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RECOLLEMENTS INDUCED BY GOOD (CO)SILTING DG-MODULES

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Abstract. Let U be a dg-A-module, B the endomorphism dg-algebra of U. We know that if U is a good silting object, then there exist a dg-algebra C and a recollement among the derived categories $\mathbf{D}(C,d)$ of C, $\mathbf{D}(B,d)$ of B and $\mathbf{D}(A,d)$ of A. We investigate the condition under which the induced dg-algebra C is weak nonpositive. In order to deal with both silting and cosilting dg-modules consistently, the notion of weak silting dg-modules is introduced. Thus, similar results for good cosilting dg-modules are obtained. Finally, some applications are given related to good 2-term silting complexes, good tilting complexes and modules.

Keywords: silting object; dg-algebra; cosilting dg-module; recollement

MSC 2020: 16E45, 16D90, 18G80

1. INTRODUCTION

Infinitely generated tilting modules over arbitrary associated rings have attracted increasing attention towards understanding derived categories and equivalences of general rings, see [4], [5], [6], [13], [14], [27]. It is shown in [4] that if T is a good tilting module over a ring A, the right derived functor $\mathbb{R}\mathrm{Hom}_A(T,-)$ induces an equivalence between the derived category $\mathbf{D}(A)$ and a subcategory of the derived category $\mathbf{D}(B)$, where B is the endomorphism algebra of T. Thus, in general, the right derived functor $\mathbb{R}\mathrm{Hom}_A(T,-)$ does not define a derived equivalence between A and B. If we considered good tilting modules of projective dimension at most one, in this case, Chen and Xi proved that the triangulated category $\mathrm{Ker}\Big(T \overset{\mathbb{L}}{\underset{B}{\otimes}} -\Big)$ is equivalent to the derived category of a ring C, and therefore, there is a recollement among the derived categories of rings A, B and C, see [13], Theorem 1.1. Recently,

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Theorem 1.1 of [14] gives a necessary and sufficient condition for good tilting modules of higher projective dimension to induce recollements of derived module categories via homological ring epimorphisms.

As an extension of tilting theory, silting theory encompasses methods for studying derived equivalences which are widely employed in many areas of research, see [1], [3]. Wei introduced in [28] the notion of semi-tilting complexes, which is a generalization of tilting complexes, and proved that semi-tilting complexes induce derived equivalences between dg-algebras. Compact silting objects were also considered in abstract triangulated categories and noncompact silting objects and their associated t-structures were studied in derived module categories, see [3], [18], [22]. In order to get the silting theorem for noncompact general silting complexes, Breaz and Modoi in [11] defined big, small and good n-silting objects in $\mathbf{D}(A,d)$, where A is a dg-algebra. Note that the notion of an n-silting object here agrees to the notion of n-semitilting complex in [28]. Let U be a good silting dg-A-module and $B = \mathrm{DgEnd}_A(U)$. Under some fairly general appropriate hypotheses, they proved that it induces derived equivalences between the derived category over A and a subcategory $\mathcal{K}$ of the derived category of dg-endomorphism algebra B of U, where $\mathcal{K} = \mathrm{Ker}\Bigl(-\overset{\mathbb{L}}{\underset{B}{\otimes}} U\Bigr)$, see [11], Theorem 2.4. Recently, this result was extended by Nicolás and Saorín (see [21]) to the context of derived categories of dg categories. Moreover, given small dg categories $\mathcal{A}$ and $\mathcal{B}$ and a $\mathcal{B}$-$\mathcal{A}$-bimodule T, they expressed $\mathbf{D}(\mathcal{B})$ as a recollement of $\mathbf{D}(\mathcal{A})$ and the derived category of another dg category.

The main purpose of this paper is to extend the interesting results (see [13], Theorem 1.1 and [14], Theorem 1.1) to good (co)silting dg-modules over dg-algebras. More precisely, we are going to provide answers for the following questions.

(1) Let U be a good silting dg-A-module. Is there a dg-algebra C such that the subcategory $\mathrm{Ker}\Bigl(-\overset{\mathbb{L}}{\underset{B}{\otimes}} U\Bigr)$ is equivalent to the derived category of C, and is there a recollement among the derived categories of dg-algebra A, B and C?

(2) Starting with the good silting dg-module U, then we have the recollement and note that the dg-algebra B has to be weak nonpositive. Under what extent does C satisfy the same property?

(3) Whether the existence of such a recollement implies goodness of silting objects?

(4) How can we get similar results for good cosilting dg-modules over dg-algebras?

Actually, answer for the question (1) is positive. According to Corollary 6.7 of [21], such a recollement exists under the hypothesis that A belongs to the smallest thick subcategory containing U, see also [29], Theorem 1. Note that this hypothesis is equivalent to the one assuming that U is good silting, see Remark 3.1 (1).

For the question (2), we give an equivalent condition for the induced dg-algebra C to be weak nonpositive, explicitly, the dg-algebra C induced in Proposition 3.3 is weak nonpositive if and only if $H^i\Big(U\overset{\mathbb{L}}{\underset{A}{\bigotimes}}\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(U,B)\Big)=0$ for $i\geqslant 2$ or, equivalently, $H^i\Big(U\overset{\mathbb{L}}{\underset{A}{\bigotimes}}\mathbb{R}\mathrm{Hom}_A(U,A)\Big)=0$ for $i\geqslant 2$, see Theorem 3.7. Moreover, we show that the existence of such a recollement indeed implies that the given silting object U is good, which gives a positive answer to the question (3), see Proposition 3.4. To deal with both tilting and cotilting modules consistently, the notion of weak tilting module was introduced, see [14], Definition 4.1. Inspired by this work, we introduce the notion of weak tilting dg-modules and good cosilting dg-modules (see Definitions 4.1 and 4.6), and show that a weak silting object always induces a recollement among derived categories of dg-algebras, see Proposition 4.3. Using this result, we get an answer to the question (4) and obtain the similar recollement induced by a good cosilting dg-module, see Theorem 4.9.

The paper is organized as follows. We start in Section 2 with some basics about the dg-algebras and their derived categories. In Section 3, we investigate under what extent does the induced dg-algebra C weak nonpositive. In Section 4, we introduce the notion of weak silting dg-modules and show that there exists a recollement induced by weak silting dg-modules. Thus, similar results for the good cosilting dg-module are obtained. In Section 5, some applications are given related to good 2-term silting complexes, good tilting complexes and modules.

2. Preliminaries

Now we introduce some notations and conventions used later in the paper.

2.1. Differential graded algebras and differential graded modules. A good reference for dg-algebras and their derived categories is the book, see [30]. Let k be a commutative ring. Recall that a dg-algebra is a $\mathbb{Z}$-graded k-algebra $A=\bigoplus_{i\in\mathbb{Z}}A^i$ endowed with a differential $d\colon A\to A$ such that $d^2=0$ which is homogeneous of degree 1, that is $d(A^i)\subseteq A^{i+1}$ for all $i\in\mathbb{Z}$, and satisfies the graded Leibniz rule

$$d(ab)=d(a)b+(-1)^i ad(b)\quad \text{for all } a\in A^i \text{ and } b\in A.$$

A (right) dg-module over A is a $\mathbb{Z}$-graded module $M=\bigoplus_{i\in Z}M^i$ endowed with a k-linear square-zero differential $d\colon M\to M$, which is homogeneous of degree 1 and satisfies the graded Leibniz rule

$$d(ma)=d(m)a+(-1)^i md(a)\quad \text{for all } m\in M^i \text{ and } a\in A.$$

Left dg-A-modules are defined similarly. A morphism of dg-A-modules is an A-linear map $f\colon M \to N$ compatible with gradings and differentials. In this way we obtain the category $\mathrm{Mod}(A,d)$ of all dg-A-modules.

If A is a dg-algebra, then the dual dg-algebra A^{op} is defined as follows: as graded k-modules $A^{\mathrm{op}} = A$, the multiplication is given by $ab = (-1)^{ij}ba$ for all $a \in A^i$ and all $b \in A^j$ and the differential $d\colon A^{\mathrm{op}} \to A^{\mathrm{op}}$ is the same as in the case of A. A dg-algebra A is called *nonpositive* if $A^i = 0$ for all $i > 0$. A dg-algebra A is called *weak nonpositive* if $H^i(A) = 0$ for all $i > 0$ and *weak positive* if $H^i(A) = 0$ for all $i < 0$.

