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ON WSQ-PRIMARY IDEALS

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Abstract. We introduce weakly strongly quasi-primary (briefly, wsq-primary) ideals in commutative rings. Let R be a commutative ring with a nonzero identity and Q a proper ideal of R. The proper ideal Q is said to be a weakly strongly quasi-primary ideal if whenever $0 \neq ab \in Q$ for some $a, b \in R$, then $a^2 \in Q$ or $b \in \sqrt{Q}$. Many examples and properties of wsq-primary ideals are given. Also, we characterize nonlocal Noetherian von Neumann regular rings, fields, nonlocal rings over which every proper ideal is wsq-primary, and zero dimensional rings over which every proper ideal is wsq-primary. Finally, we study finite union of wsq-primary ideals.

Keywords: primary ideal; weakly primary ideal; quasi-primary ideal; weakly 2-prime ideal; strongly quasi-primary ideal

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1. Introduction

Throughout this article, we focus only on commutative rings with a nonzero identity and nonzero unital modules. Let R always represent such a ring and let M represent such an R-module. The concept of prime ideals and its generalizations have a distinguished place in Commutative Algebra since not only they are used in characterizing certain class of rings but also they have some applications to other areas such as general topology, algebraic geometry, graph theory etc., see for example [6], [9], [19] and [23]. Recall from [5] ([3]) that a proper ideal Q of R is said to be a *prime* (*weakly prime*) *ideal* if whenever $ab \in Q$ $(0 \neq ab \in Q)$ for some $a, b \in R$, then either $a \in Q$ or $b \in Q$. Also, a proper ideal Q of R is said to be a *primary* (*weakly primary*) *ideal* if whenever $ab \in Q$ $(0 \neq ab \in Q)$ for some $a, b \in R$, then either $a \in Q$ or $b \in \sqrt{Q}$, where $\sqrt{Q}$ denotes the radical of the ideal Q, see [5], ([12]). In 2016, Beddani and Messirdi introduced 2-prime ideals and they used it to characterize valuation domains. Let R be an integral domain and k its quotient field, R is said to be a *valuation domain* if for each $x \in k$, either $x \in R$ or $x^{-1} \in R$, see [16]. Recall from [9] ([14]) that a proper ideal Q of R is said to be a *2-prime ideal* (*weakly 2-prime ideal*) if $ab \in Q$ ($0 \neq ab \in Q$) for some $a, b \in R$, then $a^2 \in Q$ or $b^2 \in Q$. The authors in [9] showed that an integral domain R is a valuation domain if and only if every proper ideal Q of R is a 2-prime ideal. Afterwards, Koç et al. defined strongly quasi-primary ideals which is a generalization of 2-prime ideals and they characterized divided domains in terms of strongly quasi-primary ideals. Recall from [15] that a proper ideal Q of R is said to be a *strongly quasi-primary ideal* if whenever $ab \in Q$ for some $a, b \in R$, then $a^2 \in Q$ or $b \in \sqrt{Q}$. Our aim in this article is to introduce and study weakly strongly quasi-primary ideals. For the sake of completeness, now we will give some notions and notations which will be frequently used in the sequel. Let M be an R-module, N a submodule of M and K a nonempty subset of M. The residual of N by K is denoted by $(N : K) = \{a \in R\colon\ aK \subseteq N\}$. In particular, if $N = 0$, we will use $\operatorname{ann}(K)$ instead of $(0 : K)$. Also, for every ideal Q of R and every nonempty subset J of R, the residual of Q by J is defined as $(Q : J) = \{a \in R\colon\ aJ \subseteq Q\}$. We denote the set of all elements $x \in R$ such that $xy \in Q$ for some $y \notin Q$ by $\mathrm{Zd}_R(Q)$. Particularly, the set of all zero divisors of R is denoted by $\mathrm{zd}(R)$ instead of $\mathrm{Zd}_R(0)$, and the set of all regular elements of R is denoted by $\mathrm{reg}(R) = R - \mathrm{zd}(R)$. A proper ideal Q of R is said to be a *weakly strongly quasi-primary* (briefly, *wsq-primary*) *ideal* if whenever $0 \neq ab \in Q$ for some $a, b \in R$, then $a^2 \in Q$ or $b \in \sqrt{Q}$. Among other things, in this paper, we compare wsq-primary ideals and other classical ideals such as strongly quasi-primary ideals, weakly primary ideals, weakly 2-prime ideals and weakly semi-primary ideals, see, Propositions 2.1–2.4 and Examples 2.1–2.3. Also, we investigate the stability of wsq-primary ideals under intersection, under homomorphism, in factor rings, in localization of rings, in cartesian product of rings, in trivial extension of an R-module M, see Propositions 2.5–2.10, Lemma 2.1 and Theorem 2.4. Furthermore, we characterize nonlocal Noetherian von Neumann regular rings, fields, nonlocal rings over which every proper ideal is wsq-primary, and zero dimensional rings over which every proper ideal is wsq-primary, see Theorems 2.2, 2.3, 2.6, Corollary 2.2. Finally, from Lemma 2.2 to Theorem 2.9, we study the finite union of strongly quasi-primary ideals.

2. wsq-primary ideals of commutative rings

Definition 2.1. Let R be a commutative ring. A proper ideal Q of R is called a *weakly strongly quasi-primary* if whenever $a, b \in R$ and $0 \neq ab \in Q$, then either $a^2 \in Q$ or $b \in \sqrt{Q}$.

Recall from [7] that a proper ideal Q of R is said to be a *weakly semi-primary* if whenever $0 \neq ab \in Q$ for some $a, b \in R$, then $a \in \sqrt{Q}$ or $b \in \sqrt{Q}$. Also, Q is said to be a *weakly 2-absorbing primary ideal* if whenever $0 \neq abc \in Q$ for some $a, b, c \in R$, then $ab \in Q$ or $ac \in \sqrt{Q}$ or $bc \in \sqrt{Q}$, see [8]. Note that every weakly semi-primary ideal is also a weakly 2-absorbing primary ideal. However, the converse is not true in general. For instance, let $R = \mathbb{Z}$ and $Q = 6\mathbb{Z}$, namely, principal ideal of $\mathbb{Z}$ generated by 6. Then Q is a weakly 2-absorbing primary ideal which is not a weakly semi-primary ideal.

Proposition 2.1. *Let R be a ring and Q be a proper ideal of R. The following statements hold.*

(1) *If Q is a strongly quasi-primary ideal, then Q is a wsq-primary ideal.*
(2) *If Q is a weakly primary ideal, then Q is a wsq-primary ideal.*
(3) *If Q is a weakly 2-prime ideal, then Q is a wsq-primary ideal.*
(4) *If Q is a wsq-primary ideal, then Q is a weakly semi-primary ideal. In particular, every wsq-primary ideal is a weakly 2-absorbing primary ideal.*
(5) *If Q is a wsq-primary ideal of a reduced ring R, then $\sqrt{Q}$ is a weakly prime ideal of R. In particular, if Q is a nonzero wsq-primary ideal of a reduced ring R, then $\sqrt{Q}$ is a prime ideal of R.*

Proof. (1), (2) and (3) These are trivial.

