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SOME HOMOLOGICAL PROPERTIES OF AMALGAMATED MODULES ALONG AN IDEAL

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Abstract. Let R and S be commutative rings with identity, J be an ideal of S, $f\colon R\to S$ be a ring homomorphism, M be an R-module, N be an S-module, and let $\varphi\colon M\to N$ be an R-homomorphism. The amalgamation of R with S along J with respect to f denoted by $R\bowtie^f J$ was introduced by M. D'Anna et al. (2010). Recently, R. El Khalfaoui et al. (2021) introduced a special kind of $(R\bowtie^f J)$-module called the amalgamation of M and N along J with respect to φ, and denoted by $M\bowtie^{\varphi} JN$. We study some homological properties of the $(R\bowtie^f J)$-module $M\bowtie^{\varphi} JN$. Among other results, we investigate projectivity, flatness, injectivity, Cohen-Macaulayness, and prime property of the $(R\bowtie^f J)$-module $M\bowtie^{\varphi} JN$ in connection to their corresponding properties of the R-modules M and JN.

Keywords: amalgamation of ring; amalgamation of module; Cohen-Macaulay; injective module; projective(flat) module

MSC 2020: 13A15, 13C10, 13C11, 13C14, 13C15

1. Introduction

Throughout this paper all rings are considered commutative with the identity element and all modules are unital. Let $f\colon R\to S$ be a ring homomorphism, and let J be an ideal of S. In [4] and [5], D'Anna et al. introduced and studied the following subring of $R\times S$:

$$R\bowtie^f J=\{(r,f(r)+j)\colon r\in R,\ j\in J\},$$

which is called the *amalgamation* of R with S along J with respect to f. It is easily seen that $R\bowtie^f J=\{(r,s)\colon r\in R,\ s\in S,\ f(r)-s\in J\}$. Categorically, $R\bowtie^f J$ is in fact a pullback (or fiber product) of the canonical projection $\pi\colon S\to S/J$ and

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$\breve{f} := \pi \circ f$, see [4], Proposition 4.2. This point of view allows the authors in [4] and [5] to investigate various properties of $R \bowtie^f J$ in connection with properties of R, J and f. Several properties of the construction $R \bowtie^f J$ are investigated in [4], [5], and [13]. The amalgamated duplication of a ring along an ideal, introduced in [6], can be considered as a particular case of $R \bowtie^f J$, see [4], Examples 2.5 and 2.6. This construction has been studied in [3], [7], [11], and [12].

Recently, El Khalfaoui et al. in [8] introduced and studied some basic properties of the amalgamated duplication of modules along an ideal. Let M be an R-module, N be an S-module, which is an R-module induced naturally by f, and let $\varphi\colon M \to N$ be an R-homomorphism. The amalgamation of M and N along J with respect to φ, denoted by $M \bowtie^\varphi JN$, is defined as

$$M \bowtie^\varphi JN = \{(m, \varphi(m) + n)\colon\ m \in M,\ n \in JN\}.$$

For every $(r, f(r) + j) \in R \bowtie^f J$ and for every $(m, \varphi(m) + n) \in M \bowtie^\varphi JN$, the following scalar multiplication gives an $(R \bowtie^f J)$-module structure to $M \bowtie^\varphi JN$:

$$(r, f(r) + j)(m, \varphi(m) + n) = (rm, \varphi(rm) + f(r)n + j\varphi(m) + jn).$$

Note that if $M = R$, $N = S$, and $\varphi = f$, then $M \bowtie^\varphi JN$ coincides with $R \bowtie^f J$. Also, if $S = R$, $N = M$, and $\varphi = \mathrm{id}_M$, then $M \bowtie^\varphi JN$ is exactly $M \bowtie J$ which is introduced in [1]. In this paper, we study some basic properties of the ring $R \bowtie^f J$ in Section 2, which will be used to study some homological properties of the $(R \bowtie^f J)$-module $M \bowtie^\varphi JN$ in Section 3. Using the fact that the amalgamation can be studied in the frame of pullback constructions, it is shown in Theorem 3.4 that the $(R \bowtie^f J)$-module $M \bowtie^\varphi JN$ is projective (or flat) if and only if the R-module M is projective (or flat) and the $(f(R) + J)$-module $\varphi(M) + JN$ is projective (or flat, respectively). Also, we show that over a Noetherian ring R, the R-module $M \bowtie^\varphi JN$ is injective if and only if the $(R \bowtie^f J)$-module $M \bowtie^\varphi JN$ is injective provided that J is a flat R-module. The notion of a strongly cotorsion module was introduced by Xu in [15] as a generalization of the injectivity of modules. The strongly cotorsion property of a dth local cohomology module $M \bowtie^\varphi JN$ is studied in Theorem 3.15, where $d = \dim_{R \bowtie^f J}(M \bowtie^\varphi JN)$. Finally, we investigate Cohen-Macaulay and prime properties of the $(R \bowtie^f J)$-module $M \bowtie^\varphi JN$ in connection with Cohen-Macaulay and prime properties of the R-module M.

