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ON EXTENDING C^k FUNCTIONS FROM AN OPEN SET TO $\mathbb{R}$ WITH APPLICATIONS

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Abstract. For $k \in \mathbb{N} \cup \{\infty\}$ and U open in $\mathbb{R}$, let $\mathrm{C}^k(U)$ be the ring of real valued functions on U with the first k derivatives continuous. It is shown that for $f \in \mathrm{C}^k(U)$ there is $g \in \mathrm{C}^\infty(\mathbb{R})$ with $U \subseteq \operatorname{coz} g$ and $h \in \mathrm{C}^k(\mathbb{R})$ with $fg|_U = h|_U$. The function f and its k derivatives are not assumed to be bounded on U. The function g is constructed using splines based on the Mollifier function. Some consequences about the ring $\mathrm{C}^k(\mathbb{R})$ are deduced from this, in particular that $\mathrm{Q}_{\mathrm{cl}}(\mathrm{C}^k(\mathbb{R})) = \mathrm{Q}(\mathrm{C}^k(\mathbb{R}))$.

Keywords: C^k function; spline; ring of quotient; Mollifier function

MSC 2020: 26A24, 54C30, 13B30

1. Introduction

This note looks at certain subrings of $\mathrm{C}(\mathbb{R})$, the ring of continuous real valued functions on the set $\mathbb{R}$ of real numbers. These subrings are $\mathrm{C}^k(\mathbb{R})$, the ring of C^k functions on $\mathbb{R}$ for $k \in \mathbb{N}$, i.e., the ring of functions on $\mathbb{R}$ with continuous derivatives up to and including the kth, as well as $\mathrm{C}^\infty(\mathbb{R}) = \bigcap_{k\in\mathbb{N}} \mathrm{C}^k(\mathbb{R})$, the ring of functions having derivatives of all orders.

The ring of continuous functions $\mathrm{C}(X)$ on a (completely regular) topological space has been very extensively studied. There is the classic text Gillman and Jerison (see [7]) and a multitude of papers since, see for example, [2] and [8] and the references therein. As a result, much is known about the ring structure of the special case of $\mathrm{C}(\mathbb{R})$, where $\mathbb{R}$ is the real line. Ring theoretical properties of $\mathrm{C}^k(\mathbb{R})$ have received less attention. The impetus for this note was the paper (see [4]) that studied the "domain-like objects" in the category $\mathcal{S}$ of semiprime commutative rings. The domain-like objects are the objects in the limit closure of the subcategory of domains. This limit closure is a reflector into the subcategory. It turns out that $\mathrm{C}(\mathbb{R})$ (in fact all $\mathrm{C}(X)$) and $\mathrm{C}^\infty(\mathbb{R})$ are domain-like (see [4], Examples 4.4.4 and 4.4.6) but $\mathrm{C}^1(\mathbb{R})$ is not (see [4], Example 4.4.5) and indeed, all $\mathrm{C}^k(\mathbb{R})$, $k \in \mathbb{N}$ are not. A description of the domain-like closures (called the DL-closures in [4]) of $\mathrm{C}^k(\mathbb{R})$ is not pursued here, rather the nature of the $\mathrm{C}^k(\mathbb{R})$ as rings is studied, as is the role played by $\mathrm{C}^k(V)$, where V is dense open in $\mathbb{R}$. The key to these results are Theorems 2.1 and 2.4; these are analytic statements of interest apart from the ring theoretic consequences.