(4) Let $0 \neq ab \in Q$ for some $a, b \in R$. As Q is a wsq-primary ideal of R, then $a^2 \in Q$ or $b \in \sqrt{Q}$. Thus, $a \in \sqrt{Q}$ or $b \in \sqrt{Q}$. Consequently, Q is a weakly semi-primary ideal of R. The rest is clear.

(5) Let $a, b \in R$ such that $0 \neq ab \in \sqrt{Q}$. Then there is $n \in \mathbb{N}$ such that $(ab)^n \in Q$. Since R is reduced, note that $(ab)^n \neq 0$. This implies that either $(a^n)^2 \in Q$ or $(b^n)^m \in Q$ for some $m \in \mathbb{N}$. Then $a \in \sqrt{Q}$ or $b \in \sqrt{Q}$, so that $\sqrt{Q}$ is a weakly prime ideal of R. Now, suppose that Q is a nonzero wsq-primary ideal and R is reduced. Then $\sqrt{Q}$ is a weakly prime ideal of R. If $\sqrt{Q}$ is not a prime ideal of R by Theorem 1 of [3], $\sqrt{Q}^2 = (0)$, which implies that $\sqrt{Q} = (0) = Q$, which is a contradiction. □

Recall from [10] that a ring R is said to be a *UN-ring* if every nonunit element $a \in R$ is a product of a unit and a nilpotent. Note that a commutative ring R is a UN-ring if and only if $\sqrt{0}$ is a maximal ideal of R.

Remark 2.1. If R is a UN-ring with $\sqrt{0}^2 = (0)$, then the class of wsq-primary ideals and all the other classes of ideals mentioned in Proposition 2.1 coincide.

Proposition 2.2. *Let R be a ring and Q be a proper ideal of R. If $\sqrt{Q}^2 \subseteq Q$, then Q is a weakly 2-prime ideal if and only if Q is a wsq-primary ideal if and only if Q is a weakly semi-primary ideal.*

P r o o f. Suppose that Q is a weakly semi-primary ideal of R and let $0 \neq ab \in Q$ for some $a, b \in R$. Then $a \in \sqrt{Q}$ or $b \in \sqrt{Q}$. As $\sqrt{Q}^2 \subseteq Q$, we conclude that $a^2 \in Q$ or $b^2 \in Q$. Thus, Q is a weakly 2-prime ideal of R. The other implications follow from Proposition 2.1. $\square$

The following examples show that the concept of wsq-primary ideals and the other concepts of ideals are totally different.

Example 2.1. Consider the zero ideal $Q := (\overline{0})$ of $R := \mathbb{Z}/12\mathbb{Z}$. It is clear that Q is a wsq-primary ideal. However, Q is not strongly quasi-primary ideal. Indeed, we have $\overline{3}.\overline{4} \in Q$, but neither $\overline{3}^2 \in Q$ nor $\overline{4} \in \sqrt{Q}$. Here $\overline{x}$ means $x + 12\mathbb{Z}$, that is, the class of x modulo 12 for any integer $x \in \mathbb{Z}$.

Example 2.2. Let $R = k[X, Y]/I$, where k is a field and $I = (X^2)$. Assume that $Q = (\overline{xy}, \overline{x}^2)$, where $\overline{a} = a + I$ for some $a \in k[X, Y]$. Then note that $\sqrt{Q} = (\overline{x})$ is a prime ideal of R. Since $\overline{0} \neq \overline{xy} \in Q$, $\overline{x} \notin Q$ and $\overline{y} \notin \sqrt{Q}$ it follows that Q is not a weakly primary ideal. Let $\overline{0} \neq \overline{fg} \in Q \subseteq \sqrt{Q} = (\overline{x})$ for some $f, g \in k[X, Y]$. Then $X \mid fg$ so that $X \mid f$ or $X \mid g$. This implies that $X^2 \mid f^2$ or $X^2 \mid g^2$. Then we have $\overline{f}^2 \in Q$ or $\overline{g} \in \sqrt{Q}$. Thus, Q is a wsq-primary ideal of R.

Example 2.3. Set $R = k[X, Y]$, where X and Y are indeterminates and k is a field. By Example 2.2 of [15], $Q = (X^3, XY, Y^3)$ is a strongly quasi-primary ideal of R, and so wsq-primary. Moreover, $0 \neq XY \in Q$ but neither $X^2 \in Q$ nor $Y^2 \in Q$. Thus, Q is not weakly 2-prime of R.

Proposition 2.3. *Let R be a ring and Q be a wsq-primary ideal of R. If Q is not strongly quasi-primary, then $Q^2 = (0)$ and consequently $\sqrt{Q} = \sqrt{0}$.*

P r o o f. Suppose that $Q^2 \neq (0)$, we show that Q is strongly quasi-primary. Let $ab \in Q$ for some a, $b \in R$ such that $a^2 \notin Q$. If $ab \neq 0$, then $b \in \sqrt{Q}$. So suppose that $ab = 0$. If $aQ \neq (0)$, then there is $q \in Q$ such that $aq \neq 0$, so $0 \neq a(b + q) = aq \in Q$, then $b + q \in \sqrt{Q}$, and then $b \in \sqrt{Q}$. If $bQ \neq (0)$, then there is $q' \in Q$ such that $bq' \neq 0$, so $0 \neq (a + q')b = bq' \in Q$, since $(a + q')^2 \notin Q$, and then $b \in \sqrt{Q}$. So assume that $aQ = bQ = (0)$. Since $Q^2 \neq (0)$, there exists c, $d \in Q$ such that $cd \neq 0$. Then $0 \neq (a + c)(b + d) = cd \in Q$, since $(a + c)^2 \notin Q$, then $b + d \in \sqrt{Q}$, and so $b \in \sqrt{Q}$. Thus, we conclude that Q is strongly quasi-primary ideal of R. Consequently, as $Q^2 = (0)$ and $\sqrt{Q} = \sqrt{Q^2}$, we conclude that $\sqrt{Q} = \sqrt{0}$. $\square$

The following example shows that a proper ideal Q of R with $Q^2 = (0)$ need not be a wsq-primary ideal of R. We have the following example.

Example 2.4. Let $R := \mathbb{Z}/12\mathbb{Z}$. Then $Q := 6\mathbb{Z}/12\mathbb{Z}$ is an ideal of R and clearly $Q^2 = (0)$. However, Q is not a wsq-primary ideal of R. Indeed, we have $0 \neq \overline{2}.\overline{3} \in Q$, but $\overline{2^2} \notin Q$ and $\overline{3^n} \notin Q$ for every integer $n \geqslant 1$.