2. Amalgamation of rings

Throughout this section, $f\colon R \to S$ is a ring homomorphism, and J is an ideal of S. The *amalgamation of R with S along J with respect to f*, introduced in [5], denoted by $R \bowtie^f J$, is the following subring of $R \times S$:

$$R \bowtie^f J = \{(r, f(r) + j)\colon\ r \in R,\ j \in J\}.$$

In the case when $S = R$, we can consider the identity map $\mathrm{id} := \mathrm{id}_R\colon R \to R$, and construct $R \bowtie^{\mathrm{id}} J$. This construction is also called an *amalgamated duplication of R along J* instead of an amalgamation of R with R along J with respect to id. Also, we use the notation $R \bowtie J$ instead of $R \bowtie^{\mathrm{id}} J$. This section contains some properties of the amalgamation of rings which will be used in the sequel. In the following proposition, we recall some properties of $R \bowtie^f J$ from [4], Propositions 5.1 and 5.7 and [5], Proposition 2.6.

Proposition 2.1. *The following statements hold.*

(i) *Let $\mathfrak{p} \in \operatorname{Spec}(R)$, and $\mathfrak{q} \in \operatorname{Spec}(S)$. Set*

$$\begin{aligned}\acute{\mathfrak{p}}^f &= \{(p, f(p) + j)\colon\ p \in \mathfrak{p},\ j \in J\},\\ \bar{\mathfrak{q}}^f &= \{(a, f(a) + j)\colon\ a \in R,\ j \in J,\ f(a) + j \in \mathfrak{q}\}.\end{aligned}$$

Then the prime ideals of $R \bowtie^f J$ are of the type $\acute{\mathfrak{p}}^f$ or $\bar{\mathfrak{q}}^f$ for $\mathfrak{p} \in \operatorname{Spec}(R)$ and $\mathfrak{q} \in \operatorname{Spec}(S) \setminus V(J)$. In particular,

$$\operatorname{Max}(R \bowtie^f J) = \{\acute{\mathfrak{m}}^f\colon\ \mathfrak{m} \in \operatorname{Max}(R)\} \cup \{\bar{\mathfrak{n}}^f\colon\ \mathfrak{n} \in \operatorname{Max}(S) \setminus V(J)\}.$$

(ii) *Let I be an ideal of R and set $I \bowtie^f J := \{(i, f(i)+j)\colon\ i \in I,\ j \in J\}$. Then $I \bowtie^f J$ is an ideal of $R \bowtie^f J$. In addition, we have the following canonical isomorphism:*

$$\frac{R \bowtie^f J}{I \bowtie^f J} \cong \frac{R}{I}.$$

(iii) *$f^{-1}(J) \times \{0\}$ is an ideal of $R \bowtie^f J$, and the following canonical isomorphism holds:*

$$\frac{R \bowtie^f J}{f^{-1}(J) \times \{0\}} \cong f(R) + J.$$

(iv) *Let J be a Noetherian R-module. Then $R \bowtie^f J$ is Noetherian if and only if R is Noetherian.*

(v) *$R \bowtie^f J$ is isomorphic as an R-module to $R \oplus J$.*

Remark 2.2. With the notation of Proposition 2.1 for every $j \in J$ we have $(0, j) \in \bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \acute{\mathfrak{p}}^f$. Also for every $\mathfrak{q} \in \mathrm{Spec}(S) \setminus V(J)$ there exists $j \in J$ such that $(0, j) \notin \bar{\mathfrak{q}}^f$. Therefore, $\acute{\mathfrak{p}}^f \not\subseteq \bar{\mathfrak{q}}^f$ for every $\mathfrak{p} \in \mathrm{Spec}(R)$ and every $\mathfrak{q} \in \mathrm{Spec}(S) \setminus V(J)$.

Fact 2.3. With the notation of Proposition 2.1, the following statements hold.

(i) $\mathfrak{p} \in \mathrm{Spec}(R)$ if and only if $\mathfrak{p} \bowtie^f J \in \mathrm{Spec}(R \bowtie^f J)$.

(ii) For every $\mathfrak{p} \in \mathrm{Spec}(R)$, and for every $\mathfrak{q} \in \mathrm{Spec}(S) \setminus V(J)$, we have $\acute{\mathfrak{p}}^f \neq \bar{\mathfrak{q}}^f$.

(iii) For every $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathrm{Spec}(R)$, $\mathfrak{p}_1 = \mathfrak{p}_2$ if and only if $\acute{\mathfrak{p}_1}^f = \acute{\mathfrak{p}_2}^f$.

