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ANOTHER VERSION OF COSUPPORT IN D(R)

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Abstract. The goal of the article is to develop a theory dual to that of support in the derived category D(R). This is done by introducing 'big' and 'small' cosupport for complexes that are different from the cosupport in D. J. Benson, S. B. Iyengar, H. Krause (2012). We give some properties for cosupport that are similar, or rather dual, to those of support for complexes, study some relations between 'big' and 'small' cosupport and give some comparisons of support and cosupport. Finally, we investigate the dual notion of associated primes.

Keywords: cosupport; support; coassociated prime; associated prime

MSC 2020: 13D07, 13D09, 13E05

1. Introduction and preliminaries

The theory of cosupport developed by Benson, Iyengar and Krause (see [2]) in the context of compactly generated triangulated categories, was partially motivated by work of Neeman, see [11]. Despite the many ways in which cosupport is dual to the notion of support introduced by Foxby (see [6]) and Neeman (see [10]), cosupport seems to be more elusive, even in the setting of a commutative noetherian ring.

In general, the theory of cosupport is not completely satisfactory because this construction is not as well understood as support. Richardson in [12] investigated the co-localization functor ${}^{\mathfrak{p}}(-)$, which is dual to the ordinary localization functor $(-)_{\mathfrak{p}}$. For example, ${}^{\mathfrak{p}}(-)$ preserves secondary representations and attached primes (the duals of primary decompositions and associated primes (see [3], Section 7.2), and preserves artinian modules when R is complete. Richardson then defined cosupport, denoted by $\mathrm{coSupp}_R K$, of an R-module K to be the set of primes at which the module's co-localization is nonzero. From his point of view, this co-localization

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functor defines a sensible cosupport. In particular, the cosupport of a nonzero module is nonempty and $\mathrm{coSupp}_R A = \{\mathfrak{p} \in \mathrm{Spec}\, R\colon\ \mathrm{Ann}_R A \subseteq \mathfrak{p}\}$ when A is artinian.

One purpose of this paper is to extend the concept of cosupport in [12] to unbounded complexes. We focus on the functor $D_R(-) = \mathrm{Hom}_R\Big(-, \bigoplus\limits_{\mathfrak{m}} E(R/\mathfrak{m})\Big)$, the sum running over all maximal ideals $\mathfrak{m}$ of R, where $E(R/\mathfrak{m})$ is the injective envelope of $R/\mathfrak{m}$. For an R-complex M, the co-localization of M relative to a prime ideal $\mathfrak{p}$ is the $R_{\mathfrak{p}}$-complex

$$^{\mathfrak{p}}M := \mathrm{Hom}_{R_{\mathfrak{p}}}(D_R(M)_{\mathfrak{p}}, E(R/\mathfrak{p})) \simeq \mathrm{Hom}_R(D_R(M), E(R/\mathfrak{p})).$$

In Section 2, we define $\mathrm{coSupp}_R M$ of 'big' cosupport of M to be the set of prime ideals $\mathfrak{p}$ so that $^{\mathfrak{p}}M \not\simeq 0$. One of the main results of this work is a computation of cosupport:

Theorem I. *For any R-complex M, one has an equality*

$$\mathrm{coSupp}_R M = \bigcup_{i \in \mathbb{Z}} \mathrm{coSupp}_R \mathrm{H}_i(M).$$

In particular, $M \not\simeq 0$ if and only if $\mathrm{coSupp}_R M \neq \emptyset$.

We provide the following (partial) duality between 'big' cosupport and support.

Theorem II. *Let M be an R-complex.*
(1) *$\mathfrak{p} \in \mathrm{coSupp}_R M$ if and only if $\mathfrak{p} \in \mathrm{Supp}_R D_R(M)$.*
(2) *If $\mathfrak{p} \in \mathrm{Supp}_R M$, then $\mathfrak{p} \in \mathrm{coSupp}_R D_R(M)$. The converse holds when $M \in \mathrm{D}^{\mathrm{n}}(R)$, i.e., each $\mathrm{H}_i(M)$ is noetherian).*

By examples we show that the above notion is not the same as the one in [13] and the converse of (2) in the above theorem does not hold in general.

Section 3 investigates 'small' cosupport of complexes

$$\mathrm{cosupp}_R M := \{\mathfrak{p} \in \mathrm{Spec}\, R\colon\ \mathrm{RHom}_R(R/\mathfrak{p}, {}^{\mathfrak{p}}M) \not\simeq 0\},$$

and some properties for 'small' cosupport that are similar to those of 'small' support are provided for $M \in \mathrm{D}_-(R)$. We also show that $\mathrm{cosupp}_R \prod\limits_{\lambda} M_\lambda \neq \bigcup\limits_{\lambda} \mathrm{cosupp}_R M_\lambda$ in general. In Section 4, we study some relations between 'big' and 'small' cosupport, show that $\mathrm{cosupp}_R M \subseteq \mathrm{coSupp}_R M$ for $M \in \mathrm{D}_-(R)$ and the inclusion may be strict. Section 5 is provides some relations between $\mathrm{cosupp}_R M$ and $\mathrm{cosupp}_R \mathrm{H}(M)$. As an application, we give the comparison of support and cosupport.

The concept of coassociated primes of complexes is introduced in the last section, and an extension of Nakayama lemma is given. In the appendix, we provide a summary of all the different cosupports of modules. To faciliate the discussion, we record some related facts and their relations.

Unless stated to the contrary we assume throughout this paper that R is a commutative noetherian ring which is not necessarily local. Next we recall some notions and facts which will be needed later. For terminology we shall follow [5] and [12].

Complexes. The category of chain R-complexes is denoted by $\mathrm{C}(R)$. The derived category of R-complexes is denoted by $\mathrm{D}(R)$.

Let M be an object in $\mathrm{C}(R)$ and $n \in \mathbb{Z}$. The soft right-truncation, $\sigma_{\geqslant n}(M)$, of M at n and the soft left-truncation, $\sigma_{\leqslant n}(M)$, of M at n are given by

$$\sigma_{\geqslant n}(M)\colon\ \ldots \to M_{n+2} \xrightarrow{d_{n+2}} M_{n+1} \xrightarrow{d_{n+1}} \operatorname{Ker} d_n \to 0,$$
$$\sigma_{\leqslant n}(M)\colon\ 0 \to \operatorname{Coker} d_{n+1} \xrightarrow{\bar{d}_n} M_{n-1} \xrightarrow{d_{n-1}} M_{n-2} \to \ldots$$

The differential $\bar{d}_n$ is the induced morphism on residue classes.

An R-complex M is called *bounded* above if $\mathrm{H}_n(M) = 0$ for all $n \gg 0$, bounded below if $\mathrm{H}_n(M) = 0$ for all $n \ll 0$, and bounded if it is both bounded above and bounded below. The full triangulated subcategories consisting of bounded above, bounded below and bounded R-complexes are denoted by $\mathrm{D}_-(R), \mathrm{D}_+(R)$ and $\mathrm{D}_{\mathrm{b}}(R)$, respectively. We denote by $\mathrm{D}^{\mathrm{n}}(R)$ the full triangulated subcategory of $\mathrm{D}(R)$ consisting of R-complexes M such that $\mathrm{H}_i(M)$ are noetherian R-modules for all i, and denote by $\mathrm{D}^{\mathrm{a}}(R)$ the full triangulated subcategory of $\mathrm{D}(R)$ consisting of R-complexes M such that $\mathrm{H}_i(M)$ are artinian R-modules for all i. For $M \in \mathrm{D}(R)$,

$$\inf M := \inf\{n \in \mathbb{Z}\colon\ \mathrm{H}_n(M) \neq 0\}, \quad \sup M := \sup\{n \in \mathbb{Z}\colon\ \mathrm{H}_n(M) \neq 0\}.$$

We write $\operatorname{Spec} R$ for the set of prime ideals of R and $\operatorname{Max} R$ for the set of maximal ideals of R. For an ideal $\mathfrak{a}$ in R and $\mathfrak{p} \in \operatorname{Spec} R$, we set

$$\mathrm{U}(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} R\colon\ \mathfrak{q} \subseteq \mathfrak{p}\} \quad \text{and} \quad \mathrm{V}(\mathfrak{a}) = \{\mathfrak{q} \in \operatorname{Spec} R\colon\ \mathfrak{a} \subseteq \mathfrak{q}\}.$$

Denote $D_{\mathfrak{m}}(-) = \operatorname{Hom}_R(-, E(R/\mathfrak{m}))$ for $\mathfrak{m} \in \operatorname{Max} R$. For an R-complex M, we set $M^{\sim} = \prod_{\mathfrak{m}} D_{\mathfrak{m}}(D_{\mathfrak{m}}(M))$. Let S be a multiplicatively closed subset of R. The co-localization of the complex M relative to S is the $S^{-1}R$-complex

$$S_{-1}M := D_{S^{-1}R}(S^{-1}D_R(M)).$$

If $S = R - \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Spec} R$, we write ${}^{\mathfrak{p}}M$ for $S_{-1}M$.

2. Another version of big cosupport in $\mathrm{D}(R)$

This section introduces the set $\mathrm{coSupp}_R M$ of 'big' cosupport of an R-complex M. We show that $\mathrm{coSupp}_R M$ can be detected by the cosupport $\mathrm{coSupp}_R \mathrm{H}_i(M)$, and give a (partial) duality between $\mathrm{coSupp}_R M$ and $\mathrm{Supp}_R M$.

Definition 2.1. Let M be an R-complex. The 'big' cosupport of M is defined as

$$\mathrm{coSupp}_R M := \{\mathfrak{p} \in \mathrm{Spec}\, R\colon\ {}^{\mathfrak{p}}M \not\simeq 0\}.$$

Following [5] or [13], the 'big' support of an R-complex M is the set

$$\mathrm{Supp}_R M := \{\mathfrak{p} \in \mathrm{Spec}\, R\colon\ M_{\mathfrak{p}} \not\simeq 0\}.$$

It follows from [5], equation (6.1.3.2) that $\mathrm{Supp}_R M = \bigcup\limits_{i\in\mathbb{Z}} \mathrm{Supp}_R \mathrm{H}_i(M)$. The next theorem establishes a similar fact that the big cosupport for an R-complex is completely related to the big cosupport of the homology modules of complexes.