Corollary 2.1. *Let R be a reduced ring and Q be a proper ideal R. Then Q is a wsq-primary ideal of R if and only if $Q = (0)$ or Q is a strongly quasi-primary ideal of R.*

Proposition 2.4. *Let R be a ring. Then the following statements hold:*

(1) *If Q is a radical ideal of R and Q is a wsq-primary ideal, then Q is a weakly prime ideal of R.*

(2) *If Q is a weakly primary ideal of R and Q' an ideal of R containing Q, then QQ' is a wsq-primary ideal of R.*

(3) *If Q is a weakly primary ideal of R, then Q^2 is a wsq-primary ideal of R.*

P r o o f. (1) Suppose that $Q = \sqrt{Q}$ and let $0 \neq ab \in Q$ for some $a, b \in R$. Since Q is a wsq-primary ideal of R, then $a^2 \in Q$ or $b \in \sqrt{Q}$. Hence, $a \in Q$ or $b \in Q$, and so Q is a weakly prime ideal of R.

(2) Let $0 \neq ab \in QQ' \subseteq Q$ for some $a, b \in R$. Then $a \in Q$ or $b \in \sqrt{Q}$. Since $Q \subseteq Q'$, we get $a^2 \in QQ'$ or $b \in \sqrt{Q} = \sqrt{QQ'}$. Hence, QQ' is a wsq-primary ideal of R.

(3) Follows directly from (2). $\square$

Theorem 2.1. *Let Q be a proper ideal of R. The following statements are equivalent.*

(1) *Q is a wsq-primary ideal.*

(2) *For all $x \in R$, either $(x) \subseteq (Q : x)$ or $(Q : x) \subseteq \sqrt{Q}$ or $(Q : x) \subseteq \operatorname{ann}(x)$.*

(3) *If $0 \neq xJ \subseteq Q$ for some $x \in R$ and an ideal J of R then $x^2 \in Q$ or $J \subseteq \sqrt{Q}$.*

P r o o f. Let Q be a proper ideal of R.

(1) $\Rightarrow$ (2) Suppose that Q is a wsq-primary ideal. Take $x \in R$. If $x^2 \in Q$, it is clear that $(x) \subseteq (Q : x)$. Assume $x^2 \notin Q$. Let $y \in (Q : x)$, that is, $xy \in Q$. If $0 \neq xy$, then either $x^2 \in Q$ or $y^n \in Q$ for some $n \in \mathbb{N}$. Since the first option gives us a contadiction, we conclude $y \in \sqrt{Q}$. If $0 = xy$, then $y \in \operatorname{ann}(x)$, i.e., $(Q : x) \subseteq \operatorname{ann}(x) \cup \sqrt{Q}$. This implies that $(Q : x) \subseteq \sqrt{Q}$ or $(Q : x) \subseteq \operatorname{ann}(x)$.

(2) $\Rightarrow$ (1) Choose $x, y \in R$ such that $0 \neq xy \in Q$ and $x^2 \notin Q$. Then item (2) implies that $(Q : x) \subseteq \sqrt{Q}$ or $(Q : x) \subseteq \operatorname{ann}(x)$. If $(Q : x) \subseteq \sqrt{Q}$, then $y \in (Q : x) \subseteq \sqrt{Q}$, which is desired. Let $(Q : x) \subseteq \operatorname{ann}(x)$. This means $xy = 0$, a contradiction.

(1) $\Rightarrow$ (3) Suppose that $0 \neq xJ \subseteq Q$ for some $x \in R$ and an ideal J of R. Let $x^2 \notin Q$ and $j \in J$. If $0 \neq xj \in Q$, then we have $j \in \sqrt{Q}$. Now, assume that $xj = 0$. Choose $b \in J$ such that $xb \neq 0$. Since $xb \in Q$ we have $b \in \sqrt{Q}$. On the other hand, note that $0 \neq x(b + j) \in Q$. This implies $b + j \in \sqrt{Q}$ and thus $j \in \sqrt{Q}$. Hence, $J \subseteq \sqrt{Q}$.

(3) $\Rightarrow$ (1) It is clear. $\square$

Recall from [5] that a ring R is a *local ring* if it has a unique maximal ideal. Otherwise, we say that R is a *nonlocal ring*. Also, R is said to be a *von Neumann regular ring* if every principal ideal I of R is generated by an idempotent element $e \in R$, see [21].

Theorem 2.2. *Let R be a nonlocal Noetherian ring. Then the following statements are equivalent:*
(1) *Every wsq-primary ideal of R is weakly prime.*
(2) *Every weakly 2-prime ideal of R is weakly prime.*
(3) *R is a von Neumann regular ring.*

Proof. (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) Let M be a maximal ideal of R. Clearly, M^2 is weakly 2-prime ideal of R, and so weakly prime. If M^2 is not prime, then by Theorem 1 of [3], $M^4 = (0)$, and so R is a local ring with a maximal ideal $M = \sqrt{0}$, a contradiction. Then M^2 is a prime ideal of R, and so $M^2 = \sqrt{M^2} = M$. So, by Theorem 1.8.22 of [22], M is principal and is generated by an idempotent element of R. Using Theorem 3.2 of [20], we conclude that R is a von Neumann regular ring.

(3) $\Rightarrow$ (1) Let $(0) \neq Q$ be a wsq-primary ideal of R. Since R is reduced and every ideal is radical (see [2], Theorem 3.1.), then by using Corollary 2.1, $\sqrt{Q} = Q$ is a strongly quasi-primary ideal of R. Hence, by [15], Proposition 2.1 (iii), Q is a prime ideal of R, and so is a weakly prime ideal of R. $\square$

Example 2.5. Let k be a field and $R := k[X, Y]$, where X and Y are two indeterminates. Then R is a nonlocal Noetherian ring. By Example 2 of [14], $P = (X^2, XY)$ is a weakly 2-prime ideal of R, and so wsq-primary. However, P is not a weakly prime ideal of R, since $0 \neq XY \in P$ but $X \notin P$ and $Y \notin P$.

Theorem 2.3. *Let R be a ring. Then the following statements are equivalent:*
(1) *Every wsq-primary ideal of R is prime.*
(2) *R is a domain and every strongly quasi-primary ideal of R is prime.*
(3) *R is a field.*

Proof. (1) $\Rightarrow$ (2) Since (0) is a wsq-primary of R, and every strongly quasi-primary ideal of R is wsq-primary, we conclude that R is a domain and every strongly quasi-primary ideal of R is prime.

(2) $\Rightarrow$ (3) Following Proposition 2.1 of [15] and our hypothesis, it is clear that every primary ideal of R is prime. Then, by Theorem 3.1 of [1], R is von Neumann regular. Accordingly, R is a field since it is a domain.