Continuous functions on cozero sets and their extensions were first explored in [5], Proposition 1.1. They showed that in a space X, if $U = \operatorname{coz} g$ and $f \in \mathrm{C}(U)$ is bounded, then there is $h \in \mathrm{C}(X)$ such that $fg|_U = h|_U$. In the present context, the real line $\mathbb{R}$ is the space to be studied and more than continuity will come into play. Consider an open subset $U \subseteq \mathbb{R}$ and a C^k function f on U. If f and its k derivatives are all bounded on U, then for any $g \in \mathrm{C}^k(\mathbb{R})$, where $U = \operatorname{coz} g$ and the k derivatives of g are well behaved, extending $fg|_U$ by making it zero in $\mathbb{R} \setminus U$ seems feasible. However, f and its k derivatives need not be bounded on U, making the task more difficult. Section 2 of this article is to show that such g can always be found, and that it is even possible to construct $g \in \mathrm{C}^\infty(\mathbb{R})$. This means that $fg|_U = h|_U$ for some $h \in \mathrm{C}^k(\mathbb{R})$ and $g|_U$ is invertible in $\mathrm{C}^k(U)$. This surprising fact then has ring theoretic consequences explored in Section 3. Once these results have been established, it can be seen that each $\mathrm{C}^k(U)$, U open in $\mathbb{R}$, is a localization of $\mathrm{C}^k(\mathbb{R})$ at a multiplicatively closed set of nonzero divisors. With this in hand, using U dense, it will follow that the classical ring of quotients (the ring of fractions) and the complete ring of quotients of $\mathrm{C}^k(\mathbb{R})$ coincide, are von Neumann regular rings and flat extensions, see Theorem 3.3. These are properties that $\mathrm{C}^k(\mathbb{R})$ shares with $\mathrm{C}(\mathbb{R})$ (and indeed, all $\mathrm{C}(X)$, X a metric space, see [6], Section 3.3). Moreover, all the classical rings of quotients of $\mathrm{C}^k(\mathbb{R})$ with $k \in \mathbb{N} \cup \{\infty\}$ are distinct, see Proposition 3.5.

An essential tool in the analytic part is using the Mollifier, that will give a C^∞ spline, called an *M-spline* in the sequel; see for example, [9], Lemma 1.2.3 but here used in dimension one. The construction is detailed in the next section. The notation for functions in $\mathrm{C}(\mathbb{R})$ follows that of the text of [7]. In particular, for $f \in \mathrm{C}(\mathbb{R})$, $\operatorname{coz} f = \{x \in \mathbb{R} \colon f(x) \neq 0\}$, the cozero set, an open set, and its complement is $\mathrm{z}(f)$, a zero set.

2. $\mathrm{C}^k(U)$ vs $\mathrm{C}^k(\mathbb{R})$, U open: using the Mollifier

2.1. Definition of the M-spline. The basis for the Mollifier-spline, or *M-spline* is the Mollifier function

$$\sigma(x) = \begin{cases} \exp \dfrac{-1}{1-x^2} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geqslant 1. \end{cases}$$

Let $C = \int_{-1}^{1} \sigma(x)\,\mathrm{d}x$. Put $\varphi(x) = (1/C)\sigma(x)$. Thus, $\int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x$ exists and is equal to 1. Now define $\Phi(x) = \int_{-\infty}^{x} \varphi(t)\,\mathrm{d}t$. It follows that $(\mathrm{d}/\mathrm{dx})\Phi(x) = \varphi(x)$ and from that, all the derivatives of $\Phi(x)$ are zero at ± 1, and between -1 and 1, $\Phi(x)$ is a C^{∞} function increasing from 0 to 1.

If now the idea is to have a C^{∞} spline between the points (a, b) and (c, d) in $\mathbb{R}^2$, with $a < c$, first define $\Phi_{a,c}(x) = \Phi((2x - (a + c))/(c - a))$ and then $\gamma(x) = b + (d - b)\Phi_{a,c}(x)$ is defined on the interval $[a, c]$. All the right derivatives of $\gamma(x)$ at a and all the left derivatives at b exist and are zero. If for example $b > d$, then $\gamma(x)$ decreases as x goes from a to c. The size of the absolute value of the kth derivative of $\Phi_{a,c}(x)$ will have to be taken into account when b and d are defined.

The function $\gamma(x)$ is called the *M-spline* between the points (a, b) and (c, d).

2.2. The main analytic result for $k = \infty$. The first theorem, and its companion for k finite, pave the way for the ring theoretical results in Section 3. The proof for $k = \infty$ will be the model for that for k finite. For the purposes of induction, the 0th derivative of a continuous function is taken to be the function itself.

The method used is to show that even if $f \in \mathrm{C}^{\infty}(U)$ and its derivatives are unbounded in U, these functions can be "tamed" by an appropriate choice of constants in the definition of g.