Theorem 2.2. *Let M be an R-complex. One has that*

$$\mathrm{coSupp}_R M = \bigcup_{i\in\mathbb{Z}} \mathrm{coSupp}_R \mathrm{H}_i(M).$$

P r o o f. One has the equivalences

$$\begin{aligned}
\mathfrak{p} \in \mathrm{coSupp}_R M &\Leftrightarrow \mathrm{H}_i({}^{\mathfrak{p}}M) \neq 0 \quad \text{for some } i\\
&\Leftrightarrow \mathrm{Hom}_R(D_R(\mathrm{H}_i(M)), E(R/\mathfrak{p})) \neq 0 \quad \text{for some } i\\
&\Leftrightarrow {}^{\mathfrak{p}}\mathrm{H}_i(M) \neq 0 \quad \text{for some } i\\
&\Leftrightarrow \mathfrak{p} \in \bigcup_{i\in\mathbb{Z}} \mathrm{coSupp}_R \mathrm{H}_i(M),
\end{aligned}$$

where the second equivalence is by injectivity of $\bigoplus\limits_{\mathfrak{m}} E(R/\mathfrak{m})$ and $E(R/\mathfrak{p})$. $\square$

Corollary 2.3. *For an R-complex M, one has $M \not\simeq 0$ if and only if*

$$\mathrm{coSupp}_R M \neq \emptyset.$$

P r o o f. One has that $\mathrm{coSupp}_R M \neq \emptyset$ if and only if $\mathrm{coSupp}_R \mathrm{H}_i(M) \neq \emptyset$ for some i if and only if $\mathrm{H}_i(M) \neq 0$ for some i if and only if $M \not\simeq 0$, where the first equivalence is by Theorem 2.2, the second one is by Lemma A.2. $\square$

Following [4], the annihilator for $M \in \mathrm{D}(R)$ is defined by intersecting the corresponding sets for the homology modules $\mathrm{H}_i(M)$, i.e.,

$$\mathrm{Ann}_R M := \bigcap_{i\in\mathbb{Z}} \mathrm{Ann}_R \mathrm{H}_i(M).$$

If $0 \not\simeq M \in \mathrm{D}_{\mathrm{b}}^{\mathrm{n}}(R)$, then $\mathrm{Supp}_R M = \mathrm{V}(\mathrm{Ann}_R M)$. The next corollary is dual to this.

Corollary 2.4. *For any* $0 \not\simeq M \in \mathrm{D}_{\mathrm{b}}^{\mathrm{a}}(R)$, *one has that*

$$\mathrm{coSupp}_R M = \mathrm{V}(\mathrm{Ann}_R M) = \mathrm{Supp}_R(R/\mathrm{Ann}_R M).$$

Proof. Set $i = \inf M$ and $s = \sup M$. We have

$$\begin{aligned}\mathrm{coSupp}_R M &= \bigcup_{j=i}^{s} \mathrm{coSupp}_R \mathrm{H}_j(M) = \bigcup_{j=i}^{s} \mathrm{V}(\mathrm{Ann}_R \mathrm{H}_j(M)) \\ &= \mathrm{V}\biggl(\bigcap_{j=i}^{s} \mathrm{Ann}_R \mathrm{H}_j(M)\biggr) = \mathrm{V}(\mathrm{Ann}_R M),\end{aligned}$$

where the second equality is by Lemma A.2. □

The following result plays an important role in the rest of the paper.

Theorem 2.5. *Let M be an R-complex. The following are equivalent:*

(1) $\mathfrak{p} \in \mathrm{coSupp}_R M$;

(2) $\mathfrak{p} \in \mathrm{Supp}_R D_R(M)$.

If in addition R is semi-local, then (1) *and* (2) *are equivalent to*

(3) $\mathfrak{p} \in \mathrm{Supp}_R D_{\mathfrak{m}}(M)$ *for some* $\mathfrak{m} \in \mathrm{Max}\, R \cap \mathrm{V}(\mathfrak{p})$;

(4) $\mathrm{RHom}_R(R_{\mathfrak{p}}, M^{\sim}) \not\simeq 0$.

Proof. (1) $\Leftrightarrow$ (2) One has the equivalences

$$\begin{aligned}\mathfrak{p} \in \mathrm{coSupp}_R M &\Leftrightarrow \mathfrak{p} \in \mathrm{coSupp}_R \mathrm{H}_i(M) \quad \text{for some } i \\ &\Leftrightarrow \mathfrak{p} \in \mathrm{Supp}_R D_R(\mathrm{H}_i(M)) \quad \text{for some } i \\ &\Leftrightarrow \mathfrak{p} \in \mathrm{Supp}_R \mathrm{H}_{-i}(D_R(M)) \quad \text{for some } i \\ &\Leftrightarrow \mathfrak{p} \in \mathrm{Supp}_R D_R(M),\end{aligned}$$

where the first one is by Theorem 2.2, the second one is by Lemma A.2, the third one is by injectivity of $\bigoplus_{\mathfrak{m}} E(R/\mathfrak{m})$.

Next assume that R is semi-local.

(2) ⇔ (3) One has the equivalences

$$\begin{aligned}\mathfrak{p} \in \operatorname{Supp}_R D_{\mathfrak{m}}(M) \quad &\text{for some } \mathfrak{m} \in \operatorname{Max} R \cap \mathrm{V}(\mathfrak{p}) \\ &\Leftrightarrow \mathfrak{p} \in \operatorname{Supp}_R D_{\mathfrak{m}}(\mathrm{H}_i(M)) \quad \text{for some } i \text{ and } \mathfrak{m} \in \operatorname{Max} R \cap \mathrm{V}(\mathfrak{p}) \\ &\Leftrightarrow \mathfrak{p} \in \operatorname{coSupp}_R \mathrm{H}_i(M) \quad \text{for some } i \\ &\Leftrightarrow \mathfrak{p} \in \operatorname{Supp}_R D_R(\mathrm{H}_i(M)) \quad \text{for some } i \\ &\Leftrightarrow \mathfrak{p} \in \operatorname{Supp}_R D_R(M),\end{aligned}$$

where the first and the fourth ones are by [5], equation (6.1.3.2), the second one is by Lemmas A.1 and A.4, the third one is by Lemma A.2.

(1) ⇔ (4) For any $i \in \mathbb{Z}$, one has the isomorphisms

$$\begin{aligned}\mathrm{H}_i(\operatorname{RHom}_R(R_{\mathfrak{p}}, M^{\sim})) &\cong \prod_{\mathfrak{m}} \mathrm{H}_i(D_{\mathfrak{m}}(D_{\mathfrak{m}}(M)_{\mathfrak{p}})) \cong \prod_{\mathfrak{m}} D_{\mathfrak{m}}(D_{\mathfrak{m}}(\mathrm{H}_i(M))_{\mathfrak{p}}) \\ &\cong \prod_{\mathfrak{m}} \operatorname{Hom}_R(R_{\mathfrak{p}}, D_{\mathfrak{m}}(D_{\mathfrak{m}}(\mathrm{H}_i(M)))) \cong \operatorname{Hom}_R(R_{\mathfrak{p}}, \mathrm{H}_i(M)^{\sim}),\end{aligned}$$

where the first and the third ones are by adjointness, the second one is by injectivity of $E(R/\mathfrak{m})$ and flatness of $R_{\mathfrak{p}}$. Therefore, we have the equivalences

$$\begin{aligned}\operatorname{RHom}_R(R_{\mathfrak{p}}, M^{\sim}) \not\simeq 0 &\Leftrightarrow \operatorname{Hom}_R(R_{\mathfrak{p}}, \mathrm{H}_i(M)^{\sim}) \neq 0 \quad \text{for some } i \\ &\Leftrightarrow \mathfrak{p} \in \operatorname{coSupp}_R \mathrm{H}_i(M) \quad \text{for some } i \\ &\Leftrightarrow \mathfrak{p} \in \operatorname{coSupp}_R M,\end{aligned}$$

where the first one is by the above isomorphism, the second one is by Lemmas A.1 and A.4 and the third one is by Theorem 2.2. □

Let $\mathcal{U}$ be a subset of $\operatorname{Spec} R$. The specialization closure of $\mathcal{U}$ is the set

$$\operatorname{cl}\mathcal{U} = \{\mathfrak{p} \in \operatorname{Spec} R\colon \text{ there is } \mathfrak{q} \in \mathcal{U} \text{ with } \mathfrak{q} \subseteq \mathfrak{p}\}.$$

The subset $\mathcal{U}$ is specialization closed if $\operatorname{cl}\mathcal{U} = \mathcal{U}$.

Remark 2.6.

(1) For any R-complex M, one has that $\operatorname{coSupp}_R M = \operatorname{coSupp}_R \Sigma M$.

(2) For an exact triangle $L \to M \to N \rightsquigarrow$ in $\mathrm{D}(R)$ we have

$$\operatorname{coSupp}_R M \subseteq \operatorname{coSupp}_R L \cup \operatorname{coSupp}_R N.$$

(3) For any R-complex M, the set $\operatorname{coSupp}_R M$ is specialization closed since

$$\operatorname{coSupp}_R M = \operatorname{Supp}_R D_R(M)$$

by Theorem 2.5 and $\operatorname{Supp}_R D_R(M)$ is specialization closed.

(4) $\mathrm{H}({}^{\mathfrak{p}}M) \cong {}^{\mathfrak{p}}\mathrm{H}(M)$ for any $\mathfrak{p} \in \operatorname{Spec} R$.

(5) (i) Let $M \in \mathrm{D}_+^{\mathrm{n}}(R)$ and $N \in \mathrm{D}_+(R)$. One has the isomorphism

$$
{}^{\mathfrak{p}}\left(M \overset{\mathrm{L}}{\underset{R}{\bigotimes}} N\right) \simeq \mathrm{Hom}_{R_{\mathfrak{p}}}(\mathrm{RHom}_R(M, D_R(N))_{\mathfrak{p}}, E(R/\mathfrak{p}))
$$

$$
\simeq \mathrm{Hom}_{R_{\mathfrak{p}}}(\mathrm{RHom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, D_R(N)_{\mathfrak{p}}), E(R/\mathfrak{p})) \simeq M_{\mathfrak{p}} \overset{\mathrm{L}}{\underset{R_{\mathfrak{p}}}{\bigotimes}} {}^{\mathfrak{p}}N,
$$

where the first one is by adjointness, the second one is by [5], Lemma 6.1.6 and the third one is by [5], Theorem 2.5.6.
(ii) Let $M \in \mathrm{D}_+^{\mathrm{n}}(R)$ and $N \in \mathrm{D}_-(R)$. One has the isomorphism

$$
{}^{\mathfrak{p}}\mathrm{RHom}_R(M, N) \simeq \mathrm{RHom}_{R_{\mathfrak{p}}}\left(M_{\mathfrak{p}} \overset{\mathrm{L}}{\underset{R_{\mathfrak{p}}}{\bigotimes}} D_R(N)_{\mathfrak{p}}, E(R/\mathfrak{p})\right) \simeq \mathrm{RHom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, {}^{\mathfrak{p}}N),
$$

where the first one is by [5], Theorem 2.5.6 and the second one is by adjointness.