(3) $\Rightarrow$ (1) Obvious. $\square$

Proposition 2.5. *Let R be a ring and let $\{Q_i\}_{i\in I}$ be a family of wsq-primary of R that are not strongly quasi-primary. Then $Q := \bigcap_{i\in I} Q_i$ is a wsq-primary ideal of R.*

Proof. First, it is easy to see that $\sqrt{\bigcap_{i\in I} Q_i} = \bigcap_{i\in I} \sqrt{Q_i} = \sqrt{0}$. On the other hand, let $0 \neq ab \in Q$ such that $b \notin \sqrt{Q}$ for some $a, b \in R$. Since $0 \neq ab \in Q_i$ and $b \notin \sqrt{Q_i}$, we have $a^2 \in Q_i$ (for all $i \in I$), and so $a^2 \in Q$. Thus, Q is a wsq-primary ideal of R. $\square$

Proposition 2.6. *Let $f\colon R \to R'$ be a ring homomorphism. Then the followings hold:*

(1) *If f is an epimorphism and Q is a wsq-primary ideal of R containing* $\mathrm{Ker}(f)$, *then $f(Q)$ is a wsq-primary ideal of R'.*

(2) *If f is a monomorphism and Q' is a wsq-primary ideal of R', then $f^{-1}(Q')$ is a wsq-primary ideal of R.*

Proof. (1) Let $a', b' \in R'$ and $0 \neq a'b' \in f(Q)$. Then there exist $a, b \in R$ such that $a' = f(a)$, $b' = f(b)$ and $0 \neq f(ab) = a'b' \in f(Q)$. Since $\mathrm{Ker}(f) \subseteq Q$, we have $0 \neq ab \in Q$. It implies that $a^2 \in Q$ or $b \in \sqrt{Q}$. It means that $f(a)^2 = f(a^2) = a'^2 \in f(Q)$ or $b' \in \sqrt{f(Q)}$. Thus, $f(Q)$ is a wsq-primary ideal of R'.

(2) Let $a, b \in R$ such that $0 \neq ab \in f^{-1}(Q')$. Since $\mathrm{Ker}(f) = (0)$, we get $0 \neq f(ab) = f(a)f(b) \in Q'$. Hence, we have $f(a)^2 = f(a^2) \in Q'$ or $f(b) \in \sqrt{Q'}$, and so $a^2 \in f^{-1}(Q')$ or $b \in f^{-1}(\sqrt{Q'}) = \sqrt{f^{-1}(Q')}$. Thus, we conclude that $f^{-1}(Q')$ is a wsq-primary ideal of R. $\square$

Proposition 2.7. *Let $I \subseteq Q$ be proper ideals of a ring R. Then the following statements hold:*

(1) *If Q is a wsq-primary ideal of R, then Q/I is a wsq-primary ideal of R/I.*

(2) *If Q/I is a wsq-primary ideal of R/I and I is a wsq-primary ideal of R, then Q is a wsq-primary ideal of R.*

(3) *If Q is a wsq-primary ideal and S is a subring of R with $S \nsubseteq Q$, then $S \cap Q$ is a wsq-primary ideal of S.*

Proof. (1) Applying Proposition 2.6 (1) to the canonical surjection $\pi\colon R \to R/I$, we conclude that Q/I is a wsq-primary ideal of R/I.

(2) Let $0 \neq ab \in Q$ for some $a, b \in R$. If $0 \neq ab \in I$, then $a^2 \in I \subseteq Q$ or $b \in \sqrt{I} \subseteq \sqrt{Q}$. If $ab \notin I$, then we have $0 \neq \overline{ab} = \overline{a}\overline{b} \in Q/I$, and so $\overline{a}^2 \in Q/I$ or $\overline{b} \in \sqrt{Q/I} = \sqrt{Q}/I$. It means that $a^2 \in Q$ or $b \in \sqrt{Q}$. Thus, Q is a wsq-primary ideal of R.

(3) Consider the injection $i\colon S \to R$, which is defined as $i(s) = s$ for all $s \in S$. Since i is monic and Q is a wsq-primary ideal, by Proposition 2.6 (2), $i^{-1}(Q) = Q \cap S$ is a wsq-primary ideal of S. $\square$

Lemma 2.1. *Let $R := R_1 \times R_2$ and let Q be a proper ideal of R_1. Then the following statements are equivalent:*

(1) *$Q \times R_2$ is a wsq-primary ideal of R.*

(2) *$Q \times R_2$ is a strongly quasi-primary ideal of R.*

(3) *Q is a strongly quasi-primary ideal of R_1.*

Proof. (1) $\Rightarrow$ (2) Since $Q \times R_2 \nsubseteq \sqrt{0}$, we conclude that $Q \times R_2$ is a strongly quasi-primary ideal of R by Proposition 2.3.

(2) $\Rightarrow$ (3) Let $ab \in Q$ for some $a, b \in R_1$. Then $(a,1)(b,1) \in Q \times R_2$. Since $Q \times R_2$ is a strongly quasi-primary ideal of R, then $(a,1)^2 = (a^2,1) \in Q \times R_2$ or $(b,1) \in \sqrt{Q \times R_2} = \sqrt{Q} \times R_2$. So, $a^2 \in Q$ or $b \in \sqrt{Q}$. Thus, Q is a strongly quasi-primary ideal of R_1.

(3) $\Rightarrow$ (1) Follows directly from [15], Lemma 2.1. $\square$

Theorem 2.4. *Let $R := R_1 \times R_2$ and $(0) \neq Q := Q_1 \times Q_2$, where Q_1 and Q_2 be ideals of R_1 and R_2, respectively. Then the following statements are equivalent:*

(1) *Q is a wsq-primary ideal of R.*

(2) *$Q_1 = R_1$ and Q_2 is a strongly quasi-primary ideal of R_2 or $Q_2 = R_2$ and Q_1 is a strongly quasi-primary ideal of R_1.*

(3) *Q is a strongly quasi-primary ideal of R.*

Proof. (1) $\Rightarrow$ (2) Assume that $(0) \neq Q := Q_1 \times Q_2$ is a wsq-primary ideal of R. Without loss of generality, we may assume that $Q_1 \neq (0)$. Choose $0 \neq a \in Q_1$. Then note that $(0,0) \neq (a,1)(1,0) \in Q_1 \times Q_2$. Since Q is a wsq-primary ideal of R, we have $(a,1)^2 = (a^2,1) \in Q_1 \times Q_2$ or $(1,0) \in \sqrt{Q_1 \times Q_2} = \sqrt{Q_1} \times \sqrt{Q_2}$. Thus, $1 \in Q_1$ or $1 \in Q_2$, that is, $Q_1 = R_1$ or $Q_2 = R_2$. If $Q_2 = R_2$, then by Lemma 2.1, Q_1 is a strongly quasi-primary ideal of R_1. In other case, one can similarly show that Q_2 is a strongly quasi-primary ideal of R_2.

(2) $\Rightarrow$ (3) Follows directly from [15], Lemma 2.1.

