)\|_{L^2(\mathbb{R}^3)} + \|h(s)\|_{L^2(\mathbb{R}^3)})\,\mathrm{d}s \\
&+ |\xi|\Big(\int_0^t \|\widetilde{u}(s)\|_{L^2(\mathbb{R}^3)}^{(14-2p)/(19-5p)}\,\mathrm{d}s\Big)^{(19-5p)/8} \\
&+ |\xi|\Big(\int_0^t \|u(s)\|_{L^2(\mathbb{R}^3)}^{(14-2p)/(19-5p)}\,\mathrm{d}s\Big)^{(19-5p)/8}\Big),
\end{aligned}
$$

and for $p \geqslant 3$,

$$
\begin{aligned}
&|\widehat{w}(\xi,t)| + |\hat{h}(\xi,t)| \\
&\quad\leqslant C\Big(|\widehat{w}_0(\xi)\mathrm{e}^{-|\xi|^2 t}| + |\hat{h}_0(\xi)\mathrm{e}^{-|\xi|^2 t}| + |\xi|\int_0^t (\|w(s)\|_{L^2(\mathbb{R}^3)} + \|h(s)\|_{L^2(\mathbb{R}^3)})\,\mathrm{d}s\Big),
\end{aligned}
$$

where C depends only on the $H^1(\mathbb{R}^3)$-norm of u_0 and b_0.

P r o o f. Applying the Fourier transformation of (1.1), we have

$$
\begin{aligned}
\widehat{w}_t + |\xi|^2\widehat{w} &=: F(\xi,t), \quad \widehat{w}_0(\xi) := \widehat{w}(\xi,0) = \widehat{w_0}, \\
\hat{h}_t + |\xi|^2\hat{h} &=: H(\xi,t), \quad \hat{h}_0(\xi) := \hat{h}(\xi,0) = \widehat{h_0},
\end{aligned} \tag{2.3}
$$

where

$$
\begin{aligned}
F(\xi,t) := \nabla\cdot(|\nabla\widetilde{u}|^{p-2}\widehat{\nabla\widetilde{u}} - |\nabla u|^{p-2}\nabla u)(\xi,t) - \widehat{(\widetilde{u}\cdot\nabla)w}(\xi,t) + \widehat{(w\cdot\nabla)u}(\xi,t), \\
+ \widehat{(\tilde{b}\cdot\nabla)h}(\xi,t) + \widehat{(h\cdot\nabla)b}(\xi,t) - \widehat{\nabla\pi}(\xi,t)
\end{aligned}
$$

and

$$
H(\xi,t) := -\widehat{(\widetilde{u}\cdot\nabla)h}(\xi,t) + \widehat{(w\cdot\nabla)b}(\xi,t) + \widehat{(\tilde{b}\cdot\nabla)w}(\xi,t) + \widehat{(h\cdot\nabla)u}(\xi,t).
$$

First of all, we note that for the divergence-free vectors $v, w \in L^\infty(0,T;L^2(\mathbb{R}^3))$,

$$
\begin{aligned}
|\widehat{(v\cdot\nabla)w}(\xi,t)| &\cong \Big|\int_{\mathbb{R}^3}\mathrm{e}^{-\mathrm{i}x\cdot\xi}\nabla\cdot(u\otimes v)\,\mathrm{d}x\Big| \\
&\lesssim |\xi|\|v\otimes w\|_{L^1} \leqslant |\xi|(||v(t)||_{L^2}^2 + ||w(t)||_{L^2}^2).
\end{aligned} \tag{2.4}
$$

Taking divergence operator to both side for the first equation in (1.1)

$$
\Delta\pi = \sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}(-\widetilde{u}_i w_j - w_i u_j + \tilde{b}_i h_j - h_i b_j + |D\widetilde{u}|^{p-2}D_{ij}(\widetilde{u}) + |Du|^{p-2}D_{ij}(u)).
$$

Hölder's and Young's inequalities with (2.4) yield

(2.5)
$$
\begin{aligned}
|\widehat{\nabla\pi}(x,t)| &\leqslant |\xi|(||D\widetilde{u}(t)|^{p-1}||_{L^1} + ||Du(t)|^{p-1}||_{L^1}) + |\xi|||\widetilde{u}(t)\otimes w(t)||_{L^1}\\
&\quad + |\xi|||w(t)\otimes u(t)||_{L^1} + |\xi|||\tilde{b}(t)\otimes h(t)||_{L^1} + |\xi|||h(t)\otimes b(t)||_{L^1}\\
&\leqslant |\xi|(\|D\widetilde{u}(t)\|^{p-1}_{L^{p-1}} + \|Du(t)\|^{p-1}_{L^{p-1}}) + |\xi|||w(t)||_{L^2}(||\widetilde{u}(t)||_{L^2} + ||u(t)||_{L^2})\\
&\quad + |\xi|||h(t)||_{L^2}(||\tilde{b}(t)||_{L^2} + ||b(t)||_{L^2})\\
&\leqslant |\xi|(\|D\widetilde{u}(t)\|^{p-1}_{L^{p-1}} + \|Du(t)\|^{p-1}_{L^{p-1}}) + |\xi|||w(t)||_{L^2} + ||h(t)||_{L^2},
\end{aligned}
$$

where we use the boundedness of $||\widetilde{u}(t)||_{L^2}$, $||u(t)||_{L^2}$, $||\tilde{b}(t)||_{L^2}$ and $||b(t)||_{L^2}$. Similarly, using (2.4), we also have

$$
\begin{aligned}
&|\widehat{(\widetilde{u}\cdot\nabla)h}(\xi,t)| + |\widehat{(w\cdot\nabla)u}(\xi,t)| + |\widehat{(\tilde{b}\cdot\nabla)w}(\xi,t)| + |\widehat{(h\cdot\nabla)u}(\xi,t)|\\
&\qquad\qquad\qquad\qquad \leqslant |\xi|(||w(t)||_{L^2} + ||h(t)||_{L^2}).
\end{aligned}
$$

Inserting (2.5) into $F(\xi,t)$, we have

$$|F(\xi,t)| \leqslant C|\xi|(||w(t)||_{L^2} + ||h(t)||_{L^2}) + C|\xi|(\|D\widetilde{u}(t)\|^{p-1}_{L^{p-1}} + \|Du(t)\|^{p-1}_{L^{p-1}}).$$