(6) By the proof of Theorem 2.5, $\mathrm{RHom}_R(R_{\mathfrak{p}}, M^{\sim}) \not\simeq 0$ implies that

$$
\mathrm{Hom}_R(R_{\mathfrak{p}}, \mathrm{H}_i(M)^{\sim}) \neq 0
$$

for some i, and hence $\mathfrak{p} \in \mathrm{Cosupp}_R \mathrm{H}_i(M) \subseteq \mathrm{coSupp}_R \mathrm{H}_i(M) \subseteq \mathrm{coSupp}_R M$ by Lemmas A.1 and A.4 and Theorem 2.2. Consequently, ${}^{\mathfrak{p}}M \not\simeq 0$.

(7) Sather-Wagstaff and Wicklein in [13] extended the notion of cosupport provided by Benson, Iyengar and Krause in [2] to complexes. They defined the 'large' cosupport of an R-complex M as the set

$$
\text{Co-supp}_R M := \{\mathfrak{p} \in \operatorname{Spec} R\colon\ \mathrm{RHom}_R(R_{\mathfrak{p}}, M) \not\simeq 0\}.
$$

However, our definition of 'big' cosupport for M is not the same as the above. For example, let $M = R = k[x]$ for any field k. Since $\mathrm{Hom}_R(R_{\{0\}}, R) \neq 0$ and $\text{Co-supp}_R M$ is specialization closed by (3), it follows that $\text{Co-supp}_R M = \operatorname{Spec} R$. But $\mathrm{coSupp}_R M = \mathrm{Supp}_R D_R(M) = \operatorname{Max} R \neq \operatorname{Spec} R$ by Theorem 2.5.

(8) Let M be an R-complex and $\mathfrak{p} \in \operatorname{Spec} R$. If each $\mathrm{H}_i(M)$ is a Matlis reflexive R-module (i.e., $\mathrm{H}_i(M) \cong D_R(D_R(\mathrm{H}_i(M)))$), then $M \simeq D_R(D_R(M))$, and hence

$$
{}^{\mathfrak{p}}D_R(M) \simeq \mathrm{Hom}_{R_{\mathfrak{p}}}(D_R(D_R(M))_{\mathfrak{p}}, E(k(\mathfrak{p}))) \simeq D_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}).
$$

So ${}^{\mathfrak{p}}D_R(M) \neq 0$ if and only if $M_{\mathfrak{p}} \neq 0$ as $E(R/\mathfrak{p})$ cogenerates the category of $\mathfrak{p}$-local R-modules, that is to say,

$$
\mathfrak{p} \in \mathrm{Supp}_R M \Leftrightarrow \mathfrak{p} \in \mathrm{coSupp}_R D_R(M).
$$

In general, we have the following result.

Proposition 2.7. *Let M be an R-complex.*
(1) *If $\mathfrak{p} \in \mathrm{Supp}_R M$, then $\mathfrak{p} \in \mathrm{coSupp}_R D_R(M)$.*
(2) *If $M \in \mathrm{D}^{\mathrm{n}}(R)$, then $\mathfrak{p} \in \mathrm{Supp}_R M$ if and only if $\mathfrak{p} \in \mathrm{coSupp}_R D_R(M)$.*

Proof. (1) Since $\mathfrak{p} \in \mathrm{Supp}_R M$, $\mathfrak{p} \in \mathrm{Supp}_R \mathrm{H}_i(M)$ for some i, and so $\mathfrak{p} \in \mathrm{coSupp}_R \mathrm{H}_{-i}(D_R(M))$ by Lemma A.5. Therefore, $\mathfrak{p} \in \mathrm{coSupp}_R D_R(M)$ by Theorem 2.2.

(2) "Only if" part by (1). "If" part: Since

$$\mathfrak{p} \in \mathrm{coSupp}_R D_R(M), \quad \mathfrak{p} \in \mathrm{coSupp}_R \mathrm{H}_i(D_R(M))$$

for some i by Theorem 2.2, i.e., $\mathfrak{p} \in \mathrm{coSupp}_R D_R(\mathrm{H}_{-i}(M))$. Hence, Lemma A.5 implies that $\mathfrak{p} \in \mathrm{Supp}_R \mathrm{H}_{-i}(M)$. Consequently, $\mathfrak{p} \in \mathrm{Supp}_R M$. □

The following example shows that the converse of (1) in the above proposition does not hold in general.

Example 2.8. ([14]). Let $(R, \mathfrak{m})$ be a local domain with $\dim R > 0$. Consider the complex

$$M = 0 \to \bigoplus_{n>0} R/\mathfrak{m}^n \to 0.$$

Then $D_{\mathfrak{m}}(M) = D_R(M)$ and $0 \in \mathrm{coSupp}_R D_R(M)$. But

$$0 \notin \mathrm{Supp}_R M \quad \text{as } \mathrm{Supp}_R M = \{\mathfrak{m}\}.$$

3. Another version of small cosupport in $\mathrm{D}(R)$

This section introduces the set $\mathrm{cosupp}_R M$ of 'small' cosupport of an R-complex M, and provide a duality between the 'small' cosupport and support as in Section 2.

Definition 3.1. Let M be an R-complex. The 'small' cosupport of M is defined as

$$\mathrm{cosupp}_R M := \{\mathfrak{p} \in \mathrm{Spec}\, R\colon\ \mathrm{RHom}_R(R/\mathfrak{p}, {}^{\mathfrak{p}}M) \not\simeq 0\}.$$

Following [5] or [13], the 'small' support of an R-complex M is the set

$$\mathrm{supp}_R M := \Big\{\mathfrak{p} \in \mathrm{Spec}\, R\colon\ k(\mathfrak{p}) \bigotimes_R^{\mathrm{L}} M \not\simeq 0\Big\} = \Big\{\mathfrak{p} \in \mathrm{Spec}\, R\colon\ R/\mathfrak{p} \bigotimes_R^{\mathrm{L}} M_{\mathfrak{p}} \not\simeq 0\Big\},$$

where $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Next, we bring an analogue of Theorem 2.5.

Theorem 3.2. *Let M be an R-complex in $\mathrm{D}_-(R)$. The following are equivalent:*

(1) $\mathfrak{p} \in \operatorname{cosupp}_R M$;

(2) $\mathrm{RHom}_R(D_R(M), k(\mathfrak{p})) \not\simeq 0$;

(3) $\mathfrak{p} \in \operatorname{supp}_R D_R(M)$;

(4) $k(\mathfrak{p}) \overset{\mathrm{L}}{\underset{R_\mathfrak{p}}{\bigotimes}} {}^{\mathfrak{p}}M \not\simeq 0$;

(5) $\mathfrak{p}R_\mathfrak{p} \in \operatorname{cosupp}_{R_\mathfrak{p}} {}^{\mathfrak{p}}M$.

If in addition R is semi-local, then (1)–(5) are equivalent to

(6) $\mathrm{RHom}_R(k(\mathfrak{p}), M^{\sim}) \not\simeq 0$;

(7) $\mathrm{Hom}_R\Big(\bigoplus_{\mathfrak{m}} D_\mathfrak{m}(M), k(\mathfrak{p})\Big) \not\simeq 0$;

(8) $\mathfrak{p} \in \operatorname{supp}_R D_\mathfrak{m}(M)$ *for some* $\mathfrak{m} \in \operatorname{Max} R \cap \mathrm{V}(\mathfrak{p})$;

(9) $k(\mathfrak{p}) \overset{\mathrm{L}}{\underset{R_\mathfrak{p}}{\bigotimes}} \mathrm{RHom}_R(R_\mathfrak{p}, M^{\sim}) \not\simeq 0$.

In particular, $M \not\simeq 0$ if and only if $\operatorname{cosupp}_R M \neq \emptyset$.

P r o o f. One has the isomorphisms in $\mathrm{D}(R)$

$$ {}^{\mathfrak{p}}\mathrm{RHom}_R(R/\mathfrak{p}, M) \simeq \mathrm{RHom}_{R_\mathfrak{p}}((R/\mathfrak{p})_\mathfrak{p}, {}^{\mathfrak{p}}M) \simeq \mathrm{RHom}_R(R/\mathfrak{p}, {}^{\mathfrak{p}}M), $$

where the first one is by Remark 2.6 (5) and the second one is by adjointness.

$$ \begin{aligned} {}^{\mathfrak{p}}\mathrm{RHom}_R(R/\mathfrak{p}, M) &\simeq \mathrm{RHom}_{R_\mathfrak{p}}(k(\mathfrak{p}), {}^{\mathfrak{p}}M) \\ &\simeq \mathrm{RHom}_{R_\mathfrak{p}}(D_R(M)_\mathfrak{p}, \mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), E(k(\mathfrak{p})))) \\ &\simeq \mathrm{RHom}_R(D_R(M), k(\mathfrak{p})), \end{aligned} $$

where the first one is by Remark 2.6 (5), the second one is by adjointness and the third one is by the isomorphism $\mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), E(k(\mathfrak{p}))) \cong k(\mathfrak{p})$.

$$ D_R(\mathrm{RHom}_R(R/\mathfrak{p}, M))_\mathfrak{p} \simeq \Big(R/\mathfrak{p} \overset{\mathrm{L}}{\underset{R}{\bigotimes}} D_R(M)\Big)_\mathfrak{p} \simeq k(\mathfrak{p}) \overset{\mathrm{L}}{\underset{R}{\bigotimes}} D_R(M), $$

where the first one is by [5], Theorem 2.5.6 and the second one is by adjointness. Hence, Theorem 2.5 implies the equivalences of (1)–(3).