For a dg-module $X \in \mathrm{Mod}(A,d)$ one introduces (functorially) the following k-modules: $Z^n(X) = \mathrm{Ker}(X^n \to X^{n+1})$, $B^n(X) = \mathrm{Im}(X^{n-1} \to X^n)$, and $H^n(X) = Z^n(X)/B^n(X)$ for all $n \in \mathbb{Z}$. We call $H^n(X)$ the nth cohomology group of X. A morphism of dg-modules is called *quasi-isomorphism* if it induces isomorphisms in cohomologies. A dg-module $X \in \mathrm{Mod}(A,d)$ is called *acyclic* if $H^n(X) = 0$ for all $n \in \mathbb{Z}$. The category obtained from $\mathrm{Mod}(A,d)$ by identifying homotopic morphisms is called the *homotopy category* of right dg-modules over A and is denoted by $\mathbf{K}(A,d)$. The category obtained from $\mathrm{Mod}(A,d)$ by formally inverting quasi-isomorphisms is called the *derived category* of right dg-modules over A and is denoted by $\mathbf{D}(A,d)$. Both $\mathbf{K}(A,d)$ and $\mathbf{D}(A,d)$ are triangulated categories.

Let now A and B be two dg-algebras and let U be a dg-B-A-bimodule. For every $X \in \mathrm{Mod}(A,d)$, we can consider the so called dg-Hom:

$$\mathrm{Hom}^\bullet_A(U,X) = \prod_{n\in\mathbb{Z}} \mathrm{Hom}^n_A(U,X)$$

with $\mathrm{Hom}^n_A(U,X) = \prod\limits_{i\in\mathbb{Z}} \mathrm{Hom}_{A^0}(U^i, X^{n+i})$, whose differentials are given by $d(f)(x) = d_Y f(x) - (-1)^n f d_X(x)$ for all $f \in \mathrm{Hom}^n_A(X,Y)$. Then $\mathrm{Hom}^\bullet_A(U,X)$ becomes a dg-B-module, so we get a functor $\mathrm{Hom}^\bullet_A(U,-)\colon \mathrm{Mod}(A,d) \to \mathrm{Mod}(B,d)$. It induces a triangle functor $\mathrm{Hom}^\bullet_A(U,-)\colon \mathbf{K}(A,d) \to \mathbf{K}(B,d)$. A dg-$A$-module P (or I) is called *$\mathcal{H}$-projective* (or *$\mathcal{H}$-injective*) if $\mathrm{Hom}_{\mathbf{K}(A,d)}(P,N) = 0$ (or $\mathrm{Hom}_{\mathbf{K}(A,d)}(N,I) = 0$, respectively) for all acyclic dg-A-modules N. For every dg-A-module X there is an $\mathcal{H}$-projective dg-A-module pX and an $\mathcal{H}$-injective dg-A-module iX such that X is quasi-isomorphic to pX and to iX. We put

$$\mathbb{R}\mathrm{Hom}_A(U,-)\colon \mathbf{D}(A,d) \to \mathbf{D}(B,d)$$

by $\mathbb{R}\mathrm{Hom}_A(U,X) = \mathrm{Hom}^\bullet_A(pU,X) \cong \mathrm{Hom}^\bullet_A(U,iX)$.

Let $Y \in \mathrm{Mod}(B,d)$. There exists a natural grading on the usual tensor product $Y \bigotimes\limits_B U$, which can be described as $Y \overset{\bullet}{\bigotimes\limits_B} U = \bigoplus\limits_{n\in\mathbb{Z}} Y \overset{n}{\bigotimes\limits_B} U$, where $Y \overset{n}{\bigotimes\limits_B} U$ is the quotient of $\bigoplus_{i\in\mathbb{Z}} Y^i \bigotimes_{B^0} U^{n-i}$ by the submodule generated by $y \bigotimes bu - yb \bigotimes u$, where $y \in Y^i$, $u \in U^j$ and $b \in B^{n-i-j}$ for all $i, j \in \mathbb{Z}$. Together with the differential $d(yu) = d(y)u + (-1)^i y d(u)$ for all $y \in Y^i$, $u \in U$, we get a dg-A-module inducing a functor $- \overset{\bullet}{\underset{B}{\bigotimes}} U\colon \operatorname{Mod}(B, d) \to \operatorname{Mod}(A, d)$ and further a triangle functor $- \overset{\bullet}{\underset{B}{\bigotimes}} U\colon \mathbf{K}(B, d) \to \mathbf{K}(A, d)$. The left derived tensor product is defined by $Y \overset{\mathbb{L}}{\underset{B}{\bigotimes}} U = pY \overset{\bullet}{\underset{B}{\bigotimes}} U \cong Y \overset{\bullet}{\underset{B}{\bigotimes}} pU$. It induces a triangle functor

$$- \overset{\mathbb{L}}{\underset{B}{\bigotimes}} U\colon \mathbf{D}(B, d) \to \mathbf{D}(A, d)$$

which is the left adjoint of $\mathbb{R}\mathrm{Hom}_A(U, -)$.

2.2. Dimensions and triangulated subcategories. Let $\mathcal{C}$ be an additive category. Throughout the paper, a full subcategory $\mathcal{B}$ of $\mathcal{C}$ is always assumed to be closed under isomorphisms. We denote by $\operatorname{add}(X)$ the full subcategory of $\mathcal{C}$ consisting of all direct summands of finite coproducts of copies of X. If $\mathcal{C}$ admits small coproducts, then we denote by $\operatorname{Add}(X)$ the full subcategory of $\mathcal{C}$ consisting of all direct summands of small coproducts of copies of X. Dually, if $\mathcal{C}$ admits small products, then $\operatorname{Prod}(X)$ denotes the full subcategory of $\mathcal{C}$ consisting of all direct summands of small products of copies of X.

Let $\mathcal{D}$ be a triangulated category with the ith shift functor denoted by $[i]$ and let $\mathcal{C}$ be a subcategory of $\mathcal{D}$. We define the full subcategories of $\mathcal{D}$:

$$\begin{aligned}
\mathcal{C}^{\perp} &:= \{X \in \mathcal{D}\colon \operatorname{Hom}_{\mathcal{D}}(C, X[i]) = 0 \quad \text{for all } i \in \mathbb{Z} \text{ and } C \in \mathcal{C}\};\\
{}^{\perp}\mathcal{C} &:= \{X \in \mathcal{D}\colon \operatorname{Hom}_{\mathcal{D}}(X, C[i]) = 0 \quad \text{for all } i \in \mathbb{Z} \text{ and } C \in \mathcal{C}\};\\
\mathcal{C}^{\perp_{>0}} &:= \{X \in \mathcal{D}\colon \operatorname{Hom}_{\mathcal{D}}(U, X[i]) = 0 \quad \text{for all } i > 0 \text{ and } C \in \mathcal{C}\};\\
{}^{\perp_{>0}}\mathcal{C} &:= \{X \in \mathcal{D}\colon \operatorname{Hom}_{\mathcal{D}}(X, C[i]) = 0 \quad \text{for all } i > 0 \text{ and } C \in \mathcal{C}\}.
\end{aligned}$$

Consider an object $X \in \mathcal{D}$. Following [28], we say that X has the $\mathcal{C}$-resolution dimension (or $\mathcal{C}$-coresolution dimension) not greater than n and we write $\dim_{\mathcal{C}} X \leqslant n$ (or $\operatorname{codim}_{\mathcal{C}} X \leqslant n$) provided that there is a sequence of triangles

$$\begin{gathered}
X_{i+1} \to C_i \to X_i \rightsquigarrow \text{ with } 0 \leqslant i \leqslant n\\
(\text{or } X_i \to C_i \to X_{i+1} \rightsquigarrow \text{ with } 0 \leqslant i \leqslant n, \text{ respectively})
\end{gathered}$$

in D, such that $C_i \in \mathcal{C}$, $X_0 = X$ and $X_{n+1} = 0$. We write $\dim_{\mathcal{C}} X < \infty$ (or $\operatorname{codim}_{\mathcal{C}} X < \infty$) if we can find a positive integer n such that $\dim_{\mathcal{C}} X \leqslant n$ (or $\operatorname{codim}_{\mathcal{C}} X \leqslant n$, respectively).

Given a class of objects $\mathcal{U}$ in $\mathcal{D}$, we denote by $\mathrm{Tria}(\mathcal{U})$ the smallest full triangulated subcategory of $\mathcal{D}$ which contains $\mathcal{U}$ and is closed under small coproducts. If $\mathcal{U}$ consists of only one single object U, then we simply write $\mathrm{Tria}(U)$ for $\mathrm{Tria}(\{U\})$. Let $\mathcal{A}$ be a full subcategory of $\mathcal{D}$. Denote by $\mathrm{thick}_{\mathcal{D}}(\mathcal{A})$ the smallest thick subcategory of $\mathcal{D}$ which contains $\mathcal{A}$.