Similarly, $H(\xi,t)$ is estimated by

$$|H(\xi,t)| \leqslant C|\xi|(||w(t)||_{L^2} + ||h(t)||_{L^2}).$$

It follows from (2.3) that

$$\widehat{u}(\xi,t) = \mathrm{e}^{-|\xi|^2 t}\widehat{u}_0(\xi) + \int_0^t F(\xi,t)\mathrm{e}^{-|\xi|^2(t-s)}\,\mathrm{d}s.$$

From (2.3), integrating in time, we get

$$
\begin{aligned}
|\widehat{w}(\xi,t)| &= \left|\mathrm{e}^{-|\xi|^2 t}\widehat{w}_0(\xi) + \int_0^t F(\xi,t)\mathrm{e}^{-|\xi|^2(t-s)}\,\mathrm{d}s\right|\\
&\leqslant C|\mathrm{e}^{-|\xi|^2 t}\widehat{w}_0(\xi)| + \int_0^t \mathrm{e}^{-|\xi|^2(t-s)}|F(\xi,t)|\,\mathrm{d}s\\
&\leqslant C|\mathrm{e}^{-|\xi|^2 t}\widehat{w}_0(\xi)| + C\int_0^t \mathrm{e}^{-|\xi|^2(t-s)}|\xi|(\|w(s)\|_{L^2(\mathbb{R}^3)} + \|h(s)\|_{L^2(\mathbb{R}^3)})\,\mathrm{d}s\\
&\quad + C|\xi|\int_0^t (\|\nabla\widetilde{u}(s)\|^{p-1}_{L^{p-1}(\mathbb{R}^3)} + \|\nabla u(s)\|^{p-1}_{L^{p-1}(\mathbb{R}^3)})\,\mathrm{d}s\\
&\leqslant C|\mathrm{e}^{-|\xi|^2 t}\widehat{w}_0(\xi)| + |\xi|\int_0^t (\|w(s)\|_{L^2} + \|h(s)\|_{L^2})\,\mathrm{d}s\\
&\quad + C|\xi|\biggl(\int_0^t \|\widetilde{u}(s)\|^{(14-2p)/(19-5p)}_{L^2(\mathbb{R}^3)}\,\mathrm{d}s\biggr)^{(19-5p)/8}\\
&\quad + C|\xi|\biggl(\int_0^t \|u(s)\|^{(14-2p)/(19-5p)}_{L^2(\mathbb{R}^3)}\,\mathrm{d}s\biggr)^{(19-5p)/8},
\end{aligned}
$$

where we use Korn's inequality and the following inequality in the last inequality: for $\frac{11}{5} \leqslant p < 3$

$$\begin{aligned}\int_0^t \|\nabla z(s)\|_{L^{p-1}(\mathbb{R}^3)}^{p-1} \,\mathrm{d}s &\leqslant \int_0^t \|z(s)\|_{L^2(\mathbb{R}^3)}^{(7-p)/4} \|\nabla^2 z(s)\|_{L^2(\mathbb{R}^3)}^{(5p-11)/4} \,\mathrm{d}s \\ &\leqslant C\Big(\int_0^t \|z(s)\|^{(14-2p)/(19-5p)} \,\mathrm{d}s\Big)^{(19-5p)/8} \\ &\quad \times \Big(\int_0^\infty \|\nabla^2 z(t)\|_{L^2(\mathbb{R}^3)}^2 \,\mathrm{d}t\Big)^{(5p-11)/8} \\ &\leqslant C\Big(\int_0^t \|z(s)\|_{L^2(\mathbb{R}^3)}^{(14-2p)/(19-5p)} \,\mathrm{d}s\Big)^{(19-5p)/8},\end{aligned} \tag{2.6}$$

and for $p \geqslant 3$

$$\begin{aligned}\int_0^t \|\nabla z(s)\|_{L^{p-1}(\mathbb{R}^3)}^{p-1} \,\mathrm{d}s &\leqslant C\int_0^t \|\nabla z(s)\|_{L^2}^{2/(p-2)} \|\nabla z\|_{L^p}^{p(p-3)/(p-2)} \,\mathrm{d}s \\ &\leqslant C\|\nabla z\|_{L^2((0,t);L^2)}^{2/(p-2)} \|\nabla z\|_{L^q((0,t);L^q)}^{p(p-3)/(p-2)} < \infty.\end{aligned} \tag{2.7}$$

For (2.6)–(2.7), we use standard Gagliardo-Nirenberg inequality and the boundednes $\nabla^2 u \in L^2(0,T;L^2(\mathbb{R}^3))$. Similarly, we show that

$$|\hat{h}(\xi,t)| \leqslant C\Big(|\mathrm{e}^{-|\xi|^2 t}\hat{h}_0(\xi)| + |\xi|\int_0^t (\|w(s)\|_{L^2} + \|h(s)\|_{L^2}) \,\mathrm{d}s\Big),$$

where we use the assumptions and thus, this completes the proof. □

2.1. Upper bounds. To obtain optimal upper bound of convergence rate of strong solutions of the system, taking the L^2-inner product of (2.1) with w and h, respectively, it yields that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|(w,h)\|_{L^2(\mathbb{R}^3)}^2 + \mu_0\|\nabla(w,h)\|_{L^2(\mathbb{R}^3)}^2 \leqslant\ & -\int_{\mathbb{R}^3} (w\cdot\nabla u)w \,\mathrm{d}x - \int_{\mathbb{R}^3} (w\cdot\nabla b)h \,\mathrm{d}x \\ & + \int_{\mathbb{R}^3} (h\cdot\nabla b)w \,\mathrm{d}x + \int_{\mathbb{R}^3} (h\cdot\nabla u)h \,\mathrm{d}x \\ :=\ & \sum_{i=1}^4 \mathcal{J}_i,\end{aligned}$$

where we use

$$\int_{\mathbb{R}^3} (|\nabla u|^{p-2}\nabla u)\nabla w - (|\nabla\widetilde{u}|^{p-2}\nabla\widetilde{u})\nabla w \,\mathrm{d}x \leqslant 0,$$

and

$$-\int_{\mathbb{R}^3} (\tilde{b}\cdot\nabla h)\cdot w \,\mathrm{d}x - \int_{\mathbb{R}^3} (\tilde{b}\cdot\nabla w)\cdot h \,\mathrm{d}x = 0.$$