(1) $\Leftrightarrow$ (4) This follows from [13], Fact 3.5 and the isomorphism

$$ \mathrm{RHom}_R(R/\mathfrak{p}, {}^{\mathfrak{p}}M) \simeq \mathrm{RHom}_{R_\mathfrak{p}}(k(\mathfrak{p}), {}^{\mathfrak{p}}M) $$

in $\mathrm{D}(R)$ by Remark 2.6 (5) and adjointness.

(1) ⇔ (5) Since $M \in \mathrm{D}_-(R)$, ${}^{\mathfrak{p}}M \in \mathrm{D}_-(R_{\mathfrak{p}})$. One has the isomorphisms in $\mathrm{D}(R)$

$$\begin{aligned}
&\mathrm{RHom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, {}^{\mathfrak{p}R_{\mathfrak{p}}}({}^{\mathfrak{p}}M)) \\
&\qquad \simeq \mathrm{Hom}_{R_{\mathfrak{p}}}\Big(k(\mathfrak{p}) \overset{\mathrm{L}}{\underset{R_{\mathfrak{p}}}{\bigotimes}} \mathrm{Hom}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}M, E(k(\mathfrak{p}))), E(k(\mathfrak{p}))\Big) \\
&\qquad \simeq \mathrm{Hom}_{R_{\mathfrak{p}}}(\mathrm{Hom}_{R_{\mathfrak{p}}}(\mathrm{RHom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), {}^{\mathfrak{p}}M), E(k(\mathfrak{p}))), E(k(\mathfrak{p}))) \\
&\qquad \simeq D_{R_{\mathfrak{p}}}(D_{R_{\mathfrak{p}}}(\mathrm{RHom}_R(R/\mathfrak{p}, {}^{\mathfrak{p}}M))),
\end{aligned}$$

where the first and the third ones are by adjointness and the second one is by [5], Theorem 2.5.6. Thus, $\mathrm{RHom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, {}^{\mathfrak{p}R_{\mathfrak{p}}}({}^{\mathfrak{p}}M)) \not\simeq 0$ if and only if $\mathrm{RHom}_R(R/\mathfrak{p}, {}^{\mathfrak{p}}M) \not\simeq 0$ as $E(k(\mathfrak{p}))$ cogenerates the category of $\mathfrak{p}$-local R-modules, as desired.

One has the isomorphisms in $\mathrm{D}(R)$

$$\begin{aligned}
\mathrm{RHom}_R(k(\mathfrak{p}), M^{\sim}) &\simeq \prod_{\mathfrak{m}} D_{\mathfrak{m}}\Big(k(\mathfrak{p}) \overset{\mathrm{L}}{\underset{R}{\bigotimes}} D_{\mathfrak{m}}(M)\Big) \simeq \prod_{\mathfrak{m}} D_{\mathfrak{m}}(D_{\mathfrak{m}}(\mathrm{RHom}_R(R/\mathfrak{p}, M))_{\mathfrak{p}}) \\
&\simeq \mathrm{RHom}_R(R_{\mathfrak{p}}, \mathrm{RHom}_R(R/\mathfrak{p}, M)^{\sim}),
\end{aligned}$$

where the first and the third ones are by adjointness, the second one is by [5], Theorem 2.5.6.

$$\begin{aligned}
\mathrm{Hom}_R\Big(\bigoplus_{\mathfrak{m}} D_{\mathfrak{m}}(M), k(\mathfrak{p})\Big) &\simeq \mathrm{Hom}_R\Big(\bigoplus_{\mathfrak{m}} D_{\mathfrak{m}}(M), \mathrm{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), E(R/\mathfrak{p}))\Big) \\
&\simeq \mathrm{Hom}_R\Big(\bigoplus_{\mathfrak{m}} \Big(R/\mathfrak{p} \overset{\mathrm{L}}{\underset{R}{\bigotimes}} D_{\mathfrak{m}}(M)\Big), E(R/\mathfrak{p})\Big) \\
&\simeq \mathrm{Hom}_R\Big(\bigoplus_{\mathfrak{m}} D_{\mathfrak{m}}(\mathrm{RHom}_R(R/\mathfrak{p}, M)), E(R/\mathfrak{p})\Big),
\end{aligned}$$

where the first one is by the isomorphism $k(\mathfrak{p}) \cong \mathrm{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), E(R/\mathfrak{p}))$, the second one is by adjointness and the third one is by [5], Theorem 2.5.6.

$$D_{\mathfrak{m}}(\mathrm{RHom}_R(R/\mathfrak{p}, M))_{\mathfrak{p}} \simeq \Big(R/\mathfrak{p} \overset{\mathrm{L}}{\underset{R}{\bigotimes}} D_{\mathfrak{m}}(M)\Big)_{\mathfrak{p}} \simeq k(\mathfrak{p}) \overset{\mathrm{L}}{\underset{R}{\bigotimes}} D_{\mathfrak{m}}(M),$$

where the first one is by [5], Theorem 2.5.6 and the second one is by adjointness. Hence, Theorem 2.5 implies the equivalences of (1) ⇔ (6) ⇔ (7) ⇔ (8).

(6) ⇔ (9) This follows from [13], Fact 3.5 and the isomorphism

$$\mathrm{RHom}_R(k(\mathfrak{p}), M^{\sim}) \simeq \mathrm{RHom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), \mathrm{RHom}_R(R_{\mathfrak{p}}, M^{\sim}))$$

in $\mathrm{D}(R)$ by adjointness. □

Corollary 3.3. *Let M be an R-complex in* $\mathrm{D}_-(R)$. *One has that*

$$\mathrm{cosupp}_R M = \min(\mathrm{cosupp}_R \mathrm{H}(M)).$$

Proof. One has the equivalences

$$\begin{aligned}\mathfrak{p} \in \mathrm{cosupp}_R M \Leftrightarrow \mathfrak{p} \in \mathrm{supp}_R D_R(M) \Leftrightarrow \mathfrak{p} \in \min(\mathrm{supp}_R \mathrm{H}(D_R(M)))\\ \Leftrightarrow \mathfrak{p} \in \min(\mathrm{supp}_R D_R(\mathrm{H}(M))) \Leftrightarrow \mathfrak{p} \in \min(\mathrm{cosupp}_R \mathrm{H}(M)),\end{aligned}$$

where the first and the last equivalences are by Theorem 3.2, the second one is by [1], Theorem 5.2 and the third one is by injectivity of $\bigoplus\limits_{\mathfrak{m}} E(R/\mathfrak{m})$. □

Proposition 3.4.

(1) *Let $M \in \mathrm{D}^{\mathrm{n}}_{+}(R)$ and $N \in \mathrm{D}_-(R)$. One has that*

$$\mathrm{cosupp}_R \mathrm{RHom}_R(M, N) = \mathrm{supp}_R M \cap \mathrm{cosupp}_R N.$$

(2) *Let $M \in \mathrm{D}^{\mathrm{n}}_{\mathrm{b}}(R)$ and $N \in \mathrm{D}_-(R)$. One has that*

$$\mathrm{cosupp}_R \left(M \overset{\mathrm{L}}{\underset{R}{\bigotimes}} N \right) = \mathrm{supp}_R M \cap \mathrm{cosupp}_R N.$$

Proof. (1) One has the equivalences

$$\begin{aligned}\mathfrak{p} \in \mathrm{cosupp}_R \mathrm{RHom}_R(M, N) &\Leftrightarrow \mathfrak{p} \in \mathrm{supp}_R D_R(\mathrm{RHom}_R(M, N))\\ &\Leftrightarrow \mathfrak{p} \in \mathrm{supp}_R \left(M \overset{\mathrm{L}}{\underset{R}{\bigotimes}} D_R(N) \right)\\ &\Leftrightarrow \mathfrak{p} \in \mathrm{supp}_R M \cap \mathrm{supp}_R D_R(N)\\ &\Leftrightarrow \mathfrak{p} \in \mathrm{supp}_R M \cap \mathrm{cosupp}_R N,\end{aligned}$$

where the first and the fourth equivalences are by Theorem 3.2, the second one is by [5], Theorem 2.5.6 and the third one is by [13], Proposition 3.12.

(2) One has the equivalences

$$\begin{aligned}\mathfrak{p} \in \mathrm{cosupp}_R \left(M \overset{\mathrm{L}}{\underset{R}{\bigotimes}} N \right) &\Leftrightarrow \mathfrak{p} \in \mathrm{supp}_R D_R \left(M \overset{\mathrm{L}}{\underset{R}{\bigotimes}} N \right)\\ &\Leftrightarrow \mathfrak{p} \in \mathrm{supp}_R \mathrm{RHom}_R(M, D_R(N))\\ &\Leftrightarrow \mathfrak{p} \in \mathrm{supp}_R M \cap \mathrm{supp}_R D_R(N)\\ &\Leftrightarrow \mathfrak{p} \in \mathrm{supp}_R M \cap \mathrm{cosupp}_R N,\end{aligned}$$

where the first and the fourth equivalences are by Theorem 3.2, the third one is by [13], Proposition 3.16. □

Remark 3.5.

(1) For each $M \in \mathrm{D}_-(R)$, one has ${}^{\mathfrak{p}}\mathrm{RHom}_R(R/\mathfrak{p}, M) \simeq \mathrm{RHom}_R(R/\mathfrak{p}, {}^{\mathfrak{p}}M)$ by Remark 2.6 (5) and adjointness. Thus,

$$\mathfrak{p} \in \mathrm{cosupp}_R M \Leftrightarrow \mathfrak{p} \in \mathrm{coSupp}_R \mathrm{RHom}_R(R/\mathfrak{p}, M).$$

(2) If M is an R-module, then

$$\mathrm{cosupp}_R M = \{\mathfrak{p} \in \mathrm{Spec}\, R\colon\ {}^{\mathfrak{p}}\mathrm{Ext}^i_R(R/\mathfrak{p}, M) \neq 0 \text{ for some } i\}.$$

(3) Let V be a specialization closed subset of Spec R. For each $M \in \mathrm{D}_-(R)$, one has

$$\begin{aligned} \mathrm{cosupp}_R M \subseteq \mathrm{V} &\Leftrightarrow \mathrm{supp}_R D_R(M) \subseteq \mathrm{V} \\ &\Leftrightarrow D_R(M)_{\mathfrak{p}} = 0 \text{ for each } \mathfrak{p} \in \mathrm{Spec}\, R \setminus \mathrm{V} \\ &\Leftrightarrow {}^{\mathfrak{p}}M = 0 \text{ for each } \mathfrak{p} \in \mathrm{Spec}\, R \setminus \mathrm{V}, \end{aligned}$$

where the first equivalence is by Theorem 3.2, the second one is by [1], Lemma 2.3 (1) and the third one is as $E(R/\mathfrak{p})$ cogenerates the category of $\mathfrak{p}$-local R-modules.