(iv) For every $\mathfrak{q}_1, \mathfrak{q}_2 \in \mathrm{Spec}(S) \setminus V(J)$, $\mathfrak{q}_1 = \mathfrak{q}_2$ implies that $\overline{\mathfrak{q}_1}^f = \overline{\mathfrak{q}_2}^f$.

(v) Let $\mathfrak{q}_1, \mathfrak{q}_2 \in \mathrm{Spec}(S) \setminus V(J)$ such that $\mathfrak{q}_1, \mathfrak{q}_2 \subseteq f(R) + J$. Then $\mathfrak{q}_1 = \mathfrak{q}_2$, provided that $\overline{\mathfrak{q}_1}^f = \overline{\mathfrak{q}_2}^f$.

3. Amalgamation of modules

Throughout this section, $f\colon R \to S$ is a ring homomorphism, J is an ideal of S, M is an R-module, N is an S-module and $\varphi\colon M \to N$ is an R-homomorphism. The *amalgamation of M and N along J with respect to φ* denoted by $M \bowtie^\varphi JN$ is

$$M \bowtie^\varphi JN = \{(m, \varphi(m) + n)\colon\ m \in M,\ n \in JN\}.$$

This notion is introduced in [8]. It can be seen that $M \bowtie^\varphi JN$ is an $(R \bowtie^f J)$-module by the following scalar multiplication:

$$(r, f(r) + j)(m, \varphi(m) + n) = (rm, \varphi(rm) + f(r)n + j\varphi(m) + jn)$$

for every $(r, f(r) + j) \in R \bowtie^f J$ and $(m, \varphi(m) + n) \in M \bowtie^\varphi JN$.

Remark 3.1. By the scalar multiplication of $(R \bowtie^f J)$-module $M \bowtie^\varphi JN$, it is easy to check that $(\{0\} \times J)(M \bowtie^\varphi JN) = \{0\} \times JN$. Also, $(f^{-1}(J) \times \{0\}) \times (M \bowtie^\varphi JN) = f^{-1}(J)M \times \{0\} = \varphi^{-1}(JN) \times \{0\}$.

Remark 3.2. Let $p_R\colon R \bowtie^f J \to R$ and $p_{f(R)+J}\colon R \bowtie^f J \to f(R) + J$ be the natural projections. Also, consider the ring homomorphism $h\colon R \to R \bowtie^f J$ such that $h(r) = (r, f(r))$. Therefore, every $(R \bowtie^f J)$-module has R-module structure via h and every R-module has $(R \bowtie^f J)$-module structure via p_R. In particular, $M \bowtie^\varphi JN$ has R-module structure. Also M and JN have $(R \bowtie^f J)$-module structure.

In the following proposition, we recall some properties of $(R \bowtie^f J)$-module $M \bowtie^\varphi JN$ from [8].

Proposition 3.3. *The following statements hold.*

(i) *Let JN be a Noetherian R-module. Then $M \bowtie^{\varphi} JN$ is a Noetherian $(R \bowtie^{f} J)$-module if and only if M is a Noetherian R-module.*

(ii) *The sequence $0 \to JN \to M \bowtie^{\varphi} JN \to M \to 0$ of $(R \bowtie^{f} J)$-modules and $(R \bowtie^{f} J)$-homomorphisms is exact, where $\iota\colon JN \to M \bowtie^{\varphi} JN$ is given by $\iota(n) = (0, n)$ and $p_M\colon M \bowtie^{\varphi} JN \to M$ is the natural projection.*

(iii) *$M \bowtie^{\varphi} JN/(\{0\} \times JN) = M$.*

(iv) *$M \bowtie^{\varphi} JN/(\varphi^{-1}(JN) \times \{0\}) = \varphi(M) + JN$.*

In [8], Remark 2.1 it is shown that $\varphi(M) + JN$ is an $(f(R) + J)$-submodule of N. Therefore, we have the following result.

Theorem 3.4. *The $(R \bowtie^{f} J)$-module $M \bowtie^{\varphi} JN$ is projective (or flat) if and only if the R-module M is projective (or flat) and the $(f(R) + J)$-module $\varphi(M) + JN$ is projective (or flat, respectively).*

P r o o f. Assume that $\pi_R\colon R \to (f(R) + J)/J$ and $\pi_{f(R)+J}\colon f(R) + J \to (f(R) + J)/J$ are natural epimorphisms. Then the following diagram is a pull-back of rings:

$$\begin{array}{ccc} R \bowtie^{f} J & \xrightarrow{\;p_R\;} & R \\ \big\downarrow{\scriptstyle p_{f(R)+J}} & & \big\downarrow{\scriptstyle \pi_R} \\ f(R) + J & \xrightarrow{\;\pi_{f(R)+J}\;} & \dfrac{f(R)+J}{J} \end{array}$$