Theorem 2.1. *Let $\emptyset \neq U$ be an open set in $\mathbb{R}$. For $f \in \mathrm{C}^{\infty}(U)$ there is $g \in \mathrm{C}^{\infty}(\mathbb{R})$ such that* (i) *$U \subseteq \operatorname{coz} g$, and* (ii) *$fg|_U$ extends to $h \in \mathrm{C}^{\infty}(\mathbb{R})$.*

P r o o f. The proof will be for dense open sets because $V = U \cup \operatorname{Int}(\mathbb{R} \setminus U)$ is dense in $\mathbb{R}$ and f can be extended to V by making it zero on $\operatorname{Int}(\mathbb{R} \setminus U)$. From now on, it is assumed that V is dense open and $f \in \mathrm{C}^{\infty}(V)$. It will be seen that in this case, $\operatorname{coz} g = V$.

The open set V is a disjoint union of open intervals indexed over N, where $N = \mathbb{N}$ or is finite, $V = \bigcup\limits_{m \in N} (u_m, v_m)$. The first step is to define g in the interval (u_m, v_m). The restriction of g to this interval will be denoted g_m. It is first assumed that the interval is finite of length L_m. To simplify, notation the interval will temporarily be called (u, v) of length L.

The interval will be subdivided with $a_1 = u + L/2$ and for $n > 1$, $a_n = a_{n-1} + (v - a_n)/2 = a_1 + L/2^n$. The idea is to connect the points $(a_n, g(a_n))$ to $(a_{n+1}, g(a_{n+1}))$ with an M-spline and with values of g chosen so that the splines with their derivatives, the product of the splines with f and its derivatives all go to zero as $n \to \infty$. Once this is done, a similar development can be done working from a_1 to the left toward u. The choice of the constants will take some effort.

Before continuing the definition of g, there needs to be a word about unbounded intervals. It might be that there are intervals such as $(-\infty, v)$ or (u, ∞) in the expression for V. The two cases are similar. Assume that the mth interval is $(-\infty, v)$. Here put $u = v - 1$ and $a_1 = v - 1/2$, and do the construction below (in Case A) to define g_m on $[a_1, v]$. To the left of a_1, let g_m be the constant $g_m(a_1)$. Since all the left derivatives of g_m at a_1 are zero, g_m is C^∞ on $(-\infty, a_2)$. It will be seen that it is C^∞ on all of $(-\infty, v)$. To define the constants, the function f and its derivatives are dealt with first. Fix $n \in \mathbb{N}$ and $i \geqslant 0$, put

$$A_{n,i} = 1 + \max_{x \in [a_1, a_{n+1}]} \Big| \frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x) \Big|.$$

By continuity of the derivatives of f in V, these constants are well-defined and for fixed i, the $A_{n,i}$ are at least 1 and are nondecreasing in n.

The next step is to deal with the M-splines. This time, for $n \in \mathbb{N}$ and $i \geqslant 0$, let

$$B_{n,i} = 1 + \max_{j=1,\ldots,n} \Big(\max_{x \in [a_j, a_{j+1}]} \Big\{ \Big| \frac{\mathrm{d}^i}{\mathrm{d}x^i} \Phi_{a_j, a_{j+1}}(x) \Big| \Big\} \Big).$$

Again, these numbers, always not less than 1, are well-defined and the sequence $\{B_{n,i}\}$ for fixed i is nondecreasing in n. In the expression $\max\limits_{x \in [a_j, a_{j+1}]} |(\mathrm{d}^i/\mathrm{d}x^i)\Phi_{a_j,a_{j+1}}(x)|$, the ith derivative at the endpoints should be taken to be one-sided derivative. Both are zero.

To be able to define the function g_m, it will be necessary to combine these constants. However, care must be taken to make sure that at any step in the process only a finite number of these constants are multiplied together. There are two cases to consider; it will be seen later why there is this distinction.

Case A: $L \geqslant 1$. In this situation, the M-spline in the interval $[a_1, a_2]$ will begin by only using the first derivative in the constant and the function f. For each $n \in \mathbb{N}$, define $S_n = \prod\limits_{i=0}^{n} A_{n,i} \prod\limits_{i=0}^{n} B_{n,i}$. (Notice that in the case of an infinite interval, L is taken to be 1.)

Case B: $L < 1$. Let $p \in \mathbb{N}$ be the least natural number with $L \leqslant 1/p$. In this case, the M-spline in the interval $[a_1, a_2]$ will use all the derivatives up to the pth in the constant. For $n \leqslant p$, define $S_n = \prod\limits_{i=0}^{p} A_{n,i} \prod\limits_{i=0}^{p} B_{n,i}$. For $n > p$, $S_n = \prod\limits_{i=0}^{n} A_{n,i} \prod\limits_{i=0}^{n} B_{n,i}$.