(3) $\Rightarrow$ (1) Clear. $\square$

Theorem 2.5. *Let R be a ring. If every proper ideal of R is a wsq-primary, then R has at most two maximal ideals.*

Proof. Suppose that R has at least three maximal ideals M_1, M_2 and M_3 of R. By assumption, $M_1 \cap M_2$ is a wsq-primary ideal of R. If $M_1 \cap M_2$ is a strongly quasi-primary ideal, then by [15], Proposition 2.1 (iii), $\sqrt{M_1 \cap M_2} = M_1 \cap M_2$ is a prime ideal of R, a contradiction. So, $M_1 \cap M_2$ is a wsq-primary ideal of R which is not strongly quasi-primary. Hence, by Proposition 2.3, $M_1^2 M_2^2 = (0) \subseteq M_3$, and so $M_1 \subseteq M_3$ or $M_2 \subseteq M_3$, a contradiction. Thus, R has at most two maximal ideals. $\square$

According to Theorem 2.5, the direct product of three or more rings contains always ideals which are not strongly quasi primary. Next, we characterize nonlocal and zero-dimensional rings over which every proper ideal is wsq-primary.

Theorem 2.6. *Let R be a nonlocal ring. Then every proper ideal of R is wsq-primary if and only if $R \cong k_1 \times k_2$ for some fields k_1 and k_2.*

Proof. ($\Rightarrow$) By using the hypothesis and Theorem 2.5, R has exactly two maximal ideals M_1 and M_2. Moreover, M_1M_2 is a wsq-primary ideal of R that is not strongly quasi-primary. By Proposition 2.3, we have $M_1^2 M_2^2 = (0)$, and so $R \cong R/M_1^2 \times R/M_2^2$. Now, we will show that R/M_1^2 is a field. Let I_1 be a nonzero ideal of R/M_1^2 and take P_2 a prime ideal of R/M_2^2. By assumption, $I_1 \times P_2$ is a wsq-primary of $R/M_1^2 \times R/M_2^2$. Then, by Theorem 2.4, $I_1 = R/M_1^2$, which implies that R/M_1^2 is a field. Similarly, one can show that R/M_2^2 is a field.

($\Leftarrow$) Clear. $\square$

Corollary 2.2. *Let R be a ring with $\dim(R) = 0$. Then every proper ideal of R is wsq-primary if and only if R is either*

(1) UN-*ring, or*

(2) *$R \cong k_1 \times k_2$ for some fields k_1 and k_2.*

Proof. ($\Rightarrow$) By Theorem 2.5, R has at most two maximal ideals. If R is nonlocal, then by Theorem 2.6, R is isomorphic to a direct product of two fields. Now, suppose that R is local. Then R is UN-ring since $\dim(R) = 0$.

($\Leftarrow$) If R is isomorphic to a direct product of two fields, then clearly every proper ideal of R is wsq-primary. Now, suppose that R is a UN-ring, then every proper ideal is primary, and so wsq-primary. $\square$

Proposition 2.8. *Let R be a ring and S a multiplicatively closed subset of R. Then the following assertions hold:*

(1) *If Q is a wsq-primary ideal of R with $Q \cap S = \emptyset$, then $S^{-1}Q$ is a wsq-primary ideal of $S^{-1}R$.*

(2) *If $S^{-1}Q$ is a wsq-primary ideal of $S^{-1}R$ such that $S \cap Z_Q(R) = \emptyset$ and $S \subseteq \mathrm{Reg}(R)$, then Q is a wsq-primary ideal of R.*

P r o o f. (1) Let $a, b \in R$, $s, t \in S$ such that $0 \neq (a/s)(b/t) \in S^{-1}Q$. Then there exists $u \in S$ such that $0 \neq uab \in Q$. Since Q is a wsq-primary ideal of R, then we have either $(ua)^2 = u^2a^2 \in Q$ or $b \in \sqrt{Q}$. Hence, $(a/s)^2 = a^2u^2/s^2u^2 \in S^{-1}Q$ or $b/t \in S^{-1}\sqrt{Q} = \sqrt{S^{-1}Q}$. Thus, $S^{-1}Q$ is a wsq-primary ideal of $S^{-1}R$.

(2) Let $a, b \in R$ such that $0 \neq ab \in Q$. Since $S \subseteq \operatorname{Reg}(R)$, we have $0 \neq ab/1 = (a/1)(b/1) \in S^{-1}Q$. This implies that $(a/1)^2 = a^2/1 \in S^{-1}Q$ or $b/1 \in \sqrt{S^{-1}Q} = S^{-1}\sqrt{Q}$. Then there is $u \in S$ such that $ua^2 \in Q$ or $ub \in \sqrt{Q}$. As $S \cap Z_Q(R) = \emptyset$, we get either $a^2 \in Q$ or $b \in \sqrt{Q}$. Thus, Q is a wsq-primary ideal of R. $\square$

Let R be a ring and M be an R-module. The additive group $R \times M$ with multiplication $(a, m)(b, m') = (ab, am' + bm)$ is a commutative ring with identity, denoted $R \propto M$ and called the *Nagata's idealization* or *trivial extension*, see [13]. If Q is an ideal of R and N is a submodule of M then $Q \propto N$ is an ideal of $R \propto M$ if and only if $QM \subseteq N$, see [4], Theorem 3.1.

Proposition 2.9. *Let R be a ring, M be an R-module and Q be an ideal of R. Then the following statements are equivalent:*

(1) *$Q \propto M$ is a wsq-primary ideal of $R \propto M$.*

(2) *Q is a wsq-primary ideal of R and if a, $b \in R$ with $ab = 0$, but $a^2 \notin Q$ and $b \notin \sqrt{Q}$, then $a \in \operatorname{ann}_R(M)$ and $b \in \operatorname{ann}_R(M)$.*

P r o o f. (1) $\Rightarrow$ (2) Assume that $Q \propto M$ is a wsq-primary ideal of $R \propto M$ and let $0 \neq ab \in Q$ for some a, $b \in R$. Then $0_{R\propto M} \neq (a, 0)(b, 0) \in Q \propto M$. Hence, $(a, 0)^2 = (a^2, 0) \in Q \propto M$ or $(b, 0) \in \sqrt{Q \propto M} = \sqrt{Q} \propto M$, since $Q \propto M$ is a wsq-primary ideal of $R \propto M$. Therefore, $a^2 \in Q$ or $b \in \sqrt{Q}$ and so Q is a wsq-primary ideal of R.

Now, assume that $ab = 0$ and $a^2 \notin Q$, $b \notin \sqrt{Q}$ for some a, $b \in R$. If $a \notin \operatorname{ann}_R(M)$, so there is some $m \in M$ such that $am \neq 0$. Then $0_{R\propto M} \neq (a, 0)(b, m) \in Q \propto M$ but $(a, 0)^2 \notin Q \propto M$ and $(b, m) \notin \sqrt{Q} \propto M = \sqrt{Q \propto M}$, a contradiction. Similarly, if we assume that $b \notin \operatorname{ann}_R(M)$, we get a contradiction. Thus, $a \in \operatorname{ann}_R(M)$ and $b \in \operatorname{ann}_R(M)$.