Also, by Proposition 2.1, Remark 3.1, and Proposition 3.3, we have:

$$\begin{aligned} (M \bowtie^{\varphi} JN) \bigotimes_{R \bowtie^{f} J} R &\cong (M \bowtie^{\varphi} JN) \bigotimes_{R \bowtie^{f} J} \frac{R \bowtie^{f} J}{\{0\} \times J} \cong \frac{M \bowtie^{\varphi} JN}{(\{0\} \times J)(M \bowtie^{\varphi} JN)} \\ &= \frac{M \bowtie^{\varphi} JN}{\{0\} \times JN} = M, \end{aligned}$$

and

$$\begin{aligned} (M \bowtie^{\varphi} JN) \bigotimes_{R \bowtie^{f} J} (f(R) + J) &\cong (M \bowtie^{\varphi} JN) \bigotimes_{R \bowtie^{f} J} \frac{R \bowtie^{f} J}{f^{-1}(J) \times \{0\}} \\ &\cong \frac{M \bowtie^{\varphi} JN}{(f^{-1}(J) \times \{0\})(M \bowtie^{\varphi} JN)} \\ &= \frac{M \bowtie^{\varphi} JN}{f^{-1}(J)M \times \{0\}} = \frac{M \bowtie^{\varphi} JN}{\varphi^{-1}(JN) \times \{0\}} \\ &= \varphi(M) + JN. \end{aligned}$$

Now the assertion follows from [10], Theorem 1. □

Proposition 3.5. *$M \bowtie^{\varphi} JN$ is isomorphic as an R-module to $M \oplus JN$.*

P r o o f. For every $m \in M$ and $n \in JN$, we define

$$g\colon M \bowtie^{\varphi} JN \to M \oplus JN$$

such that $g((m, \varphi(m) + n)) = (m, n)$. It is easy to check that g is well-defined, one to one and epimorphism. For every $r \in R$ and $(m, \varphi(m) + n) \in M \bowtie^{\varphi} JN$, we have $r(m, \varphi(m) + n) = (rm, \varphi(rm) + f(r)n)$ since $M \bowtie^{\varphi} JN$ is an R-module via the ring homomorphism $h\colon R \to R \bowtie^{f} J$, where $h(r) = (r, f(r))$. Hence, we have the following equation:

$$\begin{aligned} g(r(m, \varphi(m) + n)) &= g((rm, \varphi(rm) + f(r)n)) = (rm, f(r)n) \\ &= r(m, n) = rg((m, \varphi(m) + n)). \end{aligned}$$

□

Corollary 3.6. *The following statements hold.*

(i) *The R-module $M \bowtie^{\varphi} JN$ is projective if and only if the R-modules M and JN are projective.*

(ii) *If the $(R \bowtie^{f} J)$-modules M and JN are projective, then so is the $(R \bowtie^{f} J)$-module $M \bowtie^{\varphi} JN$.*

(iii) *If the $(R \bowtie^{f} J)$-modules $M \bowtie^{\varphi} JN$ and M are projective, then so is the $(R \bowtie^{f} J)$-module JN.*

(iv) *If R is a Noetherian ring, then the R-module $M \bowtie^{\varphi} JN$ is injective if and only if the R-modules M and JN are injective.*

(v) *Let R be a Noetherian ring, and let the $(R \bowtie^{f} J)$-modules M and JN be injective. Then so is the $(R \bowtie^{f} J)$-module $M \bowtie^{\varphi} JN$.*

(vi) *Let R be a Noetherian ring, and let the $(R \bowtie^{f} J)$-modules $M \bowtie^{\varphi} JN$ and JN be injective. Then so is the $(R \bowtie^{f} J)$-module M.*

P r o o f. The items (i) and (iv) follow from Proposition 3.5, and the others are induced by using the exact sequence $0 \to JN \to M \bowtie^{\varphi} JN \to M \to 0$ of $(R \bowtie^{f} J)$-modules and $(R \bowtie^{f} J)$-homomorphisms, see Proposition 3.3. □

Proposition 3.7. *The following statements hold.*

(i) *Let R be a Noetherian ring, and let M and JN be injective R-modules. Then so is the $(R \bowtie^{f} J)$-module $M \bowtie^{\varphi} JN$.*

(ii) *Let $M \bowtie^{\varphi} JN$ be an injective $(R \bowtie^{f} J)$-module. Then there exists an injective R-module E such that $M \bowtie^{\varphi} JN$ is a direct summand of the $(R \bowtie^{f} J)$-module $\mathrm{Hom}_R(R \bowtie^{f} J, E)$.*