(4) For each $M \in \mathrm{D}_-(R)$ it follows from Theorems 2.5, 3.2 and [1], Corollary 5.3 that

$$\mathrm{cosupp}_R M \subseteq \mathrm{cl}(\mathrm{cosupp}_R M) = \mathrm{coSupp}_R M \subseteq \mathrm{V}(\mathrm{Ann}_R M).$$

(5) Given a set of R-complexes M_i in $\mathrm{D}_-(R)$. In general, $\mathrm{cosupp}_R \prod_i M_i \neq \bigcup_i \mathrm{cosupp}_R M_i$. Indeed, let $(R, \mathfrak{m})$ be a local domain with $\dim R > 0$ and $M_i = R/\mathfrak{m}^i$ for $i > 0$. Since $\mathrm{supp}_R R/\mathfrak{m}^i \subseteq \{\mathfrak{m}\}$ for each $i > 0$,

$$\bigcup_i \mathrm{cosupp}_R D_{\mathfrak{m}}(R/\mathfrak{m}^i) = \{\mathfrak{m}\}$$

by Proposition 3.4. But

$$\mathrm{cosupp}_R \prod_{i>0} D_{\mathfrak{m}}(R/\mathfrak{m}^i) \neq \{\mathfrak{m}\}$$

since $0 \in \mathrm{cosupp}_R \prod_{s>0} D_{\mathfrak{m}}(R/\mathfrak{m}^s)$.

The following proposition is an analogue of Proposition 2.7.

Proposition 3.6. *Let M be an R-complex in $\mathrm{D}_+(R)$.*

(1) *If $\mathfrak{p} \in \mathrm{supp}_R M$, then $\mathfrak{p} \in \mathrm{cosupp}_R D_R(M)$.*

(2) *If $M \in \mathrm{D}^{\mathrm{n}}_+(R)$, then $\mathfrak{p} \in \mathrm{supp}_R M$ if and only if $\mathfrak{p} \in \mathrm{cosupp}_R D_R(M)$.*

P r o o f. (1) Let $\mathfrak{p} \in \operatorname{supp}_R M$. Then $\mathfrak{p} \in \operatorname{Supp}_R\Bigl(R/\mathfrak{p} \overset{\mathrm{L}}{\underset{R}{\otimes}} M\Bigr)$, and so $\mathfrak{p} \in \operatorname{coSupp}_R D_R\Bigl(R/\mathfrak{p} \overset{\mathrm{L}}{\underset{R}{\otimes}} M\Bigr)$ by Proposition 2.7 (1). But

$$D_R\left(R/\mathfrak{p} \overset{\mathrm{L}}{\underset{R}{\bigotimes}} M\right) \simeq \operatorname{RHom}_R(R/\mathfrak{p}, D_R(M)),$$

it follows from Remark 3.5 (1) that $\mathfrak{p} \in \operatorname{cosupp}_R D_R(M)$.

(2) This follows from Proposition 2.7 (2) since $R/\mathfrak{p} \overset{\mathrm{L}}{\underset{R}{\otimes}} M \in \mathrm{D}^{\mathrm{n}}_{+}(R)$. □

4. Relations between big and small cosupport

We devote this section to some relations between 'big' and 'small' cosupport. We show that $\operatorname{cosupp}_R M \subseteq \operatorname{coSupp}_R M$ for $M \in \mathrm{D}_{-}(R)$, and the inclusion may be strict.

Following [13], we denote by $\operatorname{co-supp}_R M$ the set

$$\operatorname{co-supp}_R M := \{\mathfrak{p} \in \operatorname{Spec} R \colon \operatorname{RHom}_R(k(\mathfrak{p}), M) \not\simeq 0\}.$$

Proposition 4.1. *For an R-complex M in $\mathrm{D}_{-}(R)$, the sets $\operatorname{supp}_R M$ and $\operatorname{cosupp}_R M$ have the same maximal elements with respect to containment, i.e., $\max(\operatorname{supp}_R M) = \max(\operatorname{cosupp}_R M)$. Moreover, $\max(\operatorname{cosupp}_R M) = \max(\operatorname{co-supp}_R M)$.*

P r o o f. We prove that $\max(\operatorname{supp}_R M) \subseteq \operatorname{cosupp}_R M$ and $\max(\operatorname{cosupp}_R M) \subseteq \operatorname{supp}_R M$.

If $\mathfrak{p} \in \max(\operatorname{supp}_R M)$, then $\operatorname{co-supp}_R\Bigl(R/\mathfrak{p} \overset{\mathrm{L}}{\underset{R}{\otimes}} D_R(M)\Bigr) = \{\mathfrak{p}\}$ by [13], Proposition 4.10. As $\operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), E(k(\mathfrak{p}))) \cong k(\mathfrak{p})$, it follows from [2], Proposition 5.4 that

$$\operatorname{RHom}_R(D_R(M), k(\mathfrak{p})) \simeq \operatorname{RHom}_R\left(R/\mathfrak{p} \overset{\mathrm{L}}{\underset{R}{\bigotimes}} D_R(M), E(R/\mathfrak{p})\right) \not\simeq 0,$$

and hence $\mathfrak{p} \in \operatorname{cosupp}_R M$ by Theorem 3.2. If $\mathfrak{p} \in \max(\operatorname{cosupp}_R M)$, then $\operatorname{cosupp}_R\Bigl(R/\mathfrak{p} \overset{\mathrm{L}}{\underset{R}{\otimes}} M\Bigr) = \{\mathfrak{p}\}$, so $\mathfrak{p} \in \max\Bigl(\operatorname{supp}_R D_R(R/\mathfrak{p} \overset{\mathrm{L}}{\underset{R}{\otimes}} M)\Bigr)$. Thus, [13], Proposition 4.7 (b) implies that $\mathfrak{p} \in \operatorname{co-supp}_R \operatorname{RHom}_R(R/\mathfrak{p}, D_R(M))$. Consequently, $\mathfrak{p} \in \operatorname{supp}_R M$ by [13], Proposition 4.10.

The second statement follows from [2], Theorem 4.13. □

Proposition 4.2. *For every $M \in \mathrm{D}_{-}(R)$, one has an inclusion $\operatorname{cosupp}_R M \subseteq \operatorname{coSupp}_R M$; equality holds if R is a semi-local complete ring and $M \in \mathrm{D}^{\mathrm{a}}_{-}(R)$.*

P r o o f. The inclusion follows from Theorems 2.5 and 3.2 since $\operatorname{supp}_R D_R(M) \subseteq \operatorname{Supp}_R D_R(M)$. Now let $M \in \mathrm{D}^{\mathrm{a}}_{-}(R)$ and $\mathfrak{p} \in \operatorname{coSupp}_R M$, $i = \sup {}^{\mathfrak{p}}M$. Then ${}^{\mathfrak{p}}M \in \mathrm{D}^{\mathrm{a}}_{-}(R_{\mathfrak{p}})$ by [12], Theorem 2.3, and so

$$\mathrm{H}_i(\mathrm{RHom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), {}^{\mathfrak{p}}M)) \cong \operatorname{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), \mathrm{H}_i({}^{\mathfrak{p}}M)) \neq 0$$

by [14], Theorem 4.3. Consequently, $\mathrm{RHom}_R(R/\mathfrak{p}, {}^{\mathfrak{p}}M) \not\simeq 0$ and $\mathfrak{p} \in \operatorname{cosupp}_R M$, as claimed. □

The next example shows that the inclusion in Proposition 4.2 may be strict:

Example 4.3 ([1], Example 9.4). Let k be a field and $R = k[[x, y]]$ the power series ring in indeterminates x, y, and set $\mathfrak{m} = (x, y)$ the maximal ideal of R. The minimal injective resolution of R has the form

$$\ldots \to 0 \to Q \to \coprod_{\mathrm{ht}\,\mathfrak{p}=1} E(R/\mathfrak{p}) \to E(R/\mathfrak{m}) \to 0 \to \ldots,$$

where Q denotes the fraction field of R. Let M denote the truncated complex

$$\ldots \to 0 \to Q \to \coprod_{\mathrm{ht}\,\mathfrak{p}=1} E(R/\mathfrak{p}) \to 0 \to \ldots$$

One has that $\operatorname{coSupp}_R D_R(M) = \operatorname{Spec} R$ since $\operatorname{Spec} R = \operatorname{Supp}_R M \subseteq \operatorname{coSupp}_R D_R(M)$. But $\mathfrak{m} \notin \operatorname{cosupp}_R D_R(M)$. In fact, if $\mathfrak{m} \in \operatorname{cosupp}_R D_R(M)$, then $\mathfrak{m} \in \operatorname{supp}_R D_R(M)$ by Proposition 4.1, and hence $\mathfrak{m} \in \operatorname{cosupp}_R M$ by Theorem 3.2. Consequently, $\mathfrak{m} \in \operatorname{supp}_R M$ by Proposition 4.1 again, which is a contradiction since $\operatorname{supp}_R M = \operatorname{Spec} R \setminus \{\mathfrak{m}\}$.

Proposition 4.4. *Let M be an R-complex in $\mathrm{D}_{-}(R)$.*

(1) *The sets $\operatorname{cosupp}_R M$ and $\operatorname{coSupp}_R M$ have the same minimal elements with respect to containment, i.e., $\min(\operatorname{cosupp}_R M) = \min(\operatorname{coSupp}_R M)$.*
(2) *For an ideal $\mathfrak{a}$ of R, $\operatorname{coSupp}_R M \subseteq \mathrm{V}(\mathfrak{a})$ if and only if $\operatorname{cosupp}_R M \subseteq \mathrm{V}(\mathfrak{a})$.*
(3) *The Zariski closures of $\operatorname{coSupp}_R M$ and $\operatorname{cosupp}_R M$ are equal.*

P r o o f. This follows from Theorems 2.5, 3.2 and [13], Propositions 3.14–3.15. □

Remark 4.5.