With this groundwork, the function g_m can be defined on $[a_1, v)$ using M-splines.

Definition 2.2 (g_m)**.** Using the above data for $x \in [a_n, a_{n+1})$ define $g_m(x) = g_m(a_n) + (g_m(a_{n+1}) - g_m(a_n))\Phi_{a_n,a_{n+1}}(x)$, where for any $l \geqslant 1$, $g_m(a_l) = L^2/(2^{2l+1}S_l)$.

The function g_m must also be defined on $[u, a_1]$. Here the process is similar with $(u, a_1]$ subdivided as a mirror image, say with $a_1 = b_1 > b_2 > \ldots$, and the constants defined in the same manner. The constants $A_{n,i}$ and $B_{n,i}$ are defined as above, although their values may differ from those on the right half.

Definition 2.3 (g and h). Now revert to the notation for the mth interval as (u_m, v_m) and g can now be defined:

$$g(x) = \begin{cases} g_m(x) & \text{if } x \in (u_m, v_m),\ m \in N, \\ 0 & \text{if } x \notin V. \end{cases}$$

By construction $\operatorname{coz} g = V$. It must now be shown that $g \in \mathrm{C}^\infty(\mathbb{R})$ and that $fg|_V$, defined on V, extends to $h \in \mathrm{C}^\infty(\mathbb{R})$.

Both parts will be done together. First define h:

$$h(x) = \begin{cases} f(x)g(x) & \text{if } x \in V, \\ 0 & \text{if } x \notin V. \end{cases}$$

By construction, both g and h are C^∞ on V. The complement of V has two sorts of points and they will have to be dealt with differently.

The proof that g and h are in $\mathrm{C}^\infty(\mathbb{R})$ will proceed by showing that they are in $\mathrm{C}^r(\mathbb{R})$ for each $r \geqslant 0$. This is done by induction on r.

The base case, for $r = 0$, only requires that g and h be continuous. The continuity at points in $\mathbb{R} \setminus V$ must be shown. There are two cases to be considered for $x_0 \notin V$ (these cases will come up again in the induction).

Case 1: x_0 is a left or is a right endpoint of an interval making up V.

Case 2: x_0 is not a left or is not a right endpoint of an interval making up V.

In Case 1, it suffices to show that these functions go to zero at the endpoints of the intervals in V. Consider the interval (u_m, v_m) and the right endpoint. From a_1 to the right, g_m is a decreasing function and $g_m(a_n) = L_m^2/(2^{2n+1}S_n)$, which goes to zero as n increases (since $S_n \geqslant 1$ and $L_m = v_m - u_m$ is fixed). Left endpoints are dealt with in the same way. The function fg_m on the interval also goes to zero as x approaches v_m because for $x \in [a_n, a_{n+1})$, $|f(x)| \leqslant A_{n,0} \leqslant S_n$. Then, $|f(x)g(x)| \leqslant |f(x)|L_m^2/(2^{2n+1}S_n) \leqslant L_m^2/2^{2n+1}$. This goes to zero as $x \to v_m$ because n will increase and L_m is fixed.

In Case 2, if $x_0 \notin V$ is, say, not a right endpoint of an interval, then by density of V, in any interval $(x_0 - \varepsilon, x_0)$ there are infinitely many intervals from V. It is here that the distinction between Cases A and B comes up. Suppose for $p \in \mathbb{N}$ that (u_m, v_m) is an interval with $L_m \leqslant 1/p$. Then $g_m(x) \leqslant g_m(a_1) = L_m^2/2^3$, which goes to zero as $L_m \to 0$. Again, if $x \in [a_n, a_{n+1})$, $|f(x)g(x)| \leqslant L_m^2/2^{2n+1}$. As $x \to x_0$ from the left, x will either be in $\mathbb{R} \setminus V$ and $f(x)g(x) = 0$ or will be in smaller and smaller intervals from V. Since in the calculation, $n \geqslant 1$, $f(x)g(x)$ will converge to zero. This shows the continuity of g and of h, and will be the starting point of the induction.

Throughout, the proof will be for h but that for g will be along the same lines and is simpler, and will not be worked out in detail.