P r o o f. (i) In the following sequence, the first R-isomorphism follows from the Hom-tensor adojointness and tensor cancellation, and the latter is induced by Proposition 3.5.

$$\begin{aligned}\mathrm{Hom}_{R\bowtie^f J}(-, \mathrm{Hom}_R(R \bowtie^f J, M \bowtie^{\varphi} JN)) &\cong \mathrm{Hom}_R(-, M \bowtie^{\varphi} JN)\\ &\cong \mathrm{Hom}_R(-, M) \oplus \mathrm{Hom}_R(-, JN).\end{aligned}$$

(ii) It follows from [13], Proposition 2.7. □

Corollary 3.8. *Let R be a Noetherian ring, and let the R-module $M \bowtie^{\varphi} JN$ be injective. Then so is the $(R \bowtie^f J)$-module $M \bowtie^{\varphi} JN$.*

P r o o f. This follows from Corollary 3.6 (iv) and Proposition 3.7. □

In the following, we show that the converse of Corollary 3.8 holds, provided that J is a flat R-module.

Proposition 3.9. *Let J be a flat R-module, and let $M \bowtie^{\varphi} JN$ be an injective $(R \bowtie^f J)$-module. Then so is the R-module $M \bowtie^{\varphi} JN$.*

P r o o f. In the following sequence, the first isomorphism follows from Hom cancelation, the second one is induced by Hom-tensor adjointness, and the third one follows from Proposition 2.1.

$$\begin{aligned}\mathrm{Hom}_R(-, M \bowtie^{\varphi} JN) &\cong \mathrm{Hom}_R(-, \mathrm{Hom}_{R\bowtie^f J}(R \bowtie^f J, M \bowtie^{\varphi} JN))\\ &\cong \mathrm{Hom}_{R\bowtie^f J}\Big(- \bigotimes_R R \bowtie^f J, M \bowtie^{\varphi} JN\Big)\\ &\cong \mathrm{Hom}_{R\bowtie^f J}\Big(- \bigotimes_R (R \oplus J), M \bowtie^{\varphi} JN\Big)\\ &\cong \mathrm{Hom}_{R\bowtie^f J}(-, M \bowtie^{\varphi} JN) \oplus \mathrm{Hom}_{R\bowtie^f J}\Big(- \bigotimes_R J, M \bowtie^{\varphi} JN\Big).\end{aligned}$$

By the assumption, the functors

$$\mathrm{Hom}_{R\bowtie^f J}(-, M \bowtie^{\varphi} JN) \quad \text{and} \quad \mathrm{Hom}_{R\bowtie^f J}\Big(- \bigotimes_R J, M \bowtie^{\varphi} JN\Big)$$

are exact. So, we get the assertion. □

Proposition 3.10. *With the notation of Proposition* 2.1, *the following statements hold.*

(i) *The ideals* $\acute{\mathfrak{p}}^f$ *and* $\bar{\mathfrak{q}}^f$ *belong to* $\mathrm{Supp}_{R\bowtie^f J}(M \bowtie^\varphi JN)$ *for every* $\mathfrak{p} \in \mathrm{Supp}_R(M)$ *and* $\mathfrak{q} \in \mathrm{Supp}_S(N) \setminus V(J)$.

(ii) $\mathrm{Supp}_{R\bowtie^f J}(M \bowtie^\varphi JN) = \mathrm{Supp}_{R\bowtie^f J}(M) \cup \mathrm{Supp}_{R\bowtie^f J}(JN)$.

(iii) $\mathrm{Supp}_R(M \bowtie^\varphi JN) = \mathrm{Supp}_R(M) \cup \mathrm{Supp}_R(JN)$.

(iv) $\mathrm{Ass}_{R\bowtie^f J}(M \bowtie^\varphi JN) \subseteq \mathrm{Ass}_{R\bowtie^f J}(M) \cup \mathrm{Ass}_{R\bowtie^f J}(JN)$.

(v) $\mathrm{Ass}_R(M \bowtie^\varphi JN) = \mathrm{Ass}_R(M) \cup \mathrm{Ass}_R(JN)$.