2.3. Recollements and TTF. In this subsection we recall the notion of a recollement of triangulated categories, see [9].

Let $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ be triangulated categories. A *recollement* of $\mathcal{T}$ relative to $\mathcal{T}'$ and $\mathcal{T}''$ is defined by six triangulated functors

$$\begin{tikzcd} \mathcal{T}' \arrow[r, "i_*"] & \mathcal{T} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{T}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

satisfying the conditions

(i) (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$ and (j^*, j_*) are adjoint pairs;

(ii) i_*, j_* and $j_!$ are full embeddings;

(iii) $\mathrm{Im}\, i_* = \mathrm{Ker}\, j^*$.

3. Recollements induced by good silting dg-modules

Let A be a dg-algebra. Recall from Section 3 of [11] that an object $U \in \mathbf{D}(A, d)$ is called *(pre)silting* provided that it satisfies (the first two of) the following conditions:

(S1) $\dim_{\mathrm{Add}(A)} U < \infty$;

(S2) $U^{(I)} \in U^{\perp_{>0}}$ for every set I;

(S3) $\mathrm{codim}_{\mathrm{Add}(U)} A < \infty$.

Recall that an object $X \in \mathbf{D}(A, d)$ is called *small* (or *compact*) if $\mathrm{Hom}_{\mathbf{D}(A,d)}(X, -)$ commutes with coproducts. Let $\mathrm{per}(A) = \mathrm{thick}_{\mathbf{D}(A,d)}(A)$, called the *perfect derived category* of dg-A-modules. It is known that an object M of $\mathbf{D}(A, d)$ is compact if and only if it belongs to $\mathrm{per}(A)$. A *small silting* object is an object which is both silting and small. A silting object is called *good* if the condition (S3) can be replaced by

(s3) $\mathrm{codim}_{\mathrm{add}(U)} A < \infty$.

A dg-A-module U is called (*good*) *silting* if U is $\mathcal{H}$-projective and as an object in $\mathbf{D}(A, d)$ is a (good) silting object. Two silting objects U and U' are equivalent if $\mathrm{Add}(U) = \mathrm{Add}(U')$. Condition (s3) is not particularly restrictive, because each n-silting object is equivalent to a good silting object, see [11], Lemma 2.3. For a silting object U and an $n \in \mathbb{N}$, the conditions $\mathrm{codim}_{\mathrm{Add}(U)} A \leqslant n$ and $\dim_{\mathrm{Add}(A)} U \leqslant n$ are equivalent. Call *n-silting* a silting object satisfying these equivalent conditions.

Remark 3.1.

(1) Let U be a silting dg-module over A. As stated in the introduction, the condition that A belongs to $\mathrm{thick}_{\mathbf{D}(A,d)}(U)$ is equivalent to $\mathrm{codim}_{\mathrm{add}(U)} A < \infty$. Indeed, if $\mathrm{codim}_{\mathrm{Add}(U)} A < \infty$, one can check that A belongs to ${}^{\perp_{>0}}\mathrm{add}(U)$. Therefore, by the 'small' version of Corollary 2.6 (1) of [28], we see that $A \in \mathrm{thick}_{\mathbf{D}(A,d)}(U)$ implies $\mathrm{codim}_{\mathrm{add}(U)} A < \infty$. The other direction is obvious.

(2) The notion of an n-silting object agrees to the n-semitilting complex in [28] and to the $(n+1)$-silting complex in [3].

(3) Let $B = \mathrm{DgEnd}_A(U)$. By Section 3.3 of [11] if U is an n-silting dg-A-module, then B is weak nonpositive.

Let k be a commutative ring and let A be a dg-algebra, U a dg-A-module. Set

$$B = \mathrm{DgEnd}_A(U) = \mathrm{Hom}^\bullet_A(U,U);$$

$$\mathbf{G} := - \overset{\mathbb{L}}{\underset{B}{\bigotimes}} U \colon \mathbf{D}(B,d) \to \mathbf{D}(A,d); \quad \mathbf{H} := \mathbb{R}\mathrm{Hom}_A(U,-) \colon \mathbf{D}(A,d) \to \mathbf{D}(B,d).$$

In the following, we recall the definition of homological epimorphisms of dg-algebras and its characterization at the level of derived categories.

Definition 3.2 ([23], Theorem 3.9)**.** Let $\lambda \colon C \to D$ be a morphism between two dg-algebras C and D. Then λ is called a *homological epimorphism of dg-algebras* if the canonical map $D \overset{\mathbb{L}}{\underset{C}{\bigotimes}} D \to D$ is an isomorphism, or equivalently, if the induced functor $\lambda_* \colon \mathbf{D}(D,d) \to \mathbf{D}(C,d)$ is fully faithful.

It is known that there is a projective model structure on the category $\mathrm{Dga}(k)$ of dg-algebras over k see [24], Theorem 4.1 and [8], Proposition 1.3.5 (1). Denote by $\mathrm{HoDga}(k)$ the homotopy category of this model category. By [21], Corollary 6.7 and [7], Proposition 2.5, we have the following result. In order to understand functors between derived categories better, we give a brief proof here.

Proposition 3.3. *Let A be a dg-algebra, U a good silting dg-A-module and $B = \mathrm{DgEnd}_A(U)$. Then there is a homological epimorphism $\lambda = f\sigma^{-1} \colon B \to C$ in $\mathrm{HoDga}(k)$, represented by homomorphisms of dg-algebras $\sigma \colon E \to B$ and $f \colon E \to C$, such that the following is a recollement of triangulated categories:*

$$\begin{tikzcd}[column sep=huge] \mathbf{D}(C,d) \arrow[r, "\lambda_*=\sigma^* f_*"] & \mathbf{D}(B,d) \arrow[l, bend right, "i^*=-\overset{\mathbb{L}}{\underset{E}{\otimes}} C"'] \arrow[l, bend left, "\mathbb{R}\mathrm{Hom}_E(C,-)"] \arrow[r, "\mathbf{G}"] & \mathbf{D}(A,d). \arrow[l, bend right] \arrow[l, bend left, "\mathbf{H}"] \end{tikzcd} \tag{3.1}$$

P r o o f. By [21], Corollary 6.7, we have the following recollement

$$\begin{tikzcd} \mathrm{Ker}(\mathbf{G}) \arrow[r, "inc"] & \mathbf{D}(B,d) \arrow[l, shift right=3ex] \arrow[l, shift left=3ex] \arrow[r, "\mathbf{G}"] & \mathbf{D}(A,d) \arrow[l, shift right=3ex, "\mathbf{L}"'] \arrow[l, shift left=3ex, "\mathbf{H}"] \end{tikzcd},$$

where $\mathbf{L}$ is the fully faithful left adjoint of $\mathbf{G}$. By [19], Proposition 4.4.3, we see that $\mathrm{Im}(\mathbf{L})$ is a smashing localizing class, see [7], Definition 1.4 or [15], Definition 3.3.2. Applying Proposition 2.5 of [7], we obtain a homological epimorphism $g = f\sigma^{-1}\colon B \to C$ in $\mathrm{HoDga}(k)$, represented by homomorphisms of dg-algebras $\sigma\colon E \to B$ and $f\colon E \to C$, such that $\sigma^* f_*\colon \mathbf{D}(C,d) \to \mathrm{Ker}(\mathbf{G}) = \mathrm{Im}(\mathbf{L})^{\perp}$ is an triangle equivalence, where σ is a quasi-isomorphism and induces a triangle equivalence $\sigma_*\colon \mathbf{D}(B,d) \to \mathbf{D}(E,d)$, whose quasi-inverse is $\sigma^* = -\overset{\mathbb{L}}{\underset{E}{\otimes}} B\colon \mathbf{D}(E,d) \to \mathbf{D}(B,d)$, see [16], Lemma 6.1 (a). Moreover, $-\overset{\mathbb{L}}{\underset{E}{\otimes}} C$ and $\mathbb{R}\mathrm{Hom}_E(C,-)$ are left and right adjoint of the functor $\sigma^* f_*$, respectively. Thus, we obtain the desired recollement. $\square$

In fact, by Corollary 6.7 of [21] and Remark 3.1 (1), the existence of such a recollement implies that the given silting object U is good.

Proposition 3.4. *Let A be a dg-algebra, U a dg-A-module and $B = \mathrm{DgEnd}_A(U)$. Suppose U is a silting object in $\mathbf{D}(A,d)$. If the triangle functor $\mathbf{G} := -\overset{\mathbb{L}}{\underset{B}{\otimes}} U\colon \mathbf{D}(B,d) \to \mathbf{D}(A,d)$ admits a fully faithful left adjoint $j_!\colon \mathbf{D}(A,d) \to \mathbf{D}(B,d)$, then the given silting dg-module is good.*

Starting with a good silting dg-module U as before, we have a recollement in Proposition 3.3 and note that the dg-algebra B has to be weak nonpositive. In the following, we consider under what extent does the induced dg-algebra C satisfy the same property.