(i) Example 4.3 shows that $\operatorname{supp}_R M$ and $\operatorname{supp}_R \mathrm{H}(M)$ need not coincide and $\operatorname{cosupp}_R M$ and $\operatorname{cosupp}_R \mathrm{H}(M)$ need not coincide.
(ii) For any $M \in \mathrm{D}_{-}(R)$, $\operatorname{cosupp}_R M$ may not be a specialization closed subset.
(iii) Example 2.8 and Proposition 4.4 (1) show that the converse of (1) in Proposition 3.6 does not hold in general.

Proposition 4.6.

(1) *If M is in $\mathrm{D}^{\mathrm{n}}_{-}(R)$, then* $\mathrm{cosupp}_R M \subseteq \text{co-supp}_R M$ *and* $\mathrm{coSupp}_R M \subseteq \text{Co-supp}_R M$.

(2) *Assume that R is a semi-local ring and $M \in \mathrm{D}_{-}(R)$. If each $\mathrm{H}_i(M)$ is a Matlis reflexive R-module, then* $\text{co-supp}_R M = \mathrm{cosupp}_R M$ *and* $\text{Co-supp}_R M = \mathrm{coSupp}_R M$.

P r o o f. (1) Since $M \in \mathrm{D}^{\mathrm{n}}_{-}(R)$, it follows that $\mathrm{cosupp}_R M \subseteq \mathrm{coSupp}_R M \subseteq \mathrm{Max}\, R$. Hence, Proposition 4.1 implies that $\mathrm{cosupp}_R M \subseteq \text{co-supp}_R M$. Note that $\mathrm{cosupp}_R M = \mathrm{coSupp}_R M$ and $\text{co-supp}_R M \subseteq \text{Co-supp}_R M$, so $\mathrm{coSupp}_R M \subseteq \text{Co-supp}_R M$.

(2) As $\mathrm{H}_i(M) \cong D_R(D_R(\mathrm{H}_i(M)))$ for all i, $M \simeq D_R(D_R(M))$. Hence,

$$\text{co-supp}_R M = \text{co-supp}_R D_R(D_R(M)) = \mathrm{supp}_R D_R(M) = \mathrm{cosupp}_R M$$

by [13], Propositions 6.1 and Theorem 3.2 and

$$\text{Co-supp}_R M = \text{Co-supp}_R D_R(D_R(M)) = \mathrm{Supp}_R D_R(M) = \mathrm{coSupp}_R M.$$

□

Corollary 4.7. *Assume that R is a semi-local complete ring. If $M \in \mathrm{D}^{\mathrm{n}}_{-}(R)$ or $M \in \mathrm{D}^{\mathrm{a}}_{-}(R)$, then* $\text{co-supp}_R M = \mathrm{cosupp}_R M$ *and* $\text{Co-supp}_R M = \mathrm{coSupp}_R M$.

The example in Remark 2.6 (7) shows that the inclusion in Proposition 4.6 may be strict.

5. Comparison of cosupport and support

This section puts emphasis on the relation between $\mathrm{cosupp}_R M$ and $\mathrm{cosupp}_R \mathrm{H}(M)$. As an application, we give the comparison of the support and cosupport.

Proposition 5.1. *Let $\mathfrak{p}$ be a point in* $\mathrm{Spec}\, R$. *One has that*

(1) $\mathrm{cosupp}_R R = \mathrm{Max}\, R$ *and* $\mathrm{supp}_R R = \mathrm{Spec}\, R$,

(2) $\mathrm{cosupp}_R k(\mathfrak{p}) = \{\mathfrak{p}\} = \mathrm{supp}_R k(\mathfrak{p})$,

(3) $\mathrm{supp}_R E(R/\mathfrak{p}) = \{\mathfrak{p}\}$ *and* $\mathrm{cosupp}_R E(R/\mathfrak{p}) = \mathrm{U}(\mathfrak{p})$.

P r o o f. (1) It follows from Theorem 3.2 and [13], Proposition 3.11 that

$$\mathrm{cosupp}_R R = \mathrm{supp}_R D_R(R) = \mathrm{supp}_R \bigoplus_{\mathfrak{m}} E(R/\mathfrak{m}) = \mathrm{Max}\, R.$$

It follows from Proposition 3.6 that

$$\mathrm{supp}_R R = \mathrm{cosupp}_R D_R(R) = \mathrm{cosupp}_R \bigoplus_{\mathfrak{m}} E(R/\mathfrak{m}) = \mathrm{Spec}\, R.$$

(2) Since $\mathrm{supp}_R k(\mathfrak{p}) = \{\mathfrak{p}\}$, it follows from Proposition 4.1 that $\mathrm{cosupp}_R k(\mathfrak{p}) \subseteq \mathrm{U}(\mathfrak{p})$. On the other hand, $\mathrm{cosupp}_R k(\mathfrak{p}) = \mathrm{cosupp}_R(R/\mathfrak{p} \otimes_R R_\mathfrak{p}) \subseteq \mathrm{V}(\mathfrak{p})$ by Proposition 3.4. Consequently, $\mathrm{cosupp}_R k(\mathfrak{p}) = \{\mathfrak{p}\}$.

(3) Since $\mathfrak{p} \in \text{co-supp}_R E(R/\mathfrak{p})$ if and only if $\mathrm{Hom}_R(k(\mathfrak{p}), E(R/\mathfrak{p})) \neq 0$ by [13], Proposition 4.4 if and only if $\mathrm{Hom}_R(k(\mathfrak{p}), E(R/\mathfrak{p})^{\sim}) \neq 0$ by [14], Theorem 2.17 if and only if $\mathfrak{p} \in \mathrm{cosupp}_R E(R/\mathfrak{p})$, it follows from [13], Proposition 6.3 that the equality holds. □

The next results study the relations between $\mathrm{cosupp}_R M$ (or $\mathrm{supp}_R M$) and $\mathrm{cosupp}_R \mathrm{H}(M)$ (or $\mathrm{supp}_R \mathrm{H}(M)$).

Proposition 5.2.

(1) *For each* $M \in \mathrm{D}_+^{\mathrm{n}}(R)$, *one has* $\mathrm{supp}_R M = \bigcup_{i \in \mathbb{Z}} \mathrm{supp}_R \mathrm{H}_i(M)$.

(2) *If R is semi-local complete, then for* $M \in \mathrm{D}_-^{\mathrm{a}}(R)$, $\mathrm{cosupp}_R M = \bigcup_{i \in \mathbb{Z}} \mathrm{cosupp}_R \mathrm{H}_i(M)$.

P r o o f. We just prove one of the statements since the other is dual.

One has the equalities

$$\mathrm{cosupp}_R M = \mathrm{coSupp}_R M = \bigcup_{i \in \mathbb{Z}} \mathrm{coSupp}_R \mathrm{H}_i(M) = \bigcup_{i \in \mathbb{Z}} \mathrm{cosupp}_R \mathrm{H}_i(M),$$

where the first and the third ones are by Proposition 4.2 and the second one is by Theorem 2.2, as desired. □

Proposition 5.3.

(1) *For each* $M \in \mathrm{D}_-(R)$, *one has* $\mathrm{supp}_R M \subseteq \bigcup_{i \in \mathbb{Z}} \mathrm{supp}_R \mathrm{H}_i(M)$.

(2) *For each* $M \in \mathrm{D}_\mathrm{b}(R)$, *one has* $\mathrm{cosupp}_R M \subseteq \bigcup_{i \in \mathbb{Z}} \mathrm{cosupp}_R \mathrm{H}_i(M)$.

P r o o f. We just prove (1) since (2) follows by duality.

First, let $M \in \mathrm{D}_\mathrm{b}(R)$. If $\inf M = \sup M = r$, then $M \simeq \Sigma^r \mathrm{H}_r(M)$ and $\mathrm{supp}_R M \subseteq \mathrm{supp}_R \mathrm{H}_r(M)$. Assume that $\sup M - \inf M > 0$. The exact triangle $\sigma_{\geqslant \inf M+1}(M) \to M \to \Sigma^{\inf M} \mathrm{H}_{\inf M}(M) \rightsquigarrow$ yields that

$$\mathrm{supp}_R M \subseteq \mathrm{supp}_R \sigma_{\geqslant \inf M+1}(M) \cup \mathrm{supp}_R \mathrm{H}_{\inf M}(M).$$

But $\mathrm{supp}_R \sigma_{\geqslant \inf M+1}(M) \subseteq \bigcup_{i \in \mathbb{Z}} \mathrm{supp}_R \mathrm{H}_i(\sigma_{\geqslant \inf M+1}(M)) = \bigcup_{i \geqslant \inf M+1} \mathrm{supp}_R \mathrm{H}_i(M)$ by induction, so $\mathrm{supp}_R M \subseteq \bigcup_{i \in \mathbb{Z}} \mathrm{supp}_R \mathrm{H}_i(M)$. Now let $M \in \mathrm{D}_-(R)$. Then $M = \overset{\lim_{\sigma \geqslant n}}{\longrightarrow} (M)$. Since $\mathrm{supp}_R M \subseteq \bigcup_{n \leqslant 0} \mathrm{supp}_R \sigma_{\geqslant n}(M)$ and $\mathrm{supp}_R \sigma_{\geqslant n}(M) \subseteq \bigcup_{i \geqslant n} \mathrm{supp}_R \mathrm{H}_i(M)$, it follows that $\mathrm{supp}_R M \subseteq \bigcup_{i \in \mathbb{Z}} \mathrm{supp}_R \mathrm{H}_i(M)$. □

Corollary 5.4.

(1) *For each* $M \in \mathrm{D}^{\mathrm{n}}_{\mathrm{b}}(R)$, *one has that* $\operatorname{cosupp}_R M \subseteq \operatorname{supp}_R M$.

(2) *If* R *is a semi-local complete ring, then for* $M \in \mathrm{D}^{\mathrm{a}}_{\mathrm{b}}(R)$, $\operatorname{supp}_R M \subseteq \operatorname{cosupp}_R M$.

Proof. We just prove (1) since (2) follows by duality.

By Proposition 5.3 (2), $\operatorname{cosupp}_R M \subseteq \bigcup\limits_{i\in\mathbb{Z}} \operatorname{cosupp}_R \mathrm{H}_i(M)$. But $\mathrm{H}_i(M)$ is noetherian and $\operatorname{coSupp}_R \mathrm{H}_i(M) \subseteq \operatorname{Max} R$, so it follows from Propositions 4.1 and 5.2 that $\bigcup\limits_{i\in\mathbb{Z}} \operatorname{cosupp}_R \mathrm{H}_i(M) \subseteq \bigcup\limits_{i\in\mathbb{Z}} \operatorname{supp}_R \mathrm{H}_i(M) = \operatorname{supp}_R M$, as claimed. □

Remark 5.5.