Proof. (i) Let $\mathfrak{p} \in \mathrm{Spec}(R)$, and $\mathfrak{q} \in \mathrm{Spec}(S) \setminus V(J)$. Then $\acute{\mathfrak{p}}^f$ and $\bar{\mathfrak{q}}^f$ belong to $\mathrm{Spec}(R \bowtie^f J)$, by Proposition 2.1. Note that $(M \bowtie^\varphi JN)_{\bar{\mathfrak{q}}^f}$ is canonically isomorphic to $N_{\mathfrak{q}}$, and $(M \bowtie^\varphi JN)_{\acute{\mathfrak{p}}^f}$ is canonically isomorphic to $M_{\mathfrak{p}}$, where $\mathfrak{p} \notin V(f^{-1}(J))$, by [8], Proposition 2.4. Also, for any $\mathfrak{p} \in \mathrm{Spec}(R)$ containing $f^{-1}(J)$, consider a multiplicative subset $T_{\mathfrak{p}} := f(R \setminus \mathfrak{p}) + J$ of S and a set $N_{T_{\mathfrak{p}}} := {T_{\mathfrak{p}}}^{-1}N$ and $J_{T_{\mathfrak{p}}} := {T_{\mathfrak{p}}}^{-1}J$. Then $(M \bowtie^\varphi JN)_{\acute{\mathfrak{p}}^f}$ is canonically isomorphic to $M_{\mathfrak{p}} \bowtie^{\varphi_{\mathfrak{p}}} J_{T_{\mathfrak{p}}} N_{T_{\mathfrak{p}}}$, where $f_{\mathfrak{p}} : R_{\mathfrak{p}} \to S_{T_{\mathfrak{p}}}$ is a ring homomorphism induced by f and $\varphi_{\mathfrak{p}} \colon M_{\mathfrak{p}} \to N_{T_{\mathfrak{p}}}$ is an $R_{\mathfrak{p}}$-homomorphism induced by φ, by [8], Proposition 2.4. So, we get the assertion.

The items (ii), (iv) and (iii), (v) follow from Proposition 3.3 (ii) and Proposition 3.5, respectively. □

In the following, we investigate the annihilator of $M \bowtie^\varphi JN$ as an $(R \bowtie^f J)$-module.

Remark 3.11. Let $r \in \mathrm{Ann}_R(M)$. For every $m \in M$, $f(r)\varphi(m) = \varphi(rm) = 0$, since N has naturally R-module structure via f. Therefore, $f(\mathrm{Ann}_R(M)) \subseteq \mathrm{Ann}_S(\varphi(M))$.

Proposition 3.12. *The following statements hold.*

(i) $\mathrm{Ann}_{R\bowtie^f J}(M \bowtie^\varphi JN) \subseteq \mathrm{Ann}_R(M) \bowtie^f J$.

(ii) *If* $JN \subseteq \varphi(M)$, *and* $J \subseteq \mathrm{Ann}_S(\varphi(M))$, *then* $\mathrm{Ann}_{R\bowtie^f J}(M \bowtie^\varphi JN) = \mathrm{Ann}_R(M) \bowtie^f J$. *Moreover,* $R \bowtie^f J/\mathrm{Ann}_{R\bowtie^f J}(M \bowtie^\varphi JN) \cong R/\mathrm{Ann}_R(M)$.

Proof. (i) The statement follows easily from the definition.

(ii) By (i), $\mathrm{Ann}_{R\bowtie^f J}(M \bowtie^\varphi JN) \subseteq \mathrm{Ann}_R(M) \bowtie^f J$. For the converse, let $(r, f(r) + j) \in \mathrm{Ann}_R(M) \bowtie^f J$. For every $(m, \varphi(m) + n) \in M \bowtie^\varphi JN$, we have $rm = 0 = \varphi(rm)$, since $r \in \mathrm{Ann}_R(M)$. By the assumption, $J \subseteq \mathrm{Ann}_S(\varphi(M)) \subseteq \mathrm{Ann}_S(JN)$, and so $j\varphi(m) = 0 = jn$. Note that $f(r)n = 0$, by Remark 3.11. Therefore, $(r, f(r) + j) \in \mathrm{Ann}_{R\bowtie^f J}(M \bowtie^\varphi JN)$. Also,

$$\frac{R \bowtie^f J}{\mathrm{Ann}_{R\bowtie^f J}(M \bowtie^\varphi JN)} = \frac{R \bowtie^f J}{\mathrm{Ann}_R(M) \bowtie^f J} \cong \frac{R}{\mathrm{Ann}_R(M)},$$

by Proposition 2.1. □

Corollary 3.13. *Let $JN \subseteq \varphi(M)$, and $J \subseteq \mathrm{Ann}_S(\varphi(M))$. Then* $\mathrm{Ann}_R(M) \in \mathrm{Spec}(R)$ *if and only if* $\mathrm{Ann}_{R\bowtie^f J}(M \bowtie^\varphi JN) \in \mathrm{Spec}(R \bowtie^f J)$.

Proof. This follows from Proposition 3.12 and Fact 2.3. □

Corollary 3.14. *Let M and JN be Noetherian R-modules. Then the following statements hold.*

(i) *Let $JN \subseteq \varphi(M)$ and $J \subseteq \mathrm{Ann}_S(\varphi(M))$. Then* $\dim_{R\bowtie^f J}(M \bowtie^\varphi JN) = \dim_R(M)$.