First, we have the following easy observation.

Lemma 3.5. *Adopt the notations from Proposition* 3.3. *Then the dg-algebra C is weak nonpositive if and only if $H^i(i^*(B)) = 0$ for every $i \geqslant 1$.*

P r o o f. We only need to prove that $i^*(B) \cong C$. In fact, $i^*(B) \cong \sigma_*(B) \overset{\mathbb{L}}{\underset{E}{\otimes}} C \cong E \overset{\mathbb{L}}{\underset{E}{\otimes}} C \cong C$, where $\sigma_*\colon \mathbf{D}(B,d) \to \mathbf{D}(E,d)$ is a triangle equivalence. $\square$

The following useful theorem is well-known.

Lemma 3.6. *Let R and S be dg-algebras and suppose that P_R is compact in $\mathbf{D}(R,d)$. Suppose that M and K are dg-S-R-bimodules, N is a dg-S-module and L is a dg-S^{op}-module. Then we have the following canonical isomorphisms:*

$$(\dagger)\qquad N_S \bigotimes_S^{\mathbb{L}} \mathbb{R}\mathrm{Hom}_R(P_R, {}_SM_R) \cong \mathbb{R}\mathrm{Hom}_R\Big(P_R, N_S \bigotimes_S^{\mathbb{L}} {}_SM_R\Big),$$

$$(\ddagger)\qquad P_R \bigotimes_R^{\mathbb{L}} \mathbb{R}\mathrm{Hom}_{S^{\mathrm{op}}}({}_SK_R, {}_SL) \cong \mathbb{R}\mathrm{Hom}_{S^{\mathrm{op}}}(\mathbb{R}\mathrm{Hom}_R(P_R, {}_SK_R), {}_SL)).$$

The following is the main result in this section.

Theorem 3.7. *Let A be a dg-algebra, U a good n-silting dg-A-module, and $B = \mathrm{DgEnd}_A(U)$. Then the dg-algebra C induced in Proposition 3.3 is weak nonpositive if and only if $H^i\Big(U \bigotimes_A^{\mathbb{L}} \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(U,B)\Big) = 0$ for $i \geqslant 2$, or equivalently, $H^i\Big(U \bigotimes_A^{\mathbb{L}} \mathbb{R}\mathrm{Hom}_A(U,A)\Big) = 0$ for $i \geqslant 2$.*

Proof. We define a dg-A-B-bimodule $U^* = \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(U,B)$. From the proof of Proposition 3.3, we know that ${}_BU \in \mathrm{per}(B^{\mathrm{op}})$. Hence, $U^* = \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(U,B) \in \mathrm{per}(B)$ and we have an isomorphism $- \bigotimes_B^{\mathbb{L}} \mathbb{R}\mathrm{Hom}_B(U^*,B) \cong \mathbb{R}\mathrm{Hom}_B(U^*,-)$ by Lemma 3.6 (†). On the other hand, since ${}_BU \in \mathrm{per}(B^{\mathrm{op}})$, we know from Lemma 3.6 (‡) that

$$U^{**} := \mathbb{R}\mathrm{Hom}_B(\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(U,B),B) \cong \mathbb{R}\mathrm{Hom}_B({}_BB_B, B_B) \bigotimes_B^{\mathbb{L}} U \cong {}_BU.$$

Then we get that $-\bigotimes_B^{\mathbb{L}} U \cong -\bigotimes_B^{\mathbb{L}} U^{**} \cong \mathbb{R}\mathrm{Hom}_B(U^*,-)$ as triangle functors from $\mathbf{D}(B,d)$ to $\mathbf{D}(A,d)$. It follows that $\Big(-\bigotimes_A^{\mathbb{L}} U^*, -\bigotimes_B^{\mathbb{L}} U\Big)$ is an adjoint pair. Therefore, the functor $\mathbf{L}$ induced in the proof of Proposition 3.3 is isomorphic to $-\bigotimes_A^{\mathbb{L}} U^*$.

Note that $\lambda\colon B \to C$ is a homological epimorphism of dg-algebras. Hence, $\mathbf{D}(C,d)$ can be regarded as a full triangulated subcategory of $\mathbf{D}(B,d)$. Since $\mathbf{LG}(B)) \cong B \bigotimes_B^{\mathbb{L}} U \bigotimes_A^{\mathbb{L}} U^* \cong U \bigotimes_A^{\mathbb{L}} \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(U,B)$, we have a triangle

$$U \bigotimes_A^{\mathbb{L}} \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(U,B) \to B \to i^*(B) \rightsquigarrow$$

in $\mathbf{D}(B,d)$ induced by the recollement (3.1). Applying the cohomology functor H^j to this triangle, since B is weak nonpositive, we have $H^j(i^*(B)) \cong H^{j+1}\Big(U \bigotimes_A^{\mathbb{L}} \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(U,B)\Big)$ for $j \geqslant 1$.

In the following we show that $\mathbb{R}\mathrm{Hom}_A(U,A) \simeq \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(U,B)$ in $\mathbf{D}(A^{\mathrm{op}},d)$. In fact, since U is a good n-silting dg-A-module, there is a sequence of triangles

$$A_i \to U_i \to A_{i+1} \rightsquigarrow \ \text{with } 0 \leqslant i \leqslant n$$

in $\mathbf{D}(A,d)$ such that $U_i \in \mathrm{add}(U)$, $A_0 = A$ and $A_{n+1} = 0$. Applying the functor $\Phi\colon \mathbb{R}\mathrm{Hom}_A(-,U_A)$ to these triangles, we obtain another sequence of triangles $\Phi(A_{i+1}) \to \Phi(U_i) \to \Phi(A_i) \rightsquigarrow$ with $0 \leqslant i \leqslant n$. Therefore, for any $X \in \mathbf{D}(A,d)$, we can construct the commutative diagram:

$$\begin{tikzcd}
\mathbb{R}\mathrm{Hom}_A(X,A_{n-1}) \arrow[r] \arrow[d] & \mathbb{R}\mathrm{Hom}_A(X,U_{n-1}) \arrow[r] \arrow[d, "\simeq"'] & \mathbb{R}\mathrm{Hom}_A(X,U_n) \arrow[r, squiggly] \arrow[d, "\simeq"'] & {} \\
\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(\Phi(A_{n-1}),\Phi(X)) \arrow[r] & \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(\Phi(U_{n-1}),\Phi(X)) \arrow[r] & \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(\Phi(U_n),\Phi(X)) \arrow[r, squiggly] & {} ,
\end{tikzcd}$$

where the isomorphisms in the second and third columns are due to $U_i \in \mathrm{add}(U)$ for $0 \leqslant i \leqslant n$. Consequently, $\mathbb{R}\mathrm{Hom}_A(X,A_{n-1}) \to \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(\Phi(A_{n-1}),\Phi(X))$ in the first column is an isomorphism. Proceeding similarly, we obtain that $\mathbb{R}\mathrm{Hom}_A(X,A) \cong \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(\Phi(A),\Phi(X))$. This implies that $\mathbb{R}\mathrm{Hom}_A(-,A) \xrightarrow{\simeq} \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(\Phi(A),\Phi(-)) \xrightarrow{\simeq} \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}({}_BU, \mathbb{R}\mathrm{Hom}_A(-,U_A))\colon \mathbf{D}(A,d) \to \mathbf{D}(A^{\mathrm{op}},d)$. Thus, $\mathbb{R}\mathrm{Hom}_A(U,A) \cong \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(\Phi(A),\Phi(U)) = \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(U,B)$. The result follows by Lemma 3.5. $\square$

If we specialize Theorem 3.7 to the case that U is a good 1-silting object, then it is easy to check that $H^i\Big(U \overset{\mathbb{L}}{\underset{A}{\bigotimes}} \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(U,B)\Big) = 0$ for $i \geqslant 2$. Then we obtain the following corollary.

Corollary 3.8. *Let A be a dg-algebra, U a good* 1*-silting dg-A-module, and let $B = \mathrm{DgEnd}_A(U)$. Then the dg-algebra C induced in Proposition* 3.3 *is weak nonpositive.*

4. Recollements induced by good cosilting dg-modules

Let A be a ring. From [14], Lemma 5.5, if a left A-module T is a good n-tilting module, then T as a right B-module is an n-weak tilting module (see [14], Definition 4.1), where B is the endomorphism ring of T. Similarly, we introduce here the notion of n-weak silting dg-module and show that if U_A is a good n-silting dg-module, then ${}_BU$ is n-weak silting whenever A is weak nonpositive, where $B = \mathrm{DgEnd}_A(U)$.