For $\mathcal{J}_1$ and $\mathcal{J}_2$,

$$\mathcal{J}_1 \leqslant \|w|_{L^4}^2\|\nabla u\|_{L^2} \leqslant \|w|_{L^2}^2\|\nabla u\|_{L^2}^2 + \frac{1}{16}\|\nabla w\|_{L^2}^2$$

and

$$\mathcal{J}_2 \leqslant \|w\|_{L^4}\|h\|_{L^4}\|\nabla b\|_{L^2} \leqslant \|(w,h)|_{L^2}^2\|\nabla b\|_{L^2}^2 + \frac{1}{16}\|(\nabla w, \nabla h)\|_{L^2}^2.$$

Similarly, we can see

$$\mathcal{J}_3 + \mathcal{J}_4 \leqslant \|(w,h)\|_{L^2}^2(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \frac{1}{16}\|(\nabla w, \nabla h)\|_{L^2}^2.$$

Combining $\mathcal{J}_1$–$\mathcal{J}_4$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\|(w,h)\|_{L^2(\mathbb{R}^3)}^2 &+ 2\min\{C_K,\mu_0\}\|\nabla(w,h)\|_{L^2(\mathbb{R}^3)}^2 \\ &\leqslant C\|(w,h)\|_{L^2}^2\|(\nabla u, \nabla b)\|_{L^2}^2,\end{aligned} \tag{2.8}$$

where C_K is Korn's constant. For the convenience of calculation, let $\min\{C_K,\mu_0\} = \frac{1}{2}$. Multiplying both sides of (2.8) by smooth function $\varrho(t)$, where $\varrho(t)$ satisfies $\varrho(0) = 1$ and $\varrho'(t) > 0$, and integrating in time on $[0,t]$, it yields that

$$\begin{aligned}\varrho(t)\|(w,h)\|_{L^2(\mathbb{R}^3)}^2 &+ \int_0^t \varrho(s)\|\nabla(w,h)\|_{L^2(\mathbb{R}^3)}^2\,\mathrm{d}s \\ &\leqslant \|(w_0,h_0)\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \varrho'(s)\|(w,h)\|_{L^2(\mathbb{R}^3)}^2\,\mathrm{d}s \\ &\quad + C\int_0^t \varrho(s)\|(w,h)\|_{L^2}^2\|(\nabla u, \nabla b)\|_{L^2}^2\,\mathrm{d}s.\end{aligned}$$

Set $S(t) = \{\xi \in \mathbb{R}^n\colon\ \varrho(t)|\xi|^2 \leqslant \varrho'(t)\}$. Then

$$\begin{aligned}\int_0^t \varrho(s)\int_{\mathbb{R}^3}|\xi|^2|(\widehat{w},\hat{h})(\xi,s)|^2\,\mathrm{d}\xi\,\mathrm{d}s &\geqslant \int_0^t \varrho(s)\int_{S(s)^c}|\xi|^2|(\widehat{w},\hat{h})(\xi,s)|^2\,\mathrm{d}\xi\,\mathrm{d}s \\ &\geqslant \int_0^t \varrho'(s)\int_{\mathbb{R}^3}|(\widehat{w},\hat{h})(\xi,s)|^2\,\mathrm{d}\xi\,\mathrm{d}s \\ &\quad - \int_0^t \varrho'(s)\int_{S(t)}|(\widehat{w},\hat{h})(\xi,s)|^2\,\mathrm{d}\xi\,\mathrm{d}s.\end{aligned}$$

Applying Plancherel's theorem,

$$\begin{aligned}\varrho(t)\|(w,h)\|_{L^2(\mathbb{R}^3)}^2 &\leqslant \|(w_0,h_0)\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \varrho'(s)\int_{S(s)}|(\widehat{w},\hat{h})(\xi,s)|^2\,\mathrm{d}\xi\,\mathrm{d}s \\ &\quad + C\int_0^t \varrho(s)\|(w,h)\|_{L^2}^2\|(\nabla u, \nabla b)\|_{L^2}^2\,\mathrm{d}s.\end{aligned}$$

Set $\varrho(t) = (1+t)^{4+\gamma}$. From Lemma 2.4 with Young's inequality and the energy estimate, we have

(2.9)

$$
\begin{aligned}
&(1+t)^{4+\gamma}\int_{\mathbb{R}^3}|(\widehat{w},\hat{h})(\xi,t)|^2\,\mathrm{d}\xi\\
&\quad\leqslant \int_{\mathbb{R}^3}|(\widehat{w}_0,\hat{h}_0)(\xi)|^2\,\mathrm{d}\xi + C\int_0^t(1+s)^{3+\gamma}\int_{S(s)}\mathrm{e}^{-|\xi|^2t}|(\widehat{w}_0,\hat{h}_0)(\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}s\\
&\qquad + C\int_0^t(1+s)^{3+\gamma}\int_{S(s)}|\xi|^2\Big(\int_0^s(\|(w,h)(\tau)\|_{L^2}\,\mathrm{d}\tau\Big)^2\,\mathrm{d}\xi\,\mathrm{d}s\\
&\qquad + C\int_0^t(1+s)^{3+\gamma}\int_{S(s)}|\xi|^2\bigg[\Big(\int_0^s\|\widetilde{u}(\tau)\|_{L^2(\mathbb{R}^3)}^{(14-2p)/(19-5p)}\,\mathrm{d}\tau\Big)^{(19-5p)/8}\\
&\qquad + \Big(\int_0^s\|u(\tau)\|_{L^2(\mathbb{R}^3)}^{(14-2p)/(19-5p)}\,\mathrm{d}\tau\Big)^{(19-5p)/8}\bigg]^2\,\mathrm{d}\xi\,\mathrm{d}s\\
&\qquad + C\int_0^t(1+s)^{4+\gamma}\|(w,h)\|_{L^2}^2\,\mathrm{d}s.
\end{aligned}
$$