(i) The assumption $M \in \mathrm{D}^{\mathrm{n}}_{\mathrm{b}}(R)$ in (1) and $M \in \mathrm{D}^{\mathrm{a}}_{\mathrm{b}}(R)$ in (2) in Corollary 5.4 are essential. For example, assume that $(R, \mathfrak{m})$ is local and not artinian. One has

$$\operatorname{supp}_R E(R/\mathfrak{m}) = \{\mathfrak{m}\} \subsetneq \operatorname{Spec} R = \operatorname{cosupp}_R E(R/\mathfrak{m}),$$
$$\operatorname{cosupp}_R R = \{\mathfrak{m}\} \subsetneq \operatorname{Spec} R = \operatorname{supp}_R R.$$

(ii) Proposition 5.1 (1) and (3) show that one can have proper containment or equality in the above corollary.

6. Coassociated prime for complexes

The aim of this section is to develop a theory dual to that of associated primes of complexes introduced by Christensen in [4], and find an extension of Nakayama lemma.

Let $(R, \mathfrak{m}, k)$ be a local ring and M an R-complex. The depth of M is

$$\operatorname{depth}_R M := -\sup \operatorname{RHom}_R(k, M).$$

Following [4], we say that $\mathfrak{p} \in \operatorname{Spec} R$ is an associated prime ideal for $M \in \mathrm{D}_-(R)$ if $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = -\sup M_\mathfrak{p} < \infty$, that is,

$$\operatorname{Ass}_R M := \{\mathfrak{p} \in \operatorname{Supp}_R M \colon \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = -\sup M_\mathfrak{p}\}.$$

For $M \not\simeq 0$ in $\mathrm{D}_-(R)$, we set $\operatorname{ass}_R M = \operatorname{Ass}_R \mathrm{H}_{\sup M}(M)$.

Let $(R, \mathfrak{m}, k)$ be a local ring and M an R-complex. The width of M is

$$\operatorname{width}_R M := \inf\Big(k \overset{\mathrm{L}}{\underset{R}{\bigotimes}} M\Big).$$

Definition 6.1. (1) We say that $\mathfrak{p} \in \operatorname{Spec} R$ is a coassociated prime ideal for $M \in \mathrm{D}_+(R)$ if $\operatorname{width}_{R_\mathfrak{p}} {}^{\mathfrak{p}}M = \inf {}^{\mathfrak{p}}M > -\infty$, that is,

$$\operatorname{Coass}_R M := \{\mathfrak{p} \in \operatorname{coSupp}_R M \colon \operatorname{width}_{R_\mathfrak{p}} {}^{\mathfrak{p}}M = \inf {}^{\mathfrak{p}}M\}.$$

(2) For an R-complex $M \not\simeq 0$ in $\mathrm{D}_+(R)$, we set $\mathrm{coass}_R M = \mathrm{Coass}_R \mathrm{H}_{\inf M}(M)$ and for $M \simeq 0$ we set $\mathrm{Coass}_R M = \emptyset$.

Theorem 6.2. *Let $M \in \mathrm{D}_+(R)$. Then $\mathfrak{p} \in \mathrm{Coass}_R M$ if and only if $\mathfrak{p} \in \mathrm{Ass}_R D_R(M)$. In particular, $M \not\simeq 0$ if and only if $\mathrm{Coass}_R M \neq \emptyset$.*

Proof. Since ${}^{\mathfrak{p}}M = \mathrm{Hom}_{R_{\mathfrak{p}}}(D_R(M)_{\mathfrak{p}}, E_{R_{\mathfrak{p}}}(k(\mathfrak{p})))$, it follows that

$$-\sup D_R(M)_{\mathfrak{p}} = \inf {}^{\mathfrak{p}}M = i$$

is finite. One has the equivalences

$$\begin{aligned}
\mathfrak{p} \in \mathrm{Coass}_R M &\Leftrightarrow \inf(k(\mathfrak{p}) \overset{\mathrm{L}}{\underset{R_{\mathfrak{p}}}{\bigotimes}} {}^{\mathfrak{p}}M) = \inf {}^{\mathfrak{p}}M = i \Leftrightarrow k(\mathfrak{p}) \underset{R_{\mathfrak{p}}}{\bigotimes} \mathrm{H}_i({}^{\mathfrak{p}}M) \neq 0 \\
&\Leftrightarrow k(\mathfrak{p}) \underset{R_{\mathfrak{p}}}{\bigotimes} \mathrm{Hom}_{R_{\mathfrak{p}}}(D_R(\mathrm{H}_i(M))_{\mathfrak{p}}, E(k(\mathfrak{p}))) \neq 0 \\
&\Leftrightarrow \mathrm{Hom}_{R_{\mathfrak{p}}}(\mathrm{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), D_R(\mathrm{H}_i(M))_{\mathfrak{p}}), E(k(\mathfrak{p}))) \neq 0 \\
&\Leftrightarrow \mathrm{H}_{-i}(\mathrm{RHom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), D_R(M)_{\mathfrak{p}}) = \mathrm{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), \mathrm{H}_{-i}(D_R(M)_{\mathfrak{p}})) \neq 0 \\
&\Leftrightarrow \mathfrak{p}R_{\mathfrak{p}} \in \mathrm{Ass}_{R_{\mathfrak{p}}} \mathrm{H}_{-i}(D_R(M)_{\mathfrak{p}}) \Leftrightarrow \mathfrak{p} \in \mathrm{Ass}_R D_R(M),
\end{aligned}$$

where the second one is by [5], Lemma 2.4.14, the third one is by injectivity of $E(R/\mathfrak{m})$ and $E(k(\mathfrak{p}))$ and flatness of $R_{\mathfrak{p}}$, the fourth one is by [5], Theorem 2.5.6, the fifth one is by faithful injectivty of $E(k(\mathfrak{p}))$ and [5], Lemma 2.3.19, the last one is by [4], Observations 2.4. □

Remark 6.3.

(1) Let K be an R-module. By Theorem 6.2, $\mathfrak{p} \in \mathrm{Coass}_R K$ if and only if $\mathfrak{p} \in \mathrm{Ass}_R D_R(K)$ if and only if $\mathfrak{p}R_{\mathfrak{p}} \in \mathrm{Ass}_{R_{\mathfrak{p}}} D_R(K)_{\mathfrak{p}}$ if and only if $\mathfrak{p}R_{\mathfrak{p}} \in \mathrm{Coass}_{R_{\mathfrak{p}}} {}^{\mathfrak{p}}K$ since the morphism $k(\mathfrak{p}) \to D_R(K)_{\mathfrak{p}}$ is injective if and only if the morphism ${}^{\mathfrak{p}}K \to k(\mathfrak{p}) \cong \mathrm{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), E_{R_{\mathfrak{p}}}(k(\mathfrak{p})))$ is surjective.

(2) Let $M \in \mathrm{D}_+(R)$ and $\mathfrak{p} \in \mathrm{coSupp}_R M$ and set $\inf {}^{\mathfrak{p}}M = i$. Then

$$\begin{aligned}
\mathfrak{p} \in \mathrm{Coass}_R M &\Leftrightarrow \mathfrak{p} \in \mathrm{Ass}_R D_R(M) \Leftrightarrow \mathfrak{p}R_{\mathfrak{p}} \in \mathrm{Ass}_{R_{\mathfrak{p}}} D_R(\mathrm{H}_i(M))_{\mathfrak{p}} \\
&\Leftrightarrow \mathfrak{p}R_{\mathfrak{p}} \in \mathrm{Coass}_{R_{\mathfrak{p}}} \mathrm{H}_i({}^{\mathfrak{p}}M) \Leftrightarrow \mathfrak{p}R_{\mathfrak{p}} \in \mathrm{coass}_{R_{\mathfrak{p}}} {}^{\mathfrak{p}}M \\
&\Leftrightarrow \mathfrak{p} \in \mathrm{Coass}_R \mathrm{H}_i(M),
\end{aligned}$$

where the second equivalence is by [4], Observations 2.4 and the third one is by (1). In particular, one has the following inclusion:

$$\mathrm{coass}_R M \subseteq \mathrm{Coass}_R M.$$

(3) For $M \in \mathrm{D}_+(R)$, every minimal prime in $\mathrm{coSupp}_R M$ belongs to $\mathrm{Coass}_R M$; for $N \in \mathrm{D}_{\mathrm{b}}(R)$, one has $\mathrm{Coass}_R N \subseteq \mathrm{cosupp}_R N$ by [4], Proposition 2.6, Theorems 3.2 and 6.2.
(4) For $M \in \mathrm{D}_{\mathrm{b}}^{\mathrm{a}}(R)$, the set of minimal prime in $\mathrm{coSupp}_R M$ is finite by Corollary 2.4.
(5) If M is an R-module, then $\mathrm{coass}_R M = \mathrm{Coass}_R M$.

The following result is an extension of Nakayama lemma.