Definition 4.1. Let B be a dg-algebra. A dg-B^{op}-module M is called *n-weak silting* if it, considered as an object in $\mathbf{D}(B^{\mathrm{op}}, d)$, satisfies the following conditions:
(w1) $\dim_{\mathrm{add}(B)}(M) \leqslant n$,
(w2) $M^{(I)} \in M^{\perp_{>0}}$ for every set I, and
(w3) $\mathrm{codim}_{\mathrm{Prod}(M)} B \leqslant n$.

Remark 4.2. If we regarded a ring R as a dg-algebra, then by Definition 4.1 of [14], an n-weak tilting R-module is an n-weak silting object in $\mathbf{D}(R^{\mathrm{op}})$. However, we have to warn the reader that the converse may not be true. The main reason is that an n-weak tilting module ${}_RM$ should satisfies the condition (R4): M_S is strongly S-Mittag-Leffler as a right S-module (see [14], Definition 2.3), where $S = \mathrm{End}({}_RM)$. Notice that an analog of condition (R4) is not present in our definition. The problem is that, in general, it is difficult to character the Mittag-Leffler conditions on dg-modules since many properties of silting dg-modules in this paper are defined at the level of derived categories.

If an n-weak silting dg-B^{op}-module M satisfies $\mathrm{Prod}({}_BM) = \mathrm{Add}({}_BM)$, then ${}_BM$ is a small n-silting object. Indeed, the condition $\dim_{\mathrm{add}(B)}(M) \leqslant n$ implies that $M \in \mathrm{per}(B)$. On the other hand, small n-silting dg-modules are always n-weak silting. Let

$$\begin{aligned} A &:= \mathrm{DgEnd}_{B^{\mathrm{op}}}(M), \\ \mathcal{Y} &:= \{Y \in \mathbf{D}(B^{\mathrm{op}}, d)\colon\ \mathrm{Hom}_{\mathbf{D}(R^{\mathrm{op}},d)}(M, Y[i]) = 0 \text{ for all } i \in \mathbb{Z}\}, \\ G &:= {}_BM \overset{\mathbb{L}}{\underset{A}{\bigotimes}} -\colon\ \mathbf{D}(A^{\mathrm{op}}, d) \to \mathbf{D}(B^{\mathrm{op}}, d), \\ H &:= \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(M, -)\colon\ \mathbf{D}(B^{\mathrm{op}}, d) \to \mathbf{D}(A^{\mathrm{op}}, d). \end{aligned}$$

If ${}_BM$ satisfies (w1), then ${}_BM$ is compact in $\mathbf{D}(B^{\mathrm{op}}, d)$. It follows that there exists a TTF-triple $(\mathrm{Tria}({}_BM), \mathcal{Y}, \mathcal{Z})$ in $\mathbf{D}(B^{\mathrm{op}}, d)$ by [13], Lemma 2.8 or [10], Chapter III, Theorem 2.3; Chapter IV, Proposition 1.1, where $\mathcal{Z} := \mathrm{Ker}(\mathrm{Hom}_{\mathbf{D}(B^{\mathrm{op}},d)}(\mathcal{Y}, -))$. Hence, the inclusion $\mathcal{Y} \to \mathbf{D}(B^{\mathrm{op}}, d))$ admits both a left adjoint and a right adjoint. If ${}_BM$ satisfies both (w1) and (w2), then A is weak nonpositive and the pair (G, H) induces a triangle equivalence: $\mathbf{D}(A^{\mathrm{op}}, d) \xrightarrow{\simeq} \mathrm{Tria}({}_BM)$, see [17], Chapter 5, Theorem 8.5. On the other hand, by [19], Proposition 4.4.3, we see that $\mathrm{Tria}({}_RM)$ is a smashing localizing class. Using Proposition 2.5 of [7], there is a homological epimorphism $\mu\colon B \to C$ of dg-algebras such that μ_* induces a triangle equivalence from $\mathbf{D}(C^{\mathrm{op}}, d)$ to $\mathcal{Y}$. Thus, we have the following result which shows that an n-weak silting object in $\mathbf{D}(B^{\mathrm{op}}, d)$ always induces a recollement among derived categories of dg-algebras.

Proposition 4.3. *Suppose the dg-B^{op}-module M satisfies* (w1) *and* (w2). *Then there is a homological epimorphism $\mu = g\varepsilon^{-1}\colon B \to C$ in* $\mathrm{HoDga}(k)$, *represented by* *homomorphisms of dg-algebras $\varepsilon\colon F \to B$ and $g\colon F \to C$, such that the following is a recollement of triangulated categories:*

$$\mathbf{D}(C^{\mathrm{op}}, d) \underset{\mathbb{R}\mathrm{Hom}_{F^{\mathrm{op}}}(C,-)}{\overset{p=C \overset{\mathbb{L}}{\underset{F}{\otimes}} -}{\leftleftarrows}} \!\!\xrightarrow{\mu_* = \varepsilon^* g_*} \mathbf{D}(B^{\mathrm{op}}, d) \underset{}{\overset{G}{\leftleftarrows}}\!\!\xrightarrow{H} \mathbf{D}(A^{\mathrm{op}}, d)\ .$$

Now we point out that each good silting dg-module can produce a weak silting dg-module over weak nonpositive dg-algebras.

Lemma 4.4. *Let A be a dg-algebra, U a good n-silting dg-A-module, and let $B = \mathrm{DgEnd}_A(U)$. If the dg-algebra A is weak nonpositive, then U as a dg-B^{op}-module is n-weak silting.*

Proof. By the definition, $\mathrm{codim}_{\mathrm{add}(U)} A \leqslant n$. Then from Lemma 1.1 of [11], we have $\dim_{\mathrm{add}(B)} U \leqslant n$. By assumption, A is weak nonpositive, hence

$$\mathrm{Hom}_{\mathbf{D}(B^{\mathrm{op}},d)}(U, U[i]) \cong H^i A = 0 \quad \text{for } i \geqslant 1.$$

So (w1) and (w2) hold for U. Now, we check (w3) for U. Since $\dim_{\mathrm{Add}(A)} U \leqslant n$, there is a sequence of triangles in $\mathbf{D}(A, d)$

$$V_{i+1} \to P_i \to V_i \rightsquigarrow \text{ with } 0 \leqslant i \leqslant n$$

such that $P_i \in \mathrm{Add}(A)$, $V_0 = U$ and $V_{n+1} = 0$. In fact, applying the functor $\mathbb{R}\mathrm{Hom}_A(-, U)$ to these triangles, we get triangles in $\mathbf{D}(B^{\mathrm{op}}, d)$ of the form

$$B_i \to Q_i \to B_{i+1} \rightsquigarrow \text{ with } 0 \leqslant i \leqslant n$$

such that $Q_i = \mathbb{R}\mathrm{Hom}_A(P_i, U) \in \mathrm{Prod}(U)$, $B_0 = B$ and $B_{n+1} = 0$. Thus, U satisfies (w3). $\square$

Proposition 4.5. *Let A be a weak nonpositive dg-algebra, U a good n-silting dg-A-module, and $B = \mathrm{DgEnd}_A(U)$. Then there exist a dg-algebra D and a recollement of triangulated categories*

$$\mathbf{D}(D^{\mathrm{op}}, d) \overset{\delta}{\leftleftarrows}\!\!\xrightarrow{\mu_*} \mathbf{D}(B^{\mathrm{op}}, d) \overset{G}{\leftleftarrows}\!\!\xrightarrow{H} \mathbf{D}(A^{\mathrm{op}}, d)$$

such that $\mu\colon B \to D$ is a homological epimorphism, where $G = {}_B U \overset{\mathbb{L}}{\underset{A}{\otimes}} -$ and $H = \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(U, -)$. Moreover, D is weak nonpositive if and only if $H^i(\delta(B)) = 0$ for every $i \geqslant 1$, if and only if $H^i\Big(U \overset{\mathbb{L}}{\underset{A}{\otimes}} \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(U, B)\Big) = 0$ for $i \geqslant 2$.

P r o o f. From Lemma 4.4, ${}_BU$ is an n-weak silting dg-module. By Theorem 1.4 of [11], there is a quasi-isomorphism $\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(U,U) \simeq A$. Therefore, we get the desired recollement and homological epimorphism $\mu\colon B \to D$ from Proposition 4.3. Hence, there is a triangle in $\mathbf{D}(B^{\mathrm{op}},d)$,

$$U \overset{\mathbb{L}}{\underset{A}{\bigotimes}} \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(U,B) \to B \to \delta(B) \rightsquigarrow .$$

Applying the cohomology functor H^j to this triangle, we get an exact sequence

$$\ldots \to H^i(\delta(B)) \to H^{i+1}(GH(B)) \to H^{i+1}(B) \to H^{i+1}(\delta(B)) \to \ldots$$

Since B is weak nonpositive, we have $H^j(\delta(B)) \cong H^{j+1}\Big(U \overset{\mathbb{L}}{\underset{A}{\bigotimes}} \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(U,B)\Big)$ for $j > 0$. By an argument similar to that in Lemma 3.5, we have $\delta(B) \cong D$. Thus, the equivalence follows. $\square$

In the following, we apply the above results to deal with good cosilting dg-modules. First, we construct n-weak silting objects from good n-cosilting objects, and then use Proposition 4.3 to construct the recollement.