Using the estimate

$$\int_{S(s)}|\xi|^2\,\mathrm{d}\xi \leqslant (1+s)^{-5/2}$$

and the relation

$$\frac{2(14-2p)}{(19-5p)} > 2,$$

we have

$$
\begin{aligned}
&(1+t)^{4+\gamma}\int_{\mathbb{R}^3}|(\widehat{w},\hat{h})(\xi,t)|^2\,\mathrm{d}\xi\\
&\quad= \int_{\mathbb{R}^n}|(\widehat{w}_0,\hat{h}_0)(\xi)|^2\,\mathrm{d}\xi + C\int_0^t(1+s)^{3+\gamma}\int_{S(s)}|\mathrm{e}^{-|\xi|^2t}(\widehat{w}_0,\hat{h}_0)(\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}s\\
&\qquad + C\Big(\int_0^t(1+s)^{3+\gamma}s^2\int_{S(s)}|\xi|^2\,\mathrm{d}\xi\,\mathrm{d}s\Big)\\
&\qquad + C\int_0^t(1+s)^{3+\gamma}\int_{S(s)}|\xi|^2\Big(s\int_0^s\|u(\tau)\|_{L^2(\mathbb{R}^3)}^{2(14-2p)/(19-5p)}\,\mathrm{d}\tau + C\Big)\,\mathrm{d}\xi\,\mathrm{d}s\\
&\quad= C + C\int_0^t(1+s)^{3+\gamma}\int_{S(s)}|\xi|^2\,\mathrm{d}\xi\,\mathrm{d}s + C\Big(\int_0^t(1+s)^{3+\gamma}s^2\int_{S(s)}|\xi|^2\,\mathrm{d}\xi\,\mathrm{d}s\Big)\\
&\qquad + C\int_0^t(1+s)^{4+\gamma}\|(w,h)\|_{L^2}^2\,\mathrm{d}s\\
&\quad\leqslant C + C(1+t)^4 + C(1+t)^{7/2+\gamma}\\
&\qquad + C\int_0^t(1+s)^{4+\gamma}\|(w,h)\|_{L^2}^2\|(\nabla u,\nabla b)\|_{L^2}^2\,\mathrm{d}s,
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
&(1+t)^{4+\gamma}\int_{\mathbb{R}^3}|(\widehat{w},\hat{h})(\xi,t)|^2\,\mathrm{d}\xi\\
&\qquad\leqslant C(1+t)^{7/2+\gamma}+C\int_0^t(1+s)^{4+\gamma}\|(w,h)\|_{L^2}^2\|(\nabla u,\nabla b)\|_{L^2}^2\,\mathrm{d}s.
\end{aligned}
$$

Applying Gronwall's inequality, we immediately deduce that

$$
\int_{\mathbb{R}^3}|(\widehat{w},\hat{h})(\xi,t)|^2\,\mathrm{d}\xi\leqslant C(1+t)^{-1/2}\quad\text{for large } t>1. \tag{2.10}
$$

To further improve the decay rate, we now employ the iterative methods to get the optimal decay estimates. Plugging (2.10) into (2.9) leads to

$$
\begin{aligned}
&(1+t)^{4+\gamma}\int_{\mathbb{R}^3}|(\widehat{w},\hat{h})(\xi,t)|^2\,\mathrm{d}\xi\\
&\quad\leqslant\int_{\mathbb{R}^3}|(\widehat{w}_0,\hat{h}_0)(\xi)|^2\,\mathrm{d}\xi+C\int_0^t(1+s)^{3+\gamma}\int_{S(s)}\mathrm{e}^{-|\xi|^2t}|(\widehat{w}_0,\hat{h}_0)(\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}s\\
&\qquad+C\int_0^t(1+s)^{3+\gamma}\int_{S(s)}|\xi|^2\Big(\int_0^s(1+\tau)^{-1/4}\,\mathrm{d}\tau\Big)^2\,\mathrm{d}\xi\,\mathrm{d}s\\
&\qquad+C\int_0^t(1+s)^{3+\gamma}\int_{S(s)}|\xi|^2\Big[\Big(\int_0^s\|\widetilde{u}(\tau)\|_{L^2(\mathbb{R}^3)}^{(14-2p)/(19-5p)}\,\mathrm{d}\tau\Big)^{(19-5p)/8}\\
&\qquad+\Big(\int_0^s\|u(\tau)\|_{L^2(\mathbb{R}^3)}^{(14-2p)/(19-5p)}\,\mathrm{d}\tau\Big)^{(19-5p)/8}\Big]^2\,\mathrm{d}\xi\,\mathrm{d}s\\
&\quad=C+C\int_0^t(1+s)^{3+\gamma}\int_{S(s)}|\xi|^2\,\mathrm{d}\xi\,\mathrm{d}s+C\Big(\int_0^t(1+s)^{3+\gamma}s^2\int_{S(s)}|\xi|^2\,\mathrm{d}\xi\,\mathrm{d}s\Big)\\
&\qquad+C\int_0^t(1+s)^{3+\gamma}\|(w,h)\|_{L^2}^2\|(\nabla u,\nabla b)\|_{L^2}^2\,\mathrm{d}s\\
&\quad\leqslant\ +C(1+t)^{3+\gamma}+C\int_0^t(1+s)^{4+\gamma}\|(w,h)\|_{L^2}^2\|(\nabla u,\nabla b)\|_{L^2}^2\,\mathrm{d}s.
\end{aligned}
$$

Applying Gronwall's inequality, we immediately yield that

$$
\int_{\mathbb{R}^3}|(\widehat{w},\hat{h})(\xi,t)|^2\,\mathrm{d}\xi\leqslant C(1+t)^{-1}\quad\text{for large } t>1. \tag{2.11}
$$