(2) $\Rightarrow$ (1) Let $0_{R\propto M} \neq (a, m)(b, m') \in Q \propto M$ for some $(a, m), (b, m') \in R \propto M$. So, $ab \in Q$. If $ab \neq 0$, we have $a^2 \in Q$ or $b \in \sqrt{Q}$ since Q is a wsq-primary ideal of R. Then $(a, m)^2 \in Q \propto M$ or $(b, m') \in \sqrt{Q} \propto M = \sqrt{Q \propto M}$, this means that $Q \propto M$ is a wsq-primary ideal of $R \propto M$. Now assume that $ab = 0$. If $a^2 \notin Q$ and $b \notin \sqrt{Q}$, then by the hypothesis, a, $b \in \operatorname{ann}_R(M)$, and so $(a, m)(b, m') = 0_{R\propto M}$, a contradiction. $\square$

Recall from [17] that an R-module M is said to be a *reduced module* if whenever $a^2m = 0$ for some $a \in R$, $m \in M$ then $am = 0$.

Proposition 2.10. *Let R be a ring, M be a reduced R-module. Suppose that Q is an ideal of R and N is a submodule of M such that $\sqrt{Q}M \subseteq N$. Then the following statements are equivalent:*

(1) *$Q \propto N$ is a wsq-primary ideal of $R \propto M$.*

(2) *Q is a wsq-primary ideal of R and if a, $b \in R$ with $ab = 0$, but $a^2 \notin Q$ and $b \notin \sqrt{Q}$ then $a \in \operatorname{ann}_R(N)$ and $b \in \operatorname{ann}_R(N)$.*

Proof. (1) $\Rightarrow$ (2) Assume that $Q \propto N$ is a wsq-primary ideal of $R \propto M$. Similar argument as in the proof of Proposition 2.9 shows that Q is a wsq-primary ideal of R and if a, $b \in R$ with $ab = 0$, but $a^2 \notin Q$ and $b \notin \sqrt{Q}$, then $a \in \operatorname{ann}_R(N)$ and $b \in \operatorname{ann}_R(N)$.

(2) $\Rightarrow$ (1) Let $0_{R\propto M} \neq (a, m)(b, m') \in Q \propto N$ for some $(a, m), (b, m') \in R \propto M$. So, $ab \in Q$. If $ab \neq 0$, we have $a^2 \in Q$ or $b \in \sqrt{Q}$ since Q is a wsq-primary ideal of R. If $a^2 \in Q$, then we have $a \in \sqrt{Q}$, which implies that $2am \in \sqrt{Q}M \subseteq N$. Then, $(a, m)^2 = (a^2, 2am) \in Q \propto N$. If $b \in \sqrt{Q}$, then we have $(b, m') \in \sqrt{Q} \propto M = \sqrt{Q \propto N}$, this means that $Q \propto N$ is a wsq-primary ideal of $R \propto M$. Now assume that $ab = 0$. If $a^2 \notin Q$ and $b \notin \sqrt{Q}$, then by the hypothesis, $a, b \in \operatorname{ann}_R(N)$. Since $am' + bm \in N$ and $ab = 0$, we conclude that $a(am' + bm) = 0$, which implies $a^2m' = 0$. As M is reduced, we have $am' = 0$. Likewise, $bm = 0$. Then we have $(a, m)(b, m') = 0_{R\propto M}$, a contradiction. $\square$

Prime avoidance theorem states that if an ideal Q of a ring R is contained in a finite union of ideals $P_1, P_2, \ldots, P_n$ of R, where at most two of P_i's are not prime, then Q must be contained in P_j for some $1 \leqslant j \leqslant n$, see [22], Theorem 1.4.3. Moreover, some authors considered the infinite prime avoidance theorem and they studied the class of rings satisfying infinite prime avoidance theorem (such rings were called compactly packed rings), see for example [11], [14] and [18]. From now on, we study finite union of strongly quasi-primary ideals and prove a result analogous to the prime avoidance theorem for strongly quasi-primary ideals. First, we need the following lemma.

Lemma 2.2. *Let $Q \subseteq \bigcup_{i=1}^{n} Q_i$ be an efficient covering, where $Q, Q_1, Q_2, \ldots, Q_n$ are ideals of R and $n \geqslant 2$. Further, assume that $Q \cap \sqrt{Q_i} \nsubseteq Q \cap \sqrt{Q_j}$ for all $i \neq j$. Then Q_j is not a strongly quasi-primary ideal of R for each $j = 1, 2, \ldots, n$.*

Proof. Suppose Q_j is a strongly quasi-primary ideal for some $j \in \{1, 2, \ldots, n\}$. First we will show that $Q \cap \Big(\bigcap_{i\neq j} Q_i\Big) \subseteq Q \cap Q_j$. Let $x \in Q \cap \Big(\bigcap_{i\neq j} Q_i\Big)$. By the efficient covering, there exists $y \in Q - \Big(\bigcup_{i\neq j} Q_i\Big)$, that is, $y \in Q_j$. Then $x + y \in Q$. This implies that $x + y \in Q_i$ for some $i \in \{1, 2, \ldots, n\}$. If $i \neq j$, then we have $(x + y) - x = y \in Q_i$

which is a contradiction. Thus, we have $x+y \in Q_j$ and so $x=(x+y)-y \in Q_j$. Then we have $Q \cap \Big(\bigcap\limits_{i \neq j} Q_i\Big) \subseteq Q \cap Q_j$. On the other hand, since $Q \cap \sqrt{Q_i} \nsubseteq Q \cap \sqrt{Q_j}$ for all $i \neq j$, there exists $y_i \in (Q \cap \sqrt{Q_i}) \setminus \sqrt{Q_j}$. Then $y_i^{k_i} \in (Q \cap Q_i) - \sqrt{Q_j}$ for each $i \neq j$. Put $k = \max\{k_i \colon\ i \neq j\}$. Then we have $\prod\limits_{i \neq j} y_i^k \in Q \cap \Big(\bigcap\limits_{i \neq j} Q_i\Big) \subseteq Q \cap Q_j \subseteq Q_j$. Since Q_j is a strongly quasi-primary ideal and $\prod\limits_{i \neq j} y_i^k \in Q_j$, we get $y_1^{2k} \in Q_j$ or $\prod\limits_{i \neq j,1} y_i^k \in \sqrt{Q_j}$. As $\sqrt{Q_j}$ is a prime ideal, we conclude that $y_i \in \sqrt{Q_j}$ for some $i \neq j$, a contradiction. Therefore, no Q_j is a strongly quasi-primary ideal of R. $\square$