Proposition 6.4. *Let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a} \subseteq J(R)$, the Jacobson radical of R.*
(1) *If M is in $\mathrm{D}_-(R)$ such that $\mathrm{Ass}_R M \cap \mathrm{Max}\, R \neq \emptyset$, then $\mathrm{RHom}_R(R/\mathfrak{a}, M) \not\simeq 0$.*
(2) *If M is in $\mathrm{D}_+(R)$ such that $\mathrm{Coass}_R M \cap \mathrm{Max}\, R \neq \emptyset$, then $R/\mathfrak{a} \overset{\mathrm{L}}{\underset{R}{\otimes}} M \not\simeq 0$.*

Proof. (1) Given $\mathfrak{m} \in \mathrm{Ass}_R M \cap \mathrm{Max}\, R$ and set $s = \sup M_{\mathfrak{m}}$. Then

$$\mathrm{H}_s(\mathrm{RHom}_{R_{\mathfrak{m}}}(k(\mathfrak{m}), M_{\mathfrak{m}})) \cong \mathrm{Hom}_{R_{\mathfrak{m}}}(k(\mathfrak{m}), \mathrm{H}_s(M_{\mathfrak{m}})) \neq 0$$

by [5], Lemma 2.3.19 and hence,

$$\mathrm{H}_s(\mathrm{RHom}_{R_{\mathfrak{m}}}((R/\mathfrak{a})_{\mathfrak{m}}, M_{\mathfrak{m}})) \cong \mathrm{Hom}_{R_{\mathfrak{m}}}((R/\mathfrak{a})_{\mathfrak{m}}, \mathrm{H}_s(M_{\mathfrak{m}})) \neq 0$$

since the map $(R/\mathfrak{a})_{\mathfrak{m}} \twoheadrightarrow (R/\mathfrak{m})_{\mathfrak{m}}$ is surjective, which implies that

$$\mathrm{RHom}_R(R/\mathfrak{a}, M) \not\simeq 0.$$

(2) By Theorem 6.2, $\mathrm{Ass}_R D_R(M) \cap \mathrm{Max}\, R \neq \emptyset$. Hence, $\mathrm{RHom}_R(R/\mathfrak{a}, D_R(M)) \not\simeq 0$ by (1), which implies that $R/\mathfrak{a} \overset{\mathrm{L}}{\underset{R}{\otimes}} M \not\simeq 0$. □

Appendix: different cosupport of modules

The notion of support is a fundamental concept which provides a geometric approach for studying various algebraic structures.

There have been three earlier attempts to dualize the theory of support of modules. Since $S^{-1}(-) = S^{-1}R \underset{R}{\otimes} -$, the first one was made by Melkersson and Schenzel (see [9]) by choosing $S_{-1}(-)$ to be $\mathrm{Hom}_R(S^{-1}R, -)$, and defined the cosupport of an R-module K to be the set

$$\{\mathfrak{p} \in \mathrm{Spec}\, R \colon \mathrm{Hom}_R(R_{\mathfrak{p}}, K) \neq 0\}.$$

The theory of cosupport is particularly well-behaved when restricted to the class of artinian modules. However, this theory does not work at all well for non-artinian modules. For example, if S is a multiplicatively closed set of integers which includes a nonunit, then $\mathrm{Hom}_{\mathbb{Z}}(S^{-1}\mathbb{Z}, \mathbb{Z}) = 0$, that is to say, the cosupport of $\mathbb{Z}$ is empty under this definition, which is definitely not what we want.

Next, by introducing a notion of cocyclic modules, Yassemi in [14] defined the cosupport of an R-module K as the set of prime ideals $\mathfrak{p}$ such that there exists a cocyclic homomorphic image L of K with $\mathfrak{p} \supseteq \mathrm{Ann}_R L$, the annihilator of L, and denoted this set by $\mathrm{Cosupp}_R K$. An R-module L is cocyclic if L is a submodule of $E(R/\mathfrak{m})$ for some $\mathfrak{m} \in \mathrm{Max}\, R$. He showed that for an artinian R-module this definition is equivalent with the definition provided by Melkersson and Schenzel in [9], and proved some properties for cosupport that are similar—or rather dual—to those of support.

Lemma A.1. ([14]). *For any R-module K, the following are equivalent:*

(1) $\mathfrak{p} \in \mathrm{Cosupp}_R K$;

(2) $\mathfrak{p} \in \mathrm{Supp}_R D_{\mathfrak{m}}(K)$ *for some* $\mathfrak{m} \in \mathrm{Max}\, R \cap \mathrm{V}(\mathfrak{p})$;

(3) $\mathrm{Hom}_R(R_{\mathfrak{p}}, K^{\sim}) \neq 0$, *where* $K^{\sim} = \prod_{\mathfrak{m}} D_{\mathfrak{m}}(D_{\mathfrak{m}}(K))$;

(4) $\mathrm{Hom}_R\Big(\bigoplus_{\mathfrak{m}} D_{\mathfrak{m}}(K), E(R/\mathfrak{p})\Big) \neq 0$.

Finally, for an ideal $\mathfrak{a}$ of R, if K is an $\mathfrak{a}$-torsion R-module (i.e., $\mathrm{Supp}_R K \subseteq \mathrm{V}(\mathfrak{a})$), then all the higher local cohomology modules of K vanish, where the local cohomology functors are the right derived functors of the $\mathfrak{a}$-torsion functor, see [7]. Matlis in [8] defined the local homology functors to be the left derived functors of the $\mathfrak{a}$-adic completion functor. Since torsion and completion are dual, one expects these functors to live up to their name and behave in a manner dual to local cohomology. In particular, it is natural to expect there to be vanishing theorems for local homology dual to the ones that relate the local cohomology of a module to the module's support. Therefore, Richardson in [12] created a co-localization functor $S_{-1}(-) = D_{S^{-1}R}(S^{-1}D_R(-))$ which is dual to the localization functor $S^{-1}(-)$, and then defined the cosupport of an R-module K to be the set of primes at which the module's co-localization is nonzero, i.e.,

$$\mathrm{coSupp}_R K := \{\mathfrak{p} \in \mathrm{Spec}\, R\colon\ {}^{\mathfrak{p}}K \neq 0\},$$

where ${}^{\mathfrak{p}}K := \mathrm{Hom}_R(D_R(K), E(R/\mathfrak{p}))$. Richardson also gave some properties of $\mathrm{coSupp}_R K$:

Lemma A.2. ([12]). *For any R-module K, the following statements hold:*

(1) $\operatorname{coSupp}_R K = \emptyset$ *if and only if* $K = 0$.

(2) $\operatorname{coSupp}_R K = \operatorname{Supp}_R D_R(K)$.

(3) $\operatorname{coSupp}_R K \subseteq \mathrm{V}(\operatorname{Ann}_R K)$, *equality holds if K is artinian.*

Remark A.3.

(1) By [2], Remark 4.21, the notion of cosupport of modules provided by Melkersson and Schenzel in [9] is not the same as the Richardson's definition.

(2) If K is an artinian R-module, then all three definitions are equivalent by the remark after Proposition 2.3 of [14] and Lemma A.2.

The next lemma gives a relation between the set $\operatorname{Cosupp}_R K$ and the set $\operatorname{coSupp}_R K$.

Lemma A.4. *Let K be an R-module and $\mathfrak{p}$ a point in* $\operatorname{Spec} R$.

(1) *If $\mathfrak{p} \in \operatorname{Cosupp}_R K$, then $\mathfrak{p} \in \operatorname{coSupp}_R K$.*

(2) *If R is a semi-local ring or K is a finitely generated R-module, then $\mathfrak{p} \in \operatorname{coSupp}_R K$ if and only if $\mathfrak{p} \in \operatorname{Cosupp}_R K$.*

Proof. (1) If $\mathfrak{p} \in \operatorname{Cosupp}_R K$, then $\operatorname{Hom}_R\Big(\bigoplus_{\mathfrak{m}} D_{\mathfrak{m}}(K), E(R/\mathfrak{p})\Big) \neq 0$ by Lemma A.1, and hence, $\operatorname{Hom}_R(D_{\mathfrak{m}}(K), E(R/\mathfrak{p})) \neq 0$ for some $\mathfrak{m} \in \operatorname{Max} R$. Consider the exact sequence $0 \to E(R/\mathfrak{m}) \to \bigoplus_{\mathfrak{n} \in \operatorname{Max} R} E(R/\mathfrak{n}) \to \bigoplus_{\mathfrak{m} \neq \mathfrak{m}' \in \operatorname{Max} R} E(R/\mathfrak{m}') \to 0$, which induces the following short exact sequence:

$$0 \to \operatorname{Hom}_R\Big(\operatorname{Hom}_R\Big(K, \bigoplus_{\mathfrak{m} \neq \mathfrak{m}' \in \operatorname{Max} R} E(R/\mathfrak{m}')\Big), E(R/\mathfrak{p})\Big)$$
$$\to \operatorname{Hom}_R(D_R(K), E(R/\mathfrak{p})) \to \operatorname{Hom}_R(D_{\mathfrak{m}}(K), E(R/\mathfrak{p})) \to 0.$$

Thus, $\operatorname{Hom}_R(D_R(K), E(R/\mathfrak{p})) \neq 0$, and so ${}^{\mathfrak{p}}K \neq 0$ and $\mathfrak{p} \in \operatorname{coSupp}_R K$.

(2) If R is semi-local or K is finitely generated, then

$${}^{\mathfrak{p}}K \cong \operatorname{Hom}_R\Big(\bigoplus_{\mathfrak{m}} D_{\mathfrak{m}}(K), E(R/\mathfrak{p})\Big).$$

Hence, the equivalence follows from Lemma A.1. □

Lemma A.5. *Let K be an R-module and $\mathfrak{p}$ a point in* $\operatorname{Spec} R$. *If $\mathfrak{p} \in \operatorname{Supp}_R K$, then $\mathfrak{p} \in \operatorname{coSupp}_R D_R(K)$. The converse holds when K is finitely generated.*

Proof. Since $\mathfrak{p} \in \operatorname{Supp}_R K$, $\mathfrak{p} \in \operatorname{Cosupp}_R D_{\mathfrak{m}}(K)$ for some $\mathfrak{m} \in \operatorname{Max} R \cap \mathrm{V}(\mathfrak{p})$ by [14], Lemma 2.8, it follows from the exact sequence

$$0 \to \operatorname{Hom}_R\Big(K, \bigoplus_{\mathfrak{m} \neq \mathfrak{m}' \in \operatorname{Max} R} E(R/\mathfrak{m}')\Big) \to D_R(K) \to D_{\mathfrak{m}}(K) \to 0$$

and [14], Theorem 2.7 that $\mathfrak{p} \in \mathrm{Cosupp}_R D_R(K)$. Consequently, $\mathfrak{p} \in \mathrm{coSupp}_R D_R(K)$ by Lemma A.4. Conversely, if $\mathfrak{p} \in \mathrm{coSupp}_R D_R(K)$, then

$$0 \neq {}^{\mathfrak{p}}D_R(K) \cong \mathrm{Hom}_{R_{\mathfrak{p}}}\Big(K_{\mathfrak{p}}, {}^{\mathfrak{p}}\Big(\bigoplus_{\mathfrak{m}} E(R/\mathfrak{m})\Big)\Big)$$

by [12], Proposition 2.5 as K is finitely generated, which implies that $K_{\mathfrak{p}} \neq 0$. Therefore, one has $\mathfrak{p} \in \mathrm{Supp}_R K$. □

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