Let (A,d) be a weak nonpositive dg-k-algebra. We fix a faithfully injective k-module $k^\vee$. If M is a dg right A-module we set $M^\vee = \bigoplus\limits_{n\in\mathbb{Z}} \mathrm{Hom}_k(M^{-n},k^\vee)$ as a graded k-module. Then we endow $M^\vee$ with a left dg-module structure by setting $d_{M^\vee} = (-1)^n f \circ d_M^{-n-1}$. One can check we have $d_{M^\vee}(af) = d(a)f + (-1)^m a d_{M^\vee}(f)$ for $a \in A^m$, $x \in M^{-n-m-1}$ and $f \in (M^\vee)^n$. Therefore, there exists a functor $\vee\colon \mathrm{Mod}(A,d)^{\mathrm{op}} \to \mathrm{Mod}(A^{\mathrm{op}},d)$, see [25], Tag 04JD. Set $W := A_A^\vee$. Note that W is an $\mathcal{H}$-injective dg-A^{op} module by [26], Tag 09KS.

Definition 4.6. A dg-A^{op}-module N is called *n-cosilting* if it, considered as an object in $\mathbf{D}(A^{\mathrm{op}},d)$, satisfies the following conditions:

(C1) $\mathrm{codim}_{\mathrm{Prod}(W)}({}_AN) \leqslant n$,

(C2) $N^I \in {}^{\perp_{>0}}N$ for every set I, and

(C3) $\dim_{\mathrm{Prod}(N)} W \leqslant n$.

An n-cosilting dg-A^{op}-module N is said to be *good* if it, considered as an object in $\mathbf{D}(A^{\mathrm{op}},d)$, satisfies (C1), (C2) and

(c3) $\dim_{\mathrm{add}(N)} W \leqslant n$.

We say that ${}_AN$ is a (*good*) *cosilting* dg-module if ${}_AN$ is (good) n-cosilting for some $n \in \mathbb{N}$. If N_1 and N_2 are cotilting objects, then N_1 is equivalent to N_2, provided that $\mathrm{Prod}(N_1) = \mathrm{Prod}(N_2)$. Similarly, we show that each n-cosilting object is equivalent to a good cosilting object.

Lemma 4.7. *If N is an n-cosilting object then there is a good n-silting object N' such that N and N' are equivalent.*

P r o o f. Let N be an n-cosilting object. Since $\dim_{\mathrm{Prod}(N)} W \leqslant n$, there are triangles

$$W_{i+1} \to N_i \to W_i \rightsquigarrow \text{ with } 0 \leqslant i \leqslant n$$

such that $N_i \in \mathrm{Prod}(N)$, $W_0 = W$ and $W_{n+1} = 0$. By an argument similar to that in [28], Proposition 3.8, $N' = \prod_{i=0}^{n} N_i = \bigoplus_{i=0}^{n} N_i$ is an n-cosilting object and $\mathrm{Prod}(N') = \mathrm{Prod}(N)$. It is clear that N' is good since all N_i are direct summand of N'. $\square$

From now on, we assume that N is a good n-cosilting dg-A^{op}-module with conditions (C1), (C2) and (c3), and call ${}_AN$ a good n-cosilting dg-module with respect to W.

Lemma 4.8. *Let A be a weak nonpositive dg-algebra, N a good n-cosilting dg-A^{op}-module, and let $B = \mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(N, N)$, $M = \mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(N, W)$ and $\Lambda :=$ $\mathrm{DgEnd}_{A^{\mathrm{op}}}(W)$. The following hold true.*
(1) $\dim_{\mathrm{add}(B)} M \leqslant n$.
(2) *The functor* $\mathbb{R}\mathrm{Hom}_A(N, -)\colon \mathbf{D}(A^{\mathrm{op}}, d) \to \mathbf{D}(B^{\mathrm{op}}, d)$ *induces an quasi-isomorphism of dg-algebras* $\Lambda \simeq \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(M, M)$ *and* $\mathrm{Hom}_{\mathbf{D}(B^{\mathrm{op}},d)}(M, M[i]) = 0$ *for all* $i \geqslant 1$.
(3) *The dg-B^{op}-module M satisfies* (w1)–(w3) *in Definition* 4.1.

P r o o f. (1) Since $\dim_{\mathrm{add}(N)} W \leqslant n$, there is a sequence of triangles in $\mathbf{D}(\mathbf{A}^{\mathrm{op}}, \mathbf{d})$

$$K_{i+1} \to N_i \to K_i \rightsquigarrow \text{ with } 0 \leqslant i \leqslant n \tag{4.1}$$

such that $N_i \in \mathrm{add}(N)$, $K_0 = W$ and $K_{n+1} = 0$. Applying $\mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(N, -)$ to these triangles, we get a sequence of triangles in $\mathbf{D}(B^{\mathrm{op}}, d)$ of the form

$$V_{i+1} \to B_i \to V_i \rightsquigarrow \text{ with } 0 \leqslant i \leqslant n$$

such that $B_i = \mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(N, N_i) \in \mathrm{add}(B)$, $V_0 = M$ and $V_{n+1} = 0$. Thus, $\dim_{\mathrm{add}(B)} M \leqslant n$.

(2) Let Ψ be the functor $\mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(N, -)\colon \mathbf{D}(A^{\mathrm{op}}, d) \to \mathbf{D}(B^{\mathrm{op}}, d)$. Then $\Psi(N) = B$, $\Psi(W) = M$ and $\mathbb{R}\mathrm{Hom}_A(X, W) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_B(\Psi(X), \Psi(W))$ for any $X \in \mathrm{add}({}_AN)$.

If $n = 0$, then $W = N_0 \in \mathrm{add}({}_AN)$. In this case, we have

$$\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(M, M) = \mathbb{R}\mathrm{Hom}_B(\Psi(W), \quad \Psi(W)) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_A(W, W) = \Lambda$$

and $\mathrm{Hom}_{\mathbf{D}(B^{\mathrm{op}},d)}(M, M[i]) = 0$ for all $i \geqslant 1$ by (C2).

Suppose $n \geqslant 1$. By (1), there is a sequence of triangles in $\mathbf{D}(B^{\mathrm{op}}, d)$ of the form

$$V_{i+1} \to B_i \to V_i \rightsquigarrow \text{ with } 0 \leqslant i \leqslant n$$

such that $B_i = \mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(N, N_i) \in \mathrm{add}(B)$, $V_0 = M$ and $V_{n+1} = 0$. Applying $\mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(-, W)$ to the triangles (4.1), we get a sequence of triangles

$$\mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(K_i, W) \to \mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(N_i, W) \to \mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(K_{i+1}, W) \rightsquigarrow \text{ with } 0 \leqslant i \leqslant n.$$

We can construct the commutative diagram:

$$\begin{tikzcd}
\mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(K_{n-1}, W) \arrow[r] \arrow[d] & \mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(N_{n-1}, W) \arrow[r] \arrow[d, "\simeq"'] & \mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(N_n, W) \arrow[r, rightsquigarrow] \arrow[d, "\simeq"'] & {} \\
\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(\Psi(K_{n-1}), \Psi(W)) \arrow[r] & \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(\Psi(N_{n-1}), \Psi(W)) \arrow[r] & \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(\Psi(N_n), \Psi(W)) \arrow[r, rightsquigarrow] & \cdot
\end{tikzcd}$$

This implies that $\mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(K_{n-1}, W) \cong \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(\Psi(K_{n-1}), \Psi(W))$. Proceeding similarly, we obtain that $\Lambda = \mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(W, W) \to \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(\Psi(W), \Psi(W)) = \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(M, M)$ is actually a quasi-isomorphism.

It remains to prove that $\mathrm{Hom}_{\mathbf{D}(B^{\mathrm{op}},d)}(M, M[i]) = 0$ for all $i \geqslant 1$. We claim that if A is weak nonpositive, then so is Λ. Recall that for any dg-A^{op}-module X, dg-A-module Y and dg-A-A-bimodule Z, we have the swap isomorphism:

$$\mathrm{Hom}^{\bullet}_{A^{\mathrm{op}}}(X, \mathrm{Hom}^{\bullet}_{A}(Y, Z)) \cong \mathrm{Hom}^{\bullet}_{A}(Y, \mathrm{Hom}^{\bullet}_{A^{\mathrm{op}}}(X, Z)).$$

It follows that there are isomorphisms $\Lambda = \mathrm{Hom}^{\bullet}_{A^{\mathrm{op}}}(A^{\vee}, A^{\vee}) \cong \mathrm{Hom}^{\bullet}_{A}(A, A^{\vee\vee}) \cong A^{\vee\vee}$. Therefore, the claim follows from the isomorphism $H^i(A^{\vee\vee}) \cong H^i(A)^{\vee\vee}$ by the fact that $\vee$ is an exact functor, see [25], Tag 04JD.