Again, in order to improve more the decay rate, we now employ the iterative methods to get the optimal decay estimates. Plugging (2.11) into (2.9) leads to

$$
\begin{aligned}
&(1+t)^{4+\gamma}\int_{\mathbb{R}^3}|(\widehat{w},\hat{h})(\xi,t)|^2\,\mathrm{d}\xi\\
&\quad\leqslant\int_{\mathbb{R}^3}|(\widehat{w}_0,\hat{h}_0)(\xi)|^2\,\mathrm{d}\xi+C\int_0^t(1+s)^{3+\gamma}\int_{S(s)}|\mathrm{e}^{-|\xi|^2t}(\widehat{w}_0,\hat{h}_0)(\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_0^t (1+s)^{3+\gamma}\int_{S(s)}|\xi|^2\Big(\int_0^s (1+\tau)^{-1/2}\,\mathrm{d}\tau\Big)^2\,\mathrm{d}\xi\,\mathrm{d}s\\
&+C\int_0^t (1+s)^{3+\gamma}\int_{S(s)}|\xi|^2\Big[\Big(\int_0^s \|\widetilde{u}(\tau)\|_{L^2(\mathbb{R}^3)}^{(14-2p)/(19-5p)}\,\mathrm{d}\tau\Big)^{(19-5p)/8}\\
&+\Big(\int_0^s \|u(\tau)\|_{L^2(\mathbb{R}^3)}^{(14-2p)/(19-5p)}\,\mathrm{d}\tau\Big)^{(19-5p)/8}\Big]^2\,\mathrm{d}\xi\,\mathrm{d}s\\
\leqslant\ & C+C(1+t)^{3/2}+C\int_0^t (1+s)^{3+\gamma}\|(w,h)\|_{L^2}^2\|(\nabla u,\nabla b)\|_{L^2}^2\,\mathrm{d}s.
\end{aligned}
$$

Applying Gronwall's inequality, we immediately deduce that

$$
\int_{\mathbb{R}^3}|(\widehat{w},\hat{h})(\xi,t)|^2\,\mathrm{d}\xi\leqslant C(1+t)^{-3/2}\quad\text{for large } t>1.
$$

To improve efficiency, using the previous process repeatedly, we reach for $2<\gamma<\frac{5}{2}$

$$
\int_{\mathbb{R}^3}|(\widehat{w},\hat{h})(\xi,t)|^2\,\mathrm{d}\xi\leqslant C(1+t)^{-\gamma/2}\quad\text{for large } t>1.
$$

2.2. Lower bounds. Let (w,h) be a solution of (2.1) and $\Phi(x,t)=\mathrm{e}^{\nu_0\Delta t}w_0$ and $\Psi(x,t)=\mathrm{e}^{\nu_0\Delta t}h_0$ be the solution of the heat equation. Then $\psi(x,t)=w(x,t)-\Phi(x,t)$ and $\phi(x,t)=h(x,t)-\Psi(x,t)$ satisfy the following difference system in the weak sense:

$$
\begin{cases}
\partial_t\psi-\mu_0\Delta\psi+\widetilde{u}\cdot\nabla w+w\cdot\nabla u-\tilde{b}\cdot\nabla h-h\cdot\nabla b+\nabla\pi\\
\qquad\qquad=\mu_1\nabla\cdot(|\nabla\widetilde{u}|^{p-2}\nabla\widetilde{u}-|\nabla u|^{p-2}\nabla u),\\
\partial_t\phi-\Delta\phi+\widetilde{u}\cdot\nabla h+w\cdot\nabla b-\tilde{b}\cdot\nabla w-h\cdot\nabla u=0,\\
\operatorname{div}\psi=0,\quad\text{and}\quad\operatorname{div}\phi=0,\\
\psi(x,0)=0,\quad\text{and}\quad\phi(x,0)=0.
\end{cases}\tag{2.12}
$$

In the same manner as Lemma 2.4, we obtain that, for $\frac{11}{5}\leqslant p<\infty$,

$$
\begin{aligned}
|\widehat{\psi}(\xi,t)|+|\widehat{\phi}(\xi,t)|\leqslant C\Big(&|\widehat{\psi}_0(\xi)|+|\widehat{\phi}_0(\xi)|+|\xi|\int_0^t(\|w(s)\|_{L^2(\mathbb{R}^3)}+\|h(s)\|_{L^2(\mathbb{R}^3)})\,\mathrm{d}s\\
&+|\xi|\Big(\int_0^t\|\widetilde{u}(s)\|_{L^2(\mathbb{R}^3)}^{(14-2p)/(19-5p)}\,\mathrm{d}s\Big)^{(19-5p)/8}\\
&+|\xi|\Big(\int_0^t\|u(s)\|_{L^2(\mathbb{R}^3)}^{(14-2p)/(19-5p)}\,\mathrm{d}s\Big)^{(19-5p)/8}\Big).
\end{aligned}
$$

For the proof of the estimate above, we use the uniform boundedness of (u,b):

$$
\|(u,b)\|_{L^2(\mathbb{R}^3)}+\|(\nabla u,\nabla b)\|_{L^2(\mathbb{R}^3)}<\infty.\tag{2.13}
$$

Taking the L^2-inner product of (2.12) with ψ and ϕ, respectively, and integrating them over $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\|(\psi,\phi)\|^2_{L^2(\mathbb{R}^3)}+\mu_0\|\nabla(\psi,\phi)\|^2_{L^2(\mathbb{R}^3)}\\
&\qquad=\mu_1\int_{\mathbb{R}^3}(|\nabla u|^{p-2}\nabla u)\nabla\psi-(|\nabla\widetilde{u}|^{p-2}\nabla\widetilde{u})\nabla\psi\,\mathrm{d}x-\int_{\mathbb{R}^3}(\widetilde{u}\cdot\nabla w)\psi\,\mathrm{d}x\\
&\qquad\quad-\int_{\mathbb{R}^3}(w\cdot\nabla u)\psi\,\mathrm{d}x+\int_{\mathbb{R}^3}(\tilde{b}\cdot\nabla h)\psi\,\mathrm{d}x+\int_{\mathbb{R}^3}(h\cdot\nabla b)\psi\,\mathrm{d}x\\
&\qquad\quad-\int_{\mathbb{R}^3}(\widetilde{u}\cdot\nabla h)\phi\,\mathrm{d}x-\int_{\mathbb{R}^3}(w\cdot\nabla b)\phi\,\mathrm{d}x\\
&\qquad\quad+\int_{\mathbb{R}^3}(\tilde{b}\cdot\nabla w)\phi\,\mathrm{d}x+\int_{\mathbb{R}^3}(h\cdot\nabla u)\phi\,\mathrm{d}x\\
&\qquad:=\sum_{i=1}^{9}\mathcal{K}_i.
\end{aligned}$$