The induction assumption: Assume that $r > 0$ and for $0 \leqslant l < r$:

(i) $(\mathrm{d}^l/\mathrm{d}x^l)h$ exists and is continuous on $\mathbb{R}$, and

(ii) $(\mathrm{d}^l/\mathrm{d}x^l)h$ is zero on $\mathbb{R} \setminus V$,

(iii) $(\mathrm{d}^l/\mathrm{d}x^l)g$ exists and is continuous on $\mathbb{R}$, and

(iv) $(\mathrm{d}^l/\mathrm{d}x^l)g$ is zero on $\mathbb{R} \setminus V$.

As above, it is necessary to look at points $x_0 \in \mathbb{R} \setminus V$. Just as before, these are of two types that need separate proofs.

Case 1: Assume that the interval is (u_m, v_m) and that $x_0 = v_m$. The case of a left endpoint will have a similar proof. Since $|(\mathrm{d}^{r-1}/\mathrm{d}x^{r-1})h(x_0)| = 0$, by the induction assumption, the expression

$$\text{(a)} \qquad \left|\frac{\mathrm{d}^{r-1}}{\mathrm{d}x^{r-1}} \frac{h(x)}{(x - x_0)}\right|$$

must be shown to have limit 0 as $x \to x_0$ from the left. Using the notation for the interval, it can be assumed that $x \in [a_n, a_{n+1})$ for some $n \geqslant 1$ and that $n \geqslant r$. Note that $x_0 - x \geqslant x_0 - a_{n+1} = L/2^{n+1}$. Expression (a) can be rewritten:

$$\begin{aligned}
\text{(b)} \quad & \left|\frac{1}{x - x_0} \sum_{i=0}^{r-1} \frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x) \frac{\mathrm{d}^{r-i-1}}{\mathrm{d}x^{r-i-1}} g(x)\right| \\
&= \frac{1}{x_0 - x}\left|\sum_{i=0}^{r-1} \frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x)(g_m(a_{n+1}) - g_m(a_n)) \frac{\mathrm{d}^{r-1-i}}{\mathrm{d}x^{r-1-i}} \Phi_{a_n,a_{n+1}}(x)\right. \\
&\qquad \left. + g_m(a_n) \frac{\mathrm{d}^{r-1}}{\mathrm{d}x^{r-1}} f(x)\right| \\
&\leqslant \frac{2^{n+1}}{L}\left|\sum_{i=0}^{r-1} \frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x)\Big(\frac{L^2}{2^{2n+3} S_{n+1}} - \frac{L^2}{2^{2n+1} S_n}\Big) \frac{\mathrm{d}^{r-1-i}}{\mathrm{d}x^{r-1-i}} \Phi_{a_n,a_{n+1}}(x)\right| \\
&\qquad + \frac{2^{n+1}}{L}\left|\frac{L^2}{2^{2n+1} S_n} \frac{\mathrm{d}^{r-1}}{\mathrm{d}x^{r-1}} f(x)\right| \\
&= \left|\sum_{i=0}^{r-1} \frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x)\Big(\frac{L}{2^{n+2} S_{n+1}} - \frac{L}{2^n S_n}\Big) \frac{\mathrm{d}^{r-1-i}}{\mathrm{d}x^{r-1-i}} \Phi_{a_n,a_{n+1}}\right| + \frac{L}{2^n S_n}\left|\frac{\mathrm{d}^{r-i}}{\mathrm{d}x^{r-i}} f(x)\right| \\
&\leqslant \left|\frac{L(S_n - 4S_{n+1})}{2^{n+2} S_n S_{n+1}}\right| \sum_{i=0}^{r-1}\left|\frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x) \frac{\mathrm{d}^{r-1-i}}{\mathrm{d}x^{r-1-i}} \Phi_{a_n,a_{n+1}}(x)\right| + \frac{L}{2^n S_n}\left|\frac{\mathrm{d}^{r-1}}{\mathrm{d}x^{r-1}} f(x)\right|.
\end{aligned}$$

However, $S_n \leqslant S_{n+1}$ and each term of the sum is not greater than S_n indeed,

$$\left|\frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x) \frac{\mathrm{d}^{r-1-i}}{\mathrm{d}x^{r-1-i}} \Phi_{a_n,a_{n+1}}(x)\right| \leqslant A_{n,i} B_{n,r-1-i} \leqslant S_n.$$

Also, $|(\mathrm{d}^{r-1}/\mathrm{d}x^{r-1})f(x)| \leqslant S_n$. Hence, the last expression above is not greater than $5L/(2^{n+2} S_n) r S_n + L/2^n = 5rL/2^{n+2} + L/2^n$. Since L and r are fixed, as $x \to x_0$, n increases and the original expression (a) goes to zero.