(ii) *Let $J \subseteq \mathrm{Ann}_S(N)$. Then* $\dim_{R\bowtie^f J}(M \bowtie^\varphi JN) = \dim_R(M)$.

Proof. (i) By Proposition 3.3, $M \bowtie^\varphi JN$ is a Noetherian $(R \bowtie^f J)$-module. Now the assertion follows from Proposition 3.12.

(ii) It follows from (i). □

Recall that an R-module L is called a *strongly cotorsion* module if $\mathrm{Ext}^1_R(F, L) = 0$ for all R-modules F with finite flat dimension. One can easily show that if L is a strongly cotorsion R-module, then $\mathrm{Ext}^{i\geqslant 1}_R(F, L) = 0$ for all R-modules F with finite flat dimension. The terminology of strongly cotorsion modules was introduced by Xu in [15] as a special case of cotorsion modules introduced by Enochs in [9] as a generalization of injectivity for modules. In the following, we investigate the strongly cotorsion property of the dth local cohomology module $M \bowtie^\varphi JN$, where $d = \dim_{R\bowtie^f J}(M \bowtie^\varphi JN)$, which gives a generalization of [13], Theorem 2.2.

Theorem 3.15. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let J be a finitely generated R-module such that $\mathrm{Spec}(S) = V(J)$. Then $\mathrm{H}^d_{\mathfrak{m}\bowtie^f J}(M \bowtie^\varphi JN)$ is a strongly cotorsion R-module if and only if the R-modules $\mathrm{H}^d_{\mathfrak{m}}(M)$ and $\mathrm{H}^d_{\mathfrak{m}}(JN)$ are strongly cotorsion, where $d = \dim_{R\bowtie^f J}(M \bowtie^\varphi JN)$.*

Proof. By Proposition 2.1, $R \bowtie^f J$ is a Noetherian local ring with maximal ideal $\mathfrak{m} \bowtie^f J$. In the following, the first R-isomorphism follows from [2], Theorem 4.2.1, and the second one is induced by Proposition 3.5.

$$\mathrm{H}^d_{\mathfrak{m}\bowtie^f J}(M \bowtie^\varphi JN) \cong \mathrm{H}^d_{\mathfrak{m}}(M \bowtie^\varphi JN) \cong \mathrm{H}^d_{\mathfrak{m}}(M \oplus JN) \cong \mathrm{H}^d_{\mathfrak{m}}(M) \oplus \mathrm{H}^d_{\mathfrak{m}}(JN).$$

For any R-module F with finite flat dimension, we have

$$\mathrm{Ext}^1_R(F, \mathrm{H}^d_{\mathfrak{m}\bowtie^f J}(M \bowtie^\varphi JN)) \cong \mathrm{Ext}^1_R(F, \mathrm{H}^d_{\mathfrak{m}}(M)) \oplus \mathrm{Ext}^1_R(F, \mathrm{H}^d_{\mathfrak{m}}(JN)),$$

and so the assertion holds. □

In the following, we investigate the Cohen-Macaulay property of the $(R \bowtie^f J)$-module $M \bowtie^\varphi JN$. First, we consider the depth of $M \bowtie^\varphi JN$.

Remark 3.16. Let $f\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a homomorphism of Noetherin local rings and let M be an S-module which is finitely generated as an R-module. Suppose that $\mathfrak{p} \in \mathrm{Ass}_S(M)$, and let $x \in M$ with $\mathrm{Ann}_S(x) = \mathfrak{p}$. Then f induces an embedding $R/(\mathfrak{p} \cap R) \to S/\mathfrak{p} \cong Sx$ which makes $S/\mathfrak{p}$ a finitely generated $R/(\mathfrak{p} \cap R)$-module. Therefore, $\mathfrak{p} \neq \mathfrak{n}$ implies that $\mathfrak{p} \cap R \neq \mathfrak{m}$. Using this fact, one can show that $\mathrm{depth}_R(M) = \mathrm{depth}_S(M)$.

Theorem 3.17. *Let R and S be Noetherian local rings, and let J be a Noetherian R-module. Then the following statements hold.*

(i) *Let M and JN be Noetherian R-modules. Then* $\mathrm{depth}_{R\bowtie^f J}(M \bowtie^{\varphi} JN) = \mathrm{depth}_R(M \bowtie^{\varphi} JN)$.

(ii) *Let M be a Noetherian R-module. Then* $\mathrm{depth}_{R\bowtie^f J}(M) = \mathrm{depth}_R(M)$.

(iii) *Let JN be a Noetherian R-module. Then* $\mathrm{depth}_{R\bowtie^f J}(JN) = \mathrm{depth}_R(JN) = \mathrm{depth}_S(JN)$.