Theorem 2.7 (Avoidance Theorem for strongly quasi-primary ideals)**.** *Assume that $Q_1, \ldots, Q_n$ are any ideals of R such that at least $n-2$ of them are strongly quasi-primary. Assume that $Q \subseteq \bigcup\limits_{i=1}^{n} Q_i$, where Q is an ideal of R and $Q \cap \sqrt{Q_i} \nsubseteq Q \cap \sqrt{Q_j}$ for all $i \neq j$. Then $Q \subseteq Q_j$ for some $j \in \{1, 2, \ldots, n\}$.*

P r o o f. Assume that $Q \subseteq \bigcup\limits_{i=1}^{n} Q_i$ and at least $n-2$ of Q_i's are strongly quasi-primary ideals of R. Without loss of generality, one may assume that the covering is an efficient covering. If $n=2$, then the result is obvious. Suppose that $n>2$. Since the covering is efficient and $Q \cap \sqrt{Q_i} \nsubseteq Q \cap \sqrt{Q_j}$ for all $i \neq j$, we conclude $n<2$ by Proposition 2.2. This means $n=1$. Consequently, $Q \subseteq Q_j$ for some $j \in \{1, 2, \ldots, n\}$. $\square$

In the previous theorem, the condition "$Q \cap \sqrt{Q_i} \nsubseteq Q \cap \sqrt{Q_j}$ for all $i \neq j$" is necessary. For this, we can observe from the example below that if we remove that condition, the theorem will not be satisfied.

Example 2.6. Let $R = \mathbb{Z}_2[X,Y]/(X^2, XY, Y^2)$ and $Q_1 = \{0, \overline{X}\}$, $Q_2 = \{0, \overline{Y}\}$, $Q_3 = \{0, \overline{X+Y}\}$. Consider the ideal $Q = Q_1 \cup Q_2 \cup Q_3 = \{0, \overline{X}, \overline{Y}, \overline{X+Y}\}$ of R, which is the unique maximal ideal of R. It is clear that $Q_j \subsetneq Q$ and $\sqrt{Q_j} = Q$ for each $j = 1,2,3$. So, the condition $Q \cap \sqrt{Q_i} \nsubseteq Q \cap \sqrt{Q_j}$ fails in R. Also, by Proposition 2.1 (iv) of [15], we conclude Q_j is a strongly quasi-primary ideal of R for each $j = 1,2,3$. Finally, one can see $Q \subseteq Q_1 \cup Q_2 \cup Q_3$ but $Q \nsubseteq Q_j$ for all $j \in \{1,2,3\}$.

Theorem 2.8. *Let $Q_1, \ldots, Q_n$ be strongly quasi-primary ideals of R and Q be an ideal of R such that $Q \cap \sqrt{Q_i} \nsubseteq Q \cap \sqrt{Q_j}$ for all $i \neq j$. If $rR + Q \nsubseteq \bigcup\limits_{i=1}^{n} Q_i$ for some $r \in R$, then there exists $x \in Q$ such that $r + x \notin \bigcup\limits_{i=1}^{n} Q_i$.*

P r o o f. Assume that $rR + Q \nsubseteq \bigcup\limits_{i=1}^{n} Q_i$ for some $r \in R$. Let $r \in \bigcap\limits_{i=1}^{k} Q_i$ and $r \notin \bigcup\limits_{i=k+1}^{n} Q_i$. If $k = 0$, then $r + 0 \notin \bigcup\limits_{i=1}^{n} Q_i$, as desired. Let $k \geqslant 1$. Our hypothesis $Q \cap \sqrt{Q_i} \nsubseteq Q \cap \sqrt{Q_j}$ for all $i \neq j$ implies $Q \nsubseteq \bigcup\limits_{i=1}^{k} \sqrt{Q_i}$. Then there exists $a \in Q - \bigcup\limits_{i=1}^{k} \sqrt{Q_i}$. Now, we shall show that $\bigcap\limits_{i=k+1}^{n} Q_i \nsubseteq \bigcup\limits_{i=1}^{k} \sqrt{Q_i}$. Suppose the contrary. As $\sqrt{Q_i}$ is prime, by the prime avoidence theorem, we have $\bigcap\limits_{i=k+1}^{n} Q_i \subseteq \sqrt{Q_j}$ for some $j \in \{1, 2, \ldots, k\}$. This implies $\sqrt{Q_i} \subseteq \sqrt{Q_j}$ for some $i \in \{k+1, k+2, \ldots, n\}$. Then we conclude that $Q \cap \sqrt{Q_i} \nsubseteq Q \cap \sqrt{Q_j}$, which is a contradiction. Therefore, there exists $b \in \bigcap\limits_{i=k+1}^{n} Q_i$ and $b \notin \bigcup\limits_{i=1}^{k} \sqrt{Q_i}$. Consider $x = ab \in Q$. If $r + x \in \bigcup\limits_{i=1}^{n} Q_i$, then there exists $1 \leqslant i \leqslant n$ such that $r + x \in Q_i$.

Case 1: Assume that $i \in \{1, 2, \ldots, k\}$. Since $r \in \bigcap\limits_{i=1}^{k} Q_i$, we have $(r + x) - r = x = ab \in Q_i$. As Q_i is a strongly quasi-primary ideal, we conclude that $a^2 \in Q_i$ or $b \in \sqrt{Q_i}$. Thus, we have $a \in \sqrt{Q_i} \subseteq \bigcup\limits_{i=1}^{k} \sqrt{Q_i}$ or $b \in \sqrt{Q_i} \subseteq \bigcup\limits_{i=1}^{k} \sqrt{Q_i}$, which is a contradiction.

Case 2: Assume that $i \in \{k+1, k+2, \ldots, n\}$. Since $x = ab \in \bigcap\limits_{i=k+1}^{n} Q_i$, we conclude that $(r + x) - x = r \in Q_i$ for some $i \in \{k+1, k+2, \ldots, n\}$, again a contradiction.