For $\mathcal{K}_1$, using the same proof as in [20], Equation (3.16) we know

$$\begin{aligned}
\int_{\mathbb{R}^3}&(|\nabla u|^{p-2}\nabla u)\nabla\psi-(|\nabla\widetilde{u}|^{p-2}\nabla\widetilde{u})\nabla\psi\,\mathrm{d}x\\
&\leqslant\int_{\mathbb{R}^3}(|\nabla\widetilde{u}|^{p-2}\nabla\widetilde{u})\nabla\Phi-(|\nabla u|^{p-2}\nabla u)\nabla\Phi\,\mathrm{d}x\\
&\leqslant C\|\nabla\Phi\|_{L^\infty(\mathbb{R}^3)}(\|\nabla\widetilde{u}\|^{p-1}_{L^{p-1}(\mathbb{R}^3)}+\|\nabla u\|^{p-1}_{L^{p-1}(\mathbb{R}^3)}).
\end{aligned}\tag{2.14}$$

For $\mathcal{K}_2$ and $\mathcal{K}_3$, after the integration by parts, applying Hölder's and Young's inequalities with the divergence free condition yields

$$\begin{aligned}
\mathcal{K}_2=&-\int_{\mathbb{R}^3}(\widetilde{u}\cdot\nabla w)\psi\,\mathrm{d}x=\int_{\mathbb{R}^3}(\widetilde{u}\cdot\nabla\psi)(\psi+\Phi)\,\mathrm{d}x\\
&\leqslant C\|\widetilde{u}\|^2_{L^2(\mathbb{R}^3)}\|\Phi\|^2_{L^\infty(\mathbb{R}^3)}+\frac{\mu_0}{512}\|\nabla\psi\|^2_{L^2(\mathbb{R}^3)},
\end{aligned}$$

and also we get

$$\begin{aligned}
\mathcal{K}_3=&-\int_{\mathbb{R}^3}(w\cdot\nabla u)\psi\,\mathrm{d}x=\int_{\mathbb{R}^3}(w\cdot\nabla\psi)u\,\mathrm{d}x=\int_{\mathbb{R}^3}(\psi\cdot\nabla\psi)u\,\mathrm{d}x+\int_{\mathbb{R}^3}(\Phi\cdot\nabla\psi)u\,\mathrm{d}x\\
&\leqslant\|\Phi\|_{L^\infty}\|u\|_{L^2}\|\nabla\psi\|_{L^2}+\|\psi\|^2_{L^4}\|\nabla u\|_{L^2}\\
&\leqslant\frac{\mu_0}{128}\|\nabla\psi\|^2_{L^2(\mathbb{R}^3)}+C\|u\|^2_{L^2(\mathbb{R}^3)}\|\Phi\|^2_{L^\infty(\mathbb{R}^3)}+C\|\psi\|^2_{L^2(\mathbb{R}^3)}\|\nabla u\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

That is,

$$\begin{aligned}
\mathcal{K}_2+\mathcal{K}_3\leqslant{}&C(\|u\|^2_{L^2(\mathbb{R}^3)}+\|\widetilde{u}\|^2_{L^2(\mathbb{R}^3)})\|\Phi\|^2_{L^\infty(\mathbb{R}^3)}\\
&+C\|\psi\|^2_{L^2(\mathbb{R}^3)}\|\nabla u\|^2_{L^2(\mathbb{R}^3)}+\frac{\mu_0}{64}\|\nabla\psi\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned}\tag{2.15}$$

For $\mathcal{K}_4$ and $\mathcal{K}_5$ we observe that

$$\mathcal{K}_4 + \mathcal{K}_8 = \int_{\mathbb{R}^3} (\tilde{b} \cdot \nabla \Psi)\psi \, \mathrm{d}x + \int_{\mathbb{R}^3} (\tilde{b} \cdot \nabla \Phi)\phi \, \mathrm{d}x.$$

Indeed,

$$\begin{aligned}
&\int_{\mathbb{R}^3} (\tilde{b} \cdot \nabla h)\psi \, \mathrm{d}x + \int_{\mathbb{R}^3} (\tilde{b} \cdot \nabla w)\phi \, \mathrm{d}x \\
&\qquad = \int_{\mathbb{R}^3} (\tilde{b} \cdot \nabla(\phi + \Psi))\psi \, \mathrm{d}x + \int_{\mathbb{R}^3} (\tilde{b} \cdot \nabla(\psi + \Phi))\phi \, \mathrm{d}x \\
&\qquad = \int_{\mathbb{R}^3} (\tilde{b} \cdot \nabla \phi)\psi \, \mathrm{d}x + \int_{\mathbb{R}^3} (\tilde{b} \cdot \nabla \Psi)\psi \, \mathrm{d}x \\
&\qquad\quad + \int_{\mathbb{R}^3} (\tilde{b} \cdot \nabla \psi)\phi \, \mathrm{d}x + \int_{\mathbb{R}^3} (\tilde{b} \cdot \nabla \Phi)\phi \, \mathrm{d}x \\
&\qquad = \int_{\mathbb{R}^3} (\tilde{b} \cdot \nabla \Psi)\psi \, \mathrm{d}x + \int_{\mathbb{R}^3} (\tilde{b} \cdot \nabla \Phi)\phi \, \mathrm{d}x.
\end{aligned}$$

And thus, by integration by parts, we get

$$\mathcal{K}_4 + \mathcal{K}_8 \leqslant C\|\tilde{b}\|_{L^2}^2 (\|\Psi\|_{L^\infty}^2 + \|\Phi\|_{L^\infty}^2) + \frac{\mu_0}{64}(\|\nabla\psi\|_{L^2}^2 + \|\nabla\phi\|_{L^2}^2).$$