Proof. We just prove item (i). The proofs of the other items are similar.

(i) By Proposition 2.1, $R \bowtie^f J$ is a Noetherian local ring. Also, $M \bowtie^{\varphi} JN$ is a finitely generated $(R \bowtie^f J)$-module by [8], Proposition 3.2. So, the assertion follows from Remark 3.16. □

Theorem 3.18. *Let R and S be Noetherian local rings, and let M and J be Noetherian R-modules such that $J \subseteq \mathrm{Ann}_S(N)$. Then the $(R \bowtie^f J)$-module $M \bowtie^{\varphi} JN$ is Cohen-Macaulay if and only if the R-module M is Cohen-Macaulay.*

Proof. By Theorem 3.17, $\mathrm{depth}_{R\bowtie^f J}(M \bowtie^{\varphi} JN) = \mathrm{depth}_R(M \bowtie^{\varphi} JN)$. Also, $\mathrm{depth}_R(M \bowtie^{\varphi} JN) = \min\{\mathrm{depth}_R(M), \mathrm{depth}_R(JN)\} = \mathrm{depth}_R(M)$, by Proposition 3.5. Using Corollary 3.14 (ii), we get the assertion. □

Remark 3.19. Let $JN = 0$. Then $M \bowtie^{\varphi} JN = \{(m,\varphi(m))\colon m \in M\}$. It is easy to check that $M \bowtie^{\varphi} JN$ is isomorphic to M as both $(R \bowtie^f J)$-modules and R-modules. Now, assume that R and S are Noetherian local rings, and let M and J be Noetherian R-modules. Then the $(R \bowtie^f J)$-module $M \bowtie^{\varphi} JN$ is Cohen-Macaulay if and only if the R-module $M \bowtie^{\varphi} JN$ is Cohen-Macaulay. Also, the R-module M is Cohen-Macaulay if and only if the $(R \bowtie^f J)$-module M is Cohen-Macaulay, by Theorem 3.18.

Corollary 3.20. *Let R and S be Noetherian local rings, and let M and J be Noetherian R-modules such that $J \subseteq \mathrm{Ann}_S(N)$. Then* $\dim_{R\bowtie^f J}(M \bowtie^{\varphi} JN) = \dim_R(M \bowtie^{\varphi} JN)$, *provided that M is a Cohen-Macaulay R-module.*

Proof. By Theorem 3.18, $\operatorname{depth}_{R\bowtie^f J}(M \bowtie^{\varphi} JN) = \dim_{R\bowtie^f J}(M \bowtie^{\varphi} JN)$. Also, $\operatorname{depth}_{R\bowtie^f J}(M \bowtie^{\varphi} JN) = \operatorname{depth}_R(M \bowtie^{\varphi} JN)$, by Theorem 3.17. Now, Remark 3.19 proves the claim. □

Corollary 3.21. *Let $JN = 0$. Then $\operatorname{Ann}_{R\bowtie^f J}(M) = \operatorname{Ann}_{R\bowtie^f J}(M \bowtie^{\varphi} JN) = \operatorname{Ann}_R(M) \bowtie^f J$. In addition, $\operatorname{Ann}_R(M) \in \operatorname{Spec}(R)$ if and only if $\operatorname{Ann}_{R\bowtie^f J}(M) \in \operatorname{Spec}(R \bowtie^f J)$.*

Proof. The assertions are induced by Proposition 3.12, Remark 3.19, and Fact 2.3. □

Recall that an R-module M is called *prime* if for all $r \in R$ and $m \in M$, the condition $rm = 0$ implies that $m = 0$ or $rM = 0$. For more details, see [14]. In the following, we study the prime property of the $(R \bowtie^f J)$-module $M \bowtie^{\varphi} JN$.

Proposition 3.22. *Let $JN = 0$. Then $M \bowtie^{\varphi} JN$ is a prime $(R \bowtie^f J)$-module if and only if M is a prime R-module.*

Proof. Let $M \bowtie^{\varphi} JN = \{(m, \varphi(m)) \colon m \in M\}$ be a prime $(R \bowtie^f J)$-module, and let $0 \neq r \in R$, $0 \neq m \in M$ such that $rm = 0$. Then

$$(0,0) \neq (r, f(r)) \in R \bowtie^f J \quad \text{and} \quad (0,0) \neq (m, \varphi(m)) \in M \bowtie^{\varphi} JN.$$

Also, $(r, f(r))(m, \varphi(m)) = (rm, \varphi(rm)) = (0,0)$. Therefore, for all $(a, \varphi(a)) \in M \bowtie^{\varphi} JN$ we have

$$(r, f(r))(a, \varphi(a)) = (ra, \varphi(ra)) = (0,0).$$

Hence, $rM = 0$, as desired. The converse is also established in a similar way. □

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