Therefore, $r + x \notin \bigcup\limits_{i=1}^{n} Q_i$, which completes the proof. □

Theorem 2.9. *Let $Q_1, \ldots, Q_n$ be strongly quasi-primary ideals of R and $Q = (q_1, q_2, \ldots, q_s)$ be a finitely generated ideal of R. If $Q \nsubseteq \sqrt{Q_i}$ for every $i \in \{1, 2, \ldots, n\}$ and $Q \cap \sqrt{Q_i} \nsubseteq Q \cap \sqrt{Q_j}$ for all $i \neq j$, then there exist $b_2, b_3, \ldots, b_s \in R$ such that $\alpha = q_1 + b_2 q_2 + \ldots + b_s q_s \notin \bigcup\limits_{i=1}^{n} Q_i$.*

P r o o f. We will use induction on n. If $n = 1$, it is clear. Assume that the claim is true for $n-1$. Then there are $a_2, a_3, \ldots, a_s \in R$ such that $x = q_1 + a_2 q_2 + \ldots + a_s q_s \notin \bigcup\limits_{i=1}^{n-1} Q_i$. If $x \notin Q_n$, then $x \notin \bigcup\limits_{i=1}^{n} Q_i$, which completes the proof. Suppose $x \in Q_n$. If $q_2, q_3, \ldots, q_s \in \sqrt{Q_n}$, then we have $q_1 \in \sqrt{Q_n}$. Then we conclude that $Q \subseteq \sqrt{Q_n}$, which is a contradiction. Therefore, we have $q_i \notin \sqrt{Q_n}$ for some $i \in \{2, 3, \ldots, s\}$. Without loss of generality, suppose $q_2 \notin \sqrt{Q_n}$. Moreover, $Q \cap \sqrt{Q_i} \nsubseteq Q \cap \sqrt{Q_j}$ for all $i \neq j$ implies $\sqrt{Q_i} \nsubseteq \sqrt{Q_n}$ for all $i \neq n$. Thus, there exists $y_i \in \sqrt{Q_i} - \sqrt{Q_n}$ for all $i \neq n$. Then there is $k_i \in \mathbb{N}$ such that $y_i^{k_i} \in Q_i$ for all $i \neq n$. Let $k = \max\{k_1, k_2, \ldots, k_{n-1}\}$ and $y = \prod\limits_{i=1}^{n-1} y_i$. Then $y^k \in Q_i$ for all $i \neq n$ and $y^k \notin Q_n$. Indeed, if $y^k \in Q_n$, we would have $y \in \sqrt{Q_n}$. Since $\sqrt{Q_n}$ is a prime ideal, we conclude that $y_i \in \sqrt{Q_n}$ for some $i \in \{1, 2, \ldots, n-1\}$ which is a contradiction. Let $\alpha = q_1 + (a_2 + y^k)q_2 + a_3q_3 + \ldots + a_sq_s$. We claim $\alpha \notin \bigcup_{i=1}^{n} Q_i$. For the contrary, assume $\alpha \in \bigcup_{i=1}^{n} Q_i$. Then $\alpha \in Q_j$ for some $j \in \{1, 2, \ldots, n\}$.

Now, we have two cases.

Case 1: Let $j \in \{1, 2, \ldots, n-1\}$. Since $\alpha = x + y^k q_2 \in Q_j$ and $y^k \in Q_j$, we have $x \in Q_j \subseteq \bigcup_{i=1}^{n-1} Q_i$, a contradiction.

Case 2: Let $j = n$. As $\alpha = x + y^k q_2 \in Q_n$ and $x \in Q_n$, we have $y^k q_2 \in Q_n$. This implies that either $q_2^2 \in Q_n$ or $y^k \in \sqrt{Q_n}$, which gives us a contradiction. Consequently, we conclude $\alpha \notin \bigcup_{i=1}^{n} Q_i$. $\square$

References

[1] *F. A. A. Almahdi, M. Tamekkante, A. Mamouni*: Rings over which every semi-primary ideal is 1-absorbing primary. Commun. Algebra *48* (2020), 3838–3845. zbl MR doi

[2] *D. F. Anderson, D. E. Dobbs (eds.)*: Zero-Dimensional Commutative Rings. Lecture Notes in Pure and Applied Mathematics. 171. Marcel Dekker, New York, 1995. zbl MR

[3] *D. D. Anderson, E. Smith*: Weakly prime ideals. Houston J. Math. *29* (2003), 831–840. zbl MR

[4] *D. D. Anderson, M. Winders*: Idealization of a module. J. Commut. Algebra *1* (2009), 3–56. zbl MR doi

[5] *M. F. Atiyah, I. G. Macdonald*: Introduction to Commutative Algebra. Addison-Wesley, Reading, 1969. zbl MR

[6] *A. Badawi*: On 2-absorbing ideals of commutative rings. Bull. Aust. Math. Soc. *75* (2007), 417–429. zbl MR doi

[7] *A. Badawi, D. Sonmez, G. Yesilot*: On weakly δ-semiprimary ideals of commutative rings. Algebra Colloq. *25* (2018), 387–398. zbl MR doi

[8] *A. Badawi, U. Tekir, E. Yetkin*: On weakly 2-absorbing primary ideals of commutative rings. J. Korean Math. Soc. *52* (2015), 97–111. zbl MR doi

[9] *C. Beddani, W. Messirdi*: 2-prime ideals and their applications. J. Algebra Appl. *15* (2016), Article ID 1650051, 11 pages. zbl MR doi

[10] *G. Călugăreanu*: UN-rings. J. Algebra Appl. *15* (2016), Article ID 1650182, 9 pages. zbl MR doi

[11] *J. Chen*: Infinite prime avoidance. Beitr. Algebra Geom. *62* (2021), 587–593. zbl MR doi

[12] *S. Ebrahimi Atani, F. Farzalipour*: On weakly primary ideals. Georgian Math. J. *12* (2005), 423–429. zbl MR doi

[13] *J. A. Huckaba*: Commutative Rings with Zero Divisors. Monographs and Textbooks in Pure and Applied Mathematics 117. Marcel Dekker, New York, 1988. zbl MR

[14] *S. Koç*: On weakly 2-prime ideals in commutative rings. Commun. Algebra *49* (2021), 3387–3397. zbl MR doi

[15] *S. Koc, U. Tekir, G. Ulucak*: On strongly quasi primary ideals. Bull. Korean Math. Soc. *56* (2019), 729–743. zbl MR doi

[16] *M. D. Larsen, P. J. McCarthy*: Multiplicative Theory of Ideals. Pure and Applied Mathematics 43. Academic Press, New York, 1971. zbl MR

[17] *T.-K. Lee, Y. Zhou*: Reduced modules. Rings, Modules, Algebras and Abelian Groups. Lecture Notes in Pure and Applied Mathematics 236. Marcel Dekker, New York, 2004, pp. 365–377. zbl MR

[18] *J. V. Pakala, T. S. Shores*: On compactly packed rings. Pac. J. Math. *97* (1981), 197–201. zbl MR doi
[19] *S. P. Redmond*: An ideal-based zero-divisor graph of a commutative ring. Commun. Algebra *31* (2003), 4425–4443. zbl MR doi
[20] *M. Satyanarayana*: Rings with primary ideals as maximal ideals. Math. Scand. *20* (1967), 52–54. zbl MR doi
[21] *J. von Neumann*: On regular rings. Proc. Natl. Acad. Sci. USA *22* (1936), 707–713. zbl doi
[22] *F. Wang, H. Kim*: Foundations of Commutative Rings and Their Modules. Algebra and Applications 22. Springer, Singapore, 2016. zbl MR doi
[23] *E. Yıldız, B. A. Ersoy, Ü. Tekir, S. Koç*: On S-Zariski topology. Commun. Algebra *49* (2021), 1212–1224. zbl MR doi

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