For $\mathcal{K}_5$, after integration by parts, applying Hölder's and Young's inequalities with the divergence free condition yields

$$\begin{aligned}
\mathcal{K}_5 &= -\int_{\mathbb{R}^3} (h \cdot \nabla b)\psi \, \mathrm{d}x = \int_{\mathbb{R}^3} (h \cdot \nabla \psi) b \, \mathrm{d}x \\
&= \int_{\mathbb{R}^3} (\phi \cdot \nabla \psi) b \, \mathrm{d}x + \int_{\mathbb{R}^3} (\Psi \cdot \nabla \psi) b \, \mathrm{d}x \\
&\leqslant \|\Psi\|_{L^\infty} \|b\|_{L^2} \|\nabla\psi\|_{L^2} + \|b\|_{L^4} \|\phi\|_{L^4} \|\nabla\psi\|_{L^2} \\
&\leqslant \frac{1}{64}(\|\nabla\psi\|_{L^2}^2 + \|\nabla\phi\|_{L^2}^2) + C\|b\|_{L^2(\mathbb{R}^3)}^2 \|\Psi\|_{L^\infty(\mathbb{R}^3)}^2 \\
&\quad + C\|b\|_{L^2(\mathbb{R}^3)}^2 \|\phi\|_{L^2(\mathbb{R}^3)}^2,
\end{aligned} \tag{2.16}$$

where we use the following estimate:

$$\begin{aligned}
\|b\|_{L^4} \|\phi\|_{L^4} \|\nabla\psi\|_{L^2} &\leqslant \|b\|_{L^2}^{1/2} \|\nabla b\|_{L^2}^{3/2} \|\phi\|_{L^2}^{1/2} \|\nabla\phi\|_{L^2}^{3/2} + \frac{1}{128}\|\nabla\psi\|_{L^2}^2 \\
&\leqslant C\|b\|_{L^2}^2 \|\nabla b\|_{L^2}^6 \|\phi\|_{L^2}^2 + \frac{1}{128}(\|\nabla\psi\|_{L^2}^2 + \|\nabla\phi\|_{L^2}^2) \\
&\leqslant C\|b\|_{L^2}^2 \|\phi\|_{L^2}^2 + \frac{1}{128}(\|\nabla\psi\|_{L^2}^2 + \|\nabla\phi\|_{L^2}^2)
\end{aligned}$$

due to $\nabla b \in L^\infty(0, T; L^2(\mathbb{R}^3))$.

For $\mathcal{K}_6$ and $\mathcal{K}_7$, applying Hölder's and Young's inequalities with the divergence free condition yields

$$\begin{aligned}\mathcal{K}_6 = &-\int_{\mathbb{R}^3}(\widetilde{u}\cdot\nabla h)\phi\,\mathrm{d}x = \int_{\mathbb{R}^3}(\widetilde{u}\cdot\nabla\phi)(\phi+\Psi)\,\mathrm{d}x\\ &\leqslant C\|\widetilde{u}\|^2_{L^2(\mathbb{R}^3)}\|\Psi\|^2_{L^\infty(\mathbb{R}^3)} + \frac{\mu_0}{6}\|\nabla\phi\|^2_{L^2(\mathbb{R}^3)},\end{aligned}$$

and

$$\begin{aligned}\mathcal{K}_7 = &-\int_{\mathbb{R}^3}(w\cdot\nabla b)\phi\,\mathrm{d}x = \int_{\mathbb{R}^3}(\psi\cdot\nabla\phi)b\,dx + \int_{\mathbb{R}^3}(\Phi\cdot\nabla\phi)b\,\mathrm{d}x\\ &\leqslant \|\psi\|_{L^4}\|\nabla\phi\|_{L^2}\|b\|_{L^2} + \|\Phi\|_{L^\infty}\|b\|_{L^4}\|\nabla\phi\|_{L^2}\\ &\leqslant \frac{\mu_0}{128}(\|\nabla\phi\|^2_{L^2(\mathbb{R}^3)} + \|\nabla\psi\|^2_{L^2(\mathbb{R}^3)})\\ &\quad + C\|b\|^2_{L^2(\mathbb{R}^3)}\|\Phi\|^2_{L^\infty(\mathbb{R}^3)} + C\|\psi\|^2_{L^2(\mathbb{R}^3)}\|b\|^2_{L^2(\mathbb{R}^3)},\end{aligned}$$

that is,

$$\begin{aligned}\mathcal{K}_6+\mathcal{K}_7 \leqslant &\frac{\mu_0}{64}(\|\nabla\phi\|^2_{L^2(\mathbb{R}^3)} + \|\nabla\psi\|^2_{L^2(\mathbb{R}^3)}) + C\|b\|^2_{L^2(\mathbb{R}^3)}\|\Phi\|^2_{L^\infty(\mathbb{R}^3)}\\ &+ C\|\psi\|^2_{L^2(\mathbb{R}^3)}\|b\|^2_{L^2(\mathbb{R}^3)} + C\|\widetilde{u}\|^2_{L^2(\mathbb{R}^3)}\|\Psi\|^2_{L^\infty(\mathbb{R}^3)}.\end{aligned}\tag{2.17}$$

In the same manner as $\mathcal{K}_5$, we can see

$$\mathcal{K}_9 \leqslant \frac{\mu_0}{128}\|\nabla\phi\|^2_{L^2(\mathbb{R}^3)} + C\|u\|^2_{L^2(\mathbb{R}^3)}\|\varphi\|^2_{L^2(\mathbb{R}^3)} + C\|u\|^2_{L^2(\mathbb{R}^3)}\|\Phi\|^2_{L^\infty(\mathbb{R}^3)}.\tag{2.18}$$

Combining (2.14)–(2.18) and Lemma 2.3, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\|(\psi,\phi)\|^2_{L^2(\mathbb{R}^3)} + \mu_0\|\nabla(\psi,\phi)\|^2_{L^2(\mathbb{R}^3)}\\ &\qquad\leqslant \mu_1 C\|\nabla\Phi\|_{L^\infty(\mathbb{R}^3)}(\|\nabla\tilde{u}\|^{p-1}_{L^{p-1}(\mathbb{R}^3)} + \|\nabla u\|^{p-1}_{L^{p-1}(\mathbb{R}^3)})\\ &\qquad\quad + C\|(\Phi,\Psi)\|^2_{L^\infty(\mathbb{R}^3)} + C\|(\varphi,\phi)\|^2_{L^2(\mathbb{R}^3)}\|(\nabla u,\nabla b)\|^2_{L^2(\mathbb{R}^3)}\\ &\qquad\leqslant C(1+t)^{-5/2-\gamma}(\|\nabla\widetilde{u}\|^{p-1}_{L^{p-1}(\mathbb{R}^3)} + \|\nabla u\|^{p-1}_{L^{p-1}(\mathbb{R}^3)})\\ &\qquad\quad + C(1+t)^{-3/2-\gamma} + C\|(\varphi,\phi)\|^2_{L^2(\mathbb{R}^3)}\|(\nabla u,\nabla b)\|^2_{L^2(\mathbb{R}^3)},\end{aligned}\tag{2.19}$$

where we use (2.13) and

$$\|(\Psi,\Phi)\|_{L^\infty} \leqslant \|\nabla^{3/2}(\Psi,\Phi)\|_{L^2} \leqslant C(1+t)^{-3/2-\gamma}.$$