Case 2: Once again, it is here that the distinction between Cases A and B comes up. Here $x_0 \notin V$ is not the right endpoint of an interval in V. This means that for $\varepsilon > 0$, by the density of V, the interval $(x_0 - \varepsilon, x_0)$ contains infinitely many intervals from V, all of length less than ε. For the proof it may be assumed that $\varepsilon < 1/r$. In this case as well, it must be shown that expression (a) tends to zero as $x \to x_0$ from the left. If $x \in (x_0 - \varepsilon, x_0) \setminus V$, then (a) is zero. From now on, assume $x \in (x_0 - \varepsilon, x_0) \cap V$.

If $x_0 - \varepsilon$ is the right endpoint of an interval in V, by reducing ε, it may be assumed that $x \in (u_m, v_m)$ for an interval in V of length less than $1/r$. Here, Case B will apply. In this case, no matter where x is in the interval, the constant S_n used in the definition of g will always involve the derivatives up to at least the $(r-1)$st. With this observation in hand, the proof in this case uses some of the calculations of the one for x_0 a right endpoint.

The work will be done in (u_m, v_m). A factor in (a) is $1/(x_0 - x)$. As before, there are the points $a_1 < a_2 < \ldots < v_m$ and, on the left side, $b_1 = a_1 > b_2 > \ldots > u_m$. If x falls into (u_m, v_m), there are two cases to consider. If $x \in [a_n, a_{n+1})$, then $x_0 - x > v_m - x > v_m - a_{n+1} = L/2^{n+1}$. If $x \in [b_{n+1}, b_n)$, $x_0 - x > v_m - x > x - u_m > b_{n+1} - u_m = L/2^{n+1}$. By the calculation in Case 1 in the interval (u_m, v_m), (a) $\leqslant 5rL/2^{n+2} + L/2^n$ in the first instance. In the second instance with $x \in [b_{n+1}, b_n)$, the calculation in Case 1 can still be used once it is noticed that $x_0 - x > (1/2^{n+1})L$, as indicated above. Then the calculation of (a) going down to u_m gives (a) $\leqslant 5rL/2^{n+2} + L/2^n$, again. In both cases, $n \geqslant 1$ and r is fixed. However, if $\varepsilon > 0$ and there is an interval I from V inside $(x_0 - \varepsilon, x_0)$, then the length of I is less than ε; this implies that as $x \to x_0$, $L \to 0$, showing that (a) $\to 0$. Recall that if $x \notin V$, the expression (a) $= 0$, by the induction assumption.

It has thus been shown that $(\mathrm{d}^r/\mathrm{d}x^r)h(x)$ exists for all x and has value 0 on $\mathbb{R} \setminus V$. It remains to show the continuity of $(\mathrm{d}^r/\mathrm{d}x^r)h$. This, in both cases, is very much like the calculations for the existence of the derivative. Everything proceeds as before without the factor $1/(x_0 - x)$ and with the factor $r+1$ instead of r.

As mentioned, similar calculations, without the factor r and not involving f, show the same results for g.

The induction is now complete and the functions g and h are in $\mathrm{C}^\infty(\mathbb{R})$. □

2.3. The cases $\mathrm{C}^k(\mathbb{R})$ for $k \in \mathbb{N}$. If the function f in the above is only in $\mathrm{C}^k(\mathbb{R})$ for some $k \in \mathbb{N}$, then it must be shown that there is an analogue to Theorem 2.1.