(3) Clearly, (w1) and (w2) follow from (1) and (2), respectively. It remains to show (w3) for M. In fact, by (C1), there exists a sequence of triangles

$$U_i \to I_i \to U_{i+1} \rightsquigarrow \text{ with } 0 \leqslant i \leqslant n$$

such that $I_i \in \mathrm{Prod}(W)$, $U_0 = N$ and $U_{n+1} = 0$. Applying $\mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(N, -)$ to these triangles, we get a sequence of triangles

$$\mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(N, U_i) \to \mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(N, I_i) \to \mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(N, U_{i+1}) \rightsquigarrow \text{ with } 0 \leqslant i \leqslant n.$$

Since $\mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(N, -)$ commutes with arbitrary direct products, it follows from $I_i \in \mathrm{Prod}(W)$ that $\mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(N, I_i) \in \mathrm{Prod}(\mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(N, W)) = \mathrm{Prod}(M)$ and that ${}_AM$ satisfies (w3). $\square$

By Lemma 4.8 (2), we have a quasi-isomorphism $h\colon \Lambda \to \mathrm{DgEnd}_{B^{\mathrm{op}}}(M)$. Hence, every dg-$\mathrm{DgEnd}_{B^{\mathrm{op}}}(M)$-module becomes a dg-$\Lambda$-module via the map h. As in [30], Section 12.7 we do not distinguish notationally between such a dg-module seen as dg-$\mathrm{DgEnd}_{B^{\mathrm{op}}}(M)$-module or a Λ-module. Now, we put

$$G := {}_BM_\Lambda \overset{\mathbb{L}}{\underset{\Lambda}{\bigotimes}} -\colon \mathbf{D}(\Lambda^{\mathrm{op}}, d) \to \mathbf{D}(B^{\mathrm{op}}, d)$$

and $H := \mathbb{R}\mathrm{Hom}_B(M, -)\colon \mathbf{D}(B^{\mathrm{op}}, d) \to \mathbf{D}(\Lambda^{\mathrm{op}}, d)$. Since ${}_BM$ satisfies both (w1) and (w2) of Definition 4.1, by Proposition 4.3 and the proof of Proposition 4.5, we have the following result.

Theorem 4.9. *Let A be a weak nonpositive dg-algebra, N a good n-cosilting dg-A^{op}-module, and let $B = \mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(N, N)$, $M = \mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(N, W)$ and $\Lambda :=$ $\mathrm{DgEnd}_{A^{\mathrm{op}}}(W)$. Then there exist a dg-algebra C and a recollement of triangulated categories*

$$\begin{tikzcd} \mathbf{D}(C^{\mathrm{op}}, d) \arrow[r, "\lambda_*"] & \mathbf{D}(B^{\mathrm{op}}, d) \arrow[l, shift right=3ex] \arrow[l, shift left=3ex] \arrow[r, "H"] & \mathbf{D}(\Lambda^{\mathrm{op}}, d) \arrow[l, shift right=3ex, "G"'] \arrow[l, shift left=3ex] \end{tikzcd}$$

such that $\lambda\colon B \to C$ is a homological epimorphism. Moreover, C is weak nonpositive if and only if $H^i\Big(M \overset{\mathbb{L}}{\underset{\Lambda}{\otimes}} \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(M, B)\Big) = 0$ for $i \geqslant 2$.

In the end of this section, we show that there exists an isomorphism in $\mathrm{HoDga}(k)$ from the dg-algebra C induced in Proposition 3.3 to the dg-algebra D induced in Theorem 4.5.

Proposition 4.10. *Let A be a weak nonpositive dg-algebra, $U \in \mathbf{D}(A, d)$ a good n-silting object, and let $B = \mathrm{DgEnd}_A(U)$. Then the dg-algebra C in Proposition 3.3 is isomorphic to the dg-algebra D in Proposition 4.5 in $\mathrm{HoDga}(k)$.*

P r o o f. In fact, we notice that $\lambda\colon B \to C$ is a morphism of dg-B-B-bimodules. By the proof of Theorem 3.7, together with the fact that $i^*(B) \cong C$, one has a triangle of B-B-bimodules

$$U \overset{\mathbb{L}}{\underset{A}{\bigotimes}} \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(U, B) \to B \xrightarrow{\lambda} C \rightsquigarrow,$$

see also the last paragraph of Section 4 of [20]. Similarly, by the proof of Proposition 4.5, since $\delta(B) \cong D$, we have a triangle of B-B-bimodules

$$U \overset{\mathbb{L}}{\underset{A}{\bigotimes}} \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(U, B) \to B \xrightarrow{\mu} D \rightsquigarrow .$$

Therefore, one can check that there exists an isomorphism $\varphi\colon C \to D$ in $\mathrm{HoDga}(k)$. $\square$

5. Applications

In this section, we will be concerned with some applications of the results of Section 4.

5.1. Applications to good 2-term silting complexes. In this subsection, we show that there exists a recollement induced by good 2-term silting complexes.

Let R be an ordinary ring, $\mathbb{P}$ the complex

$$\ldots \to 0 \to P^{-1} \xrightarrow{\sigma} P^{0} \to 0 \to \ldots$$

with P^{-1}, P^{0} projective. From [12], Section 3.2, $\mathbb{P}$ is called a 2-*term silting complex* if

(S1) $\mathbb{P}^{(I)} \in \mathbb{P}^{\perp_{>0}}$ for all sets I, where

$$\mathbb{P}^{\perp_{>0}} = \{Y \in \mathbf{D}(R)\colon\ \mathrm{Hom}_{\mathbf{D}(R)}(\mathbb{P}, Y[n]) = 0 \text{ for all positive integers } n\},$$

(S2) the homotopy category $\mathbf{K}^{b}(\mathrm{Proj}\, R)$ of bounded complexes of projective modules is the smallest triangulated subcategory of $\mathbf{D}(R)$ containing $\mathrm{Add}(\mathbb{P})$.

A torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathrm{Mod} - R$ is called *silting torsion theory* if there exists a silting module S such that $\mathcal{T} = \mathrm{Gen}(S)$. By Corollary 3.2.1 of [12] we know that there exists a 2-term silting complex $\mathbb{P}$ such that the silting module $S = H^{0}(\mathbb{P})$ generates the class $\mathcal{T}$ and there exists a triangle

$$R \to \mathbb{P}^{n} \to \mathbb{P}' \rightsquigarrow$$

in $\mathbf{D}(R)$ such that $\mathbb{P}' \in \mathrm{add}(\mathbb{P})$. Such a complex will be called a *good silting complex.*

Let B be the smart truncation of $\mathrm{DgEnd}_{R}(\mathbb{P})$, that is

$$B = \bigoplus_{i \in \mathbb{Z}} B^{i}, \quad \text{where } B^{i} = \begin{cases} \mathrm{DgEnd}_{R}(\mathbb{P}) & \text{if } i < 0, \\ Z^{0}(\mathrm{DgEnd}_{R}(\mathbb{P})) & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

Then B is a nonpositive dg-algebra and we have a quasi-isomorphism $B \to \mathrm{DgEnd}_{R}(\mathbb{P})$. Note that $\mathbb{P} \in \mathbf{K}^{b}(\mathrm{Proj}\, R) = \langle \mathrm{Add}(R) \rangle$ and $\mathbb{P} \in R^{\perp_{>0}}$. Hence, $\dim_{\mathrm{Add}(R)} \mathbb{P} < \infty$ by [28], Corollary 2.6 (2). Furthermore, from Proposition 3.9 of [28], we see that the good 2-term silting complex is a 1-silting object in $\mathbf{D}(R, d)$. Thus, as a consequence of Proposition 3.3 and Corollary 3.8, we obtain the following recollement.

Corollary 5.1. *Let R be a k-algebra, $\mathbb{P}$ a good 2-term silting complex in $\mathbf{D}(R)$. Then there is a homological epimorphism $g = f\sigma^{-1}\colon B \to C$ in* $\mathrm{HoDga}(k)$*, represented by homomorphisms of dg-algebras $\sigma\colon E \to B$ and $f\colon E \to C$ such that*

$$\begin{tikzcd}[column sep=large] \mathbf{D}(C,d) \arrow[r, "\sigma^* f_*"] & \mathbf{D}(B,d) \arrow[l, shift right=6ex, "C \overset{\mathbb{L}}{\underset{E}{\otimes}} -"'] \arrow[l, shift left=3ex, "\mathbb{R}\mathrm{Hom}_E(C,-)"] \arrow[r, "- \overset{\mathbb{L}}{\underset{B}{\otimes}} \mathbb{P}"] & \mathbf{D}(R) \arrow[l, shift right=6ex] \arrow[l, shift left=3ex] \end{tikzcd}$$

is a recollement of triangulated categories. Moreover, C is weak nonpositive.