To obtain the lower bound, as in the previous method, we let $\varphi(t) = (1+t)^{4+\gamma}$ with $2 < \gamma < \frac{5}{2}$, we rewrite (2.19) as

$$
\begin{aligned}
&(1+t)^{4+\gamma}\|(\psi,\phi)(t)\|^2_{L^2(\mathbb{R}^3)} \\
&\quad\leqslant C\int_0^t (1+s)^{3+\gamma}\int_{S(s)} |(\widehat{w}_0,\hat{h}_0)(\xi)|^2 \,\mathrm{d}\xi\,\mathrm{d}s \\
&\qquad + C\int_0^t (1+s)^{3+\gamma}\int_{S(s)} |\xi|^2\bigg(\int_0^s (\|w(\tau)\|_{L^2(\mathbb{R}^3)} + \|h(\tau)\|_{L^2(\mathbb{R}^3)})\,\mathrm{d}\tau\bigg)^2 \mathrm{d}\xi\,\mathrm{d}s \\
&\qquad + C\int_0^t (1+s)^{3+\gamma}\int_{S(s)} |\xi|^2\bigg[\bigg(\int_0^s \|\widetilde{u}(\tau)\|^{(14-2p)/(19-5p)}_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau\bigg)^{(19-5p)/8} \\
&\qquad + \bigg(\int_0^s \|u(\tau)\|^{(14-2p)/(19-5p)}_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau\bigg)^{(19-5p)/8}\bigg]^2 \mathrm{d}\xi\,\mathrm{d}s \\
&\qquad + C\int_0^t (1+s)^{4+\gamma}(1+s)^{-5/2-\gamma}(\|\nabla\tilde{u}\|^{p-1}_{L^{p-1}} + \|\nabla u\|^{p-1}_{L^{p-1}})\,\mathrm{d}s \\
&\qquad + C\int_0^t (1+s)^{4+\gamma}(1+s)^{-3/2-\gamma}\,\mathrm{d}s \\
&\qquad + C\int_0^t (1+s)^{4+\gamma}\|(\varphi,\phi)\|^2_{L^2(\mathbb{R}^3)}\|(\nabla u,\nabla b)\|^2_{L^2(\mathbb{R}^3)}\,\mathrm{d}s \\
&\quad\leqslant C + C\int_0^t (1+s)^{3+\gamma}\int_{S(s)} |\xi|^2\,\mathrm{d}\xi\,\mathrm{d}s + C\bigg(\int_0^t (1+s)^{3+\gamma}s^{1/2}\int_{S(s)} |\xi|^2\,\mathrm{d}\xi\,\mathrm{d}s\bigg) \\
&\qquad + C\int_0^t (1+s)^{3/4}(\|\nabla\tilde{u}\|^{p-1}_{L^{p-1}} + \|\nabla u\|^{p-1}_{L^{p-1}})\,\mathrm{d}s + C\int_0^t (1+s)^{5/2}\,\mathrm{d}s \\
&\qquad + C\int_0^t (1+s)^{4+\gamma}\|(\varphi,\phi)\|^2_{L^2(\mathbb{R}^3)}\|(\nabla u,\nabla b)\|^2_{L^2(\mathbb{R}^3)}\,\mathrm{d}s \\
&\quad\leqslant C(1+s)^{3/2+\gamma} + C\int_0^t (1+s)^{4+\gamma}\|(\varphi,\phi)\|^2_{L^2(\mathbb{R}^3)}\|(\nabla u,\nabla b)\|^2_{L^2(\mathbb{R}^3)}\,\mathrm{d}s,
\end{aligned}
$$

where we use

$$
\begin{aligned}
\int_0^t (1+t)^{3/2}\|\nabla z\|^{p-1}_{L^{p-1}}\,\mathrm{d}\tau &\leqslant (1+t)^{3/2}\int_0^t \|\nabla z(\tau)\|^{p-1}_{L^{p-1}}\,\mathrm{d}\tau \\
&\leqslant C(1+t)^{3/2}\bigg(\int_0^t \|z(\tau)\|^{2(14-2p)/(19-5p)}_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau\bigg)^{(19-5p)/4} + C(1+t)^{3/4} \\
&\leqslant C(1+t)^{3/2}\bigg(\int_0^t \|z(\tau)\|^{2(14-2p)/(19-5p)}_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau\bigg) + C(1+t)^{3/2} \\
&\leqslant C(1+t)^{\frac{3}{2}}\bigg(\int_0^t \|z(\tau)\|^2_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau\bigg) + C(1+t)^{3/2} \\
&\leqslant C(1+t)^{\frac{3}{2}}\bigg(\int_0^t (1+\tau)^{-\gamma}\,\mathrm{d}\tau\bigg) + C(1+t)^{3/2} \leqslant C(1+t)^{3/2}.
\end{aligned}
$$

Applying Gronwall's inequality, we obtain

$$\|(\psi,\phi)\|^2_{L^2(\mathbb{R}^3)} \leqslant C(1+t)^{-5/4}.$$

Due to Lemma 2.3, we know

$$\|\Phi(t)\| \geqslant C_1(1+t)^{-\gamma/2}, \quad \text{and} \quad \|\Psi(t)\| \geqslant C_1(1+t)^{-\gamma/2}. \tag{2.20}$$

Using (2.20) and the triangle inequality, we get that

$$\begin{aligned}\|(w,h)\|^2_{L^2(\mathbb{R}^3)} &= \|w\|^2_{L^2(\mathbb{R}^3)} + \|h\|^2_{L^2(\mathbb{R}^3)} \\ &\geqslant (\|\Phi\|^2_{L^2(\mathbb{R}^3)} - \|\psi\|^2_{L^2(\mathbb{R}^3)}) + (\|\Psi\|^2_{L^2(\mathbb{R}^3)} - \|\phi\|^2_{L^2(\mathbb{R}^3)}) \geqslant C(1+t)^{-\gamma/2}.\end{aligned}$$

This completes the proof for the upper and lower bound of the convergence rate through Subsections 2.1 and 2.2, and thus we finish the proof of Theorem of 1.1. □

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