5.2. Applications to good tilting complexes and modules. In this subsection, we want to show that our results generalize those of [13], [14]. In order to do that, let R be a ring and T an R-complex. Then R can be seen as a dg-algebra concentrated in degree 0. Recall that $\mathbf{K}^b(\mathrm{Proj}\,R)$ denotes the homotopy category of bounded complexes of projective modules. The complex T is called a *good tilting complex* if it satisfies the following conditions:

(T1) $T \in \mathbf{K}^b(\mathrm{Proj}\,R)$,
(T2) $\mathrm{Hom}_{\mathbf{D}(R)}(T, T^{(\alpha)}[n]) = 0$ for every set α and $n \neq 0$,
(t3) $\mathrm{codim}_{\mathrm{add}(T)} R < \infty$.

One can check that $\mathbf{K}^b(\mathrm{Proj}\,R) = \langle \mathrm{Add}(R) \rangle$. Hence, the good tilting complexes are good silting objects in $\mathbf{D}(R)$. From the condition (T2), $H^n(\mathrm{DgEnd}_R(U)) = H^n(\mathbb{R}\mathrm{Hom}_R(U,U)) \cong \mathrm{Hom}_{\mathbf{D}(R)}(T, T[n]) = 0$ for $n \neq 0$. Since $H^0(\mathrm{DgEnd}_R(U)) \cong B$, we have an equivalence between $\mathbf{D}(\mathrm{DgEnd}_R(U), d)$ and $\mathbf{D}(B)$. As a consequence of Proposition 3.3 and Theorem 3.7, we obtain the folllowing recollement.

Corollary 5.2. *Let R be a ring and T a good tilting complex, and let $B = \mathrm{End}(T)$. Then there exist a dg-algebra C and a recollement of triangulated categories*

$$\begin{tikzcd} \mathbf{D}(C,d) \arrow[r, "\lambda_*"] & \mathbf{D}(B) \arrow[l, shift right=3ex] \arrow[l, shift left=3ex] \arrow[r, "\mathbf{G}"] & \mathbf{D}(R) \arrow[l, shift right=3ex] \arrow[l, shift left=3ex] \end{tikzcd}$$

such that $\lambda\colon B \to C$ is a homological epimorphism. Moreover, C is weak nonpositive if and only if $H^i\Big(T \overset{\bullet}{\underset{R}{\otimes}} \mathrm{Hom}^{\bullet}_R(T,R)\Big) = 0$ for all $i \geqslant 2$.

Let R be a ring and T an R-module. Consider the following conditions on T:

(T1) The projective dimension of T is finite.
(T2) The module T has no self-extensions, that is $\mathrm{Ext}^i_R(T, T^{(\alpha)}) = 0$ for every $i \geqslant 1$ and every set α.

(t3) There is an exact sequence of R-modules

$$0 \to R \to T_0 \to T_1 \to \ldots \to T_n \to 0$$

such that T_i is isomorphic to a direct summand of a finite direct sum of copies of T for all $0 \leqslant i \leqslant n$.

Then T is called a *good n-tilting module*, if it satisfies (T1), (T2), (t3) and the projective dimension of T is at most n. Let B be the endomorphism ring of T. We obtain the following recollement. One can compare it with Theorem 1.1 of [14].

Theorem 5.3. *Let R be an ordinary algebra, T_R a good tilting module, and let B be the endomorphism ring of T. Then there exist a dg-algebra C and a recollement of triangulated categories*

$$\begin{tikzcd} \mathbf{D}(C,d) \arrow[r, "\lambda_*"] & \mathbf{D}(B) \arrow[l, shift right=3ex] \arrow[l, shift left=3ex] \arrow[r, "- \overset{\mathbb{L}}{\underset{B}{\otimes}} T"] & \mathbf{D}(R) \arrow[l, shift right=3ex] \arrow[l, shift left=3ex] \end{tikzcd}$$

such that $\lambda\colon B \to C$ is a homological epimorphism. Moreover, $H^i(C) = 0$ for $i \neq 0$ if and only if $H^i\Big({}_B T \bigotimes_R \mathrm{Hom}^\bullet_R(U,R)\Big) = 0$ for all $i \geqslant 2$, where the complex U is a deleted projective resolution of T.

P r o o f. Denote by U a deleted projective resolution of T. Then $T = H^0(U)$ and $U \in \mathbf{D}(R)$ is a good silting object. It is easy to see that $\mathbf{D}(\mathrm{DgEnd}_R(U),d) = \mathbf{D}(B)$ and then the recollement follows by Proposition 3.3.

On the other hand, since $U \in \mathbf{D}(R)$ is a good silting object, it is shown in Lemma 4.4 that $U \in \mathbf{D}(B^{\mathrm{op}})$ is weak silting. Therefore, by Theorem 4.5, there exist a dg-algebra D, such that $\lambda\colon B \to D$ is a homological epimorphism. Note that Theorem 3.7, Proposition 4.5 and [14], Remark 5.6 (1), imply that C is weak nonpositive if and only if D is weak nonpositive if and only if $H^i\Big({}_B U \overset{\mathbb{L}}{\underset{R}{\bigotimes}} \mathrm{Hom}^\bullet_{B^{\mathrm{op}}}(U,B)\Big) \cong H^i\Big(T \bigotimes_R \mathrm{Hom}^\bullet_R(U,R)\Big) = 0$ for all $i \geqslant 2$.

In the following, we will show that in this case, the dg-algebra C is always weak positive. By Proposition 4.10, we only need to prove that the dg-algebra D is weak positive. From [14], Definition 4.1 (3) there exists a quasi-isomorphism ${}_B B \to M$ in $\mathbf{K}(B^{\mathrm{op}})$, where $M = 0 \to M_0 \to M_1 \to \ldots \to M_n \to 0$ with $M_i \in \mathrm{Prod}({}_B T)$ for all $0 \leqslant i \leqslant n$. From [14], Lemma 5.5, T is a strongly R-Mittag-Leffler module, see [14], Definition 4.1 or [2], Definition 1.1. Therefore, by the proof of Lemma 4.3 (2) of [14] we obtain a short exact sequence of complexes

$$0 \to T \bigotimes_R \mathrm{Hom}^\bullet_{B^{\mathrm{op}}}(T,M) \to M \to L \to 0,$$

where all terms of the complex L are concentrated in degrees not less than 0. Hence, we can construct the commutative diagram in $\mathbf{D}(B^{\mathrm{op}})$:

$$\begin{tikzcd}
T \overset{\mathbb{L}}{\underset{R}{\otimes}} \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(T,B) \arrow[r] \arrow[d, "\simeq"'] & B \arrow[r] \arrow[d, "\simeq"'] & \delta(B) \arrow[r, rightsquigarrow] \arrow[d, "\simeq"'] & {} \\
T \underset{R}{\otimes} \mathrm{Hom}^{\bullet}_{B^{\mathrm{op}}}(T,M) \arrow[r] & M \arrow[r] & L \arrow[r, rightsquigarrow] & {}.
\end{tikzcd}$$

In particular, $H^j(\delta(B)) \cong H^j(L) = 0$ for all $j < 0$. Hence, by the fact that $\delta(B) \cong D$, we deduce that D is weak positive. $\square$

If we specialize Theorem 5.3 to the case that T is a good 1-tilting module, then it is easy to check that $H^i\Big(T \underset{R}{\otimes} \mathrm{Hom}^{\bullet}_R(U,R)\Big) = 0$ for all $i \geqslant 2$, and we get the following corollary. One can compare it with Theorem 1.1 of [13].

Corollary 5.4. *Let R be a ring, T_R a good 1-tilting R-module and B the endomorphism ring of T. Then there is a ring C, a homological ring epimorphism $\lambda\colon B \to C$ and a recollement among the unbounded derived categories of the rings R, B and C*

$$\begin{tikzcd}
\mathbf{D}(C) \arrow[r, "\lambda_*"] & \mathbf{D}(B) \arrow[l, shift right=8] \arrow[l, shift left=8] \arrow[r, "j^*"] & \mathbf{D}(R) \arrow[l, shift right=8] \arrow[l, shift left=8]
\end{tikzcd}$$

such that the triangle functor j^* is isomorphic to the total left-derived functor $- \overset{\mathbb{L}}{\underset{B}{\otimes}} T$.

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