Theorem 2.4. *Let $\emptyset \neq U$ be an open set in $\mathbb{R}$ and $k \in \mathbb{N}$. For $f \in \mathrm{C}^k(U)$ there is $g \in \mathrm{C}^\infty(\mathbb{R})$ such that* (i) *$U \subseteq \operatorname{coz} g$, and* (ii) *$fg|_U$ extends to $h \in \mathrm{C}^k(\mathbb{R})$.*

P r o o f. As in Theorem 2.1, U may be expanded to a dense open set V. The dense open set V is expressed as a union of disjoint intervals, $\bigcup\limits_{m\in N}(u_m, v_m)$. One such finite interval is divided so that $a_1 < a_2 < \ldots < v_m$ and $a_1 = b_1 > b_2 > \ldots > u_m$ as above, with the same device used for any infinite intervals. The proof follows much the same pattern as that for Theorem 2.1 but the constants $A_{n,i}$ require modification since only the derivatives of f up to the k th are available; they are given a different symbol. In order to define g and h, the constants $B_{n,i}$ are unchanged. The same schema is used: for $i = 1, \ldots, k$, $\mathfrak{A}_{n,i} = 1 + \max\limits_{x\in[a_1,a_{n+1}]}|(\mathrm{d}^i/\mathrm{d}x^i)f(x)|$; for $i > k$, $\mathfrak{A}_{n,i} = 1 + \max\limits_{x\in[a_1,a_{n+1}]}|(\mathrm{d}^k/\mathrm{d}x^k)f(x)|$.

The definition of the constants S_n, as before, divides into two cases according to the length L of the interval from V in question:

Case A: $L \geqslant 1$. (Recall that for an infinite interval, $L = 1$.) In this situation the M-spline in the interval $[a_1, a_2]$ will begin using the first derivative and the function f. The constant S_n is given by

$$S_n = \prod_{i=0}^{n}\mathfrak{A}_{n,i}\prod_{i=0}^{n}B_{n,i}.$$

Case B: $L < 1$. Let $p \in \mathbb{N}$ be the least natural number with $L \leqslant 1/p$. In this case the M-spline in the interval $[a_1, a_2]$ will use $A_{1,i}$ for $i = 0, \ldots, p$. For $n \leqslant p$, define $S_n = \prod\limits_{i=0}^{p}\mathfrak{A}_{n,i}\prod\limits_{i=0}^{p}B_{n,i}$. For $n > p$, $S_n = \prod\limits_{i=0}^{n}\mathfrak{A}_{n,i}\prod\limits_{i=0}^{n}B_{n,i}$.

From this point on, the proof proceeds as in Theorem 2.1 except that for h, in which case the induction stops at the kth derivative, although that for g can continue, making $g \in \mathrm{C}^\infty(\mathbb{R})$. The induction is done in parallel for g and h. This needs to be made more precise as follows; note that k is at least 1. The continuity of h and g is as in Theorem 2.1 giving the starting point of the induction.

The induction assumption:

(a) For the induction on h, assume for $0 < r \leqslant k$ and for $0 \leqslant l < r$ that
  (i) $(\mathrm{d}^l/\mathrm{d}x^l)h$ exists and is continuous on $\mathbb{R}$, and
  (ii) $(\mathrm{d}^l/\mathrm{d}x^l)h$ is zero on $\mathbb{R}\setminus V$.

(b) For the induction on g, assume for $0 < r$ and for $0 \leqslant l < r$ that
  (i) $(\mathrm{d}^l/\mathrm{d}x^l)g$ exists and is continuous on $\mathbb{R}$, and
  (ii) $(\mathrm{d}^l/\mathrm{d}x^l)g$ is zero on $\mathbb{R}\setminus V$. □

The following will be used in the next section; it is a C^k version of the fact that every open set in a metric space is a cozero set.

Corollary 2.5. *Let $k \in \mathbb{N}\cup\{\infty\}$. Then for any open set U in $\mathbb{R}$ there is $a \in \mathrm{C}^k(\mathbb{R})$ such that $\operatorname{coz} a = U$. In fact, a can be chosen to be in $\mathrm{C}^\infty(\mathbb{R})$.*

P r o o f. Let $V = U \cup \operatorname{Int}(\mathbb{R} \setminus U)$. Define $f \in \mathrm{C}^k(V)$ by $f(x) = 1$ if $x \in U$ and $f(x) = 0$ otherwise. According to Theorems 2.1 and 2.4, there are $g, h \in \mathrm{C}^k(\mathbb{R})$, $g \in \mathrm{C}^\infty(\mathbb{R})$ with $fg|_V = h|_V$. Then h is zero on $(\mathbb{R} \setminus V) \cup \operatorname{Int}(\mathbb{R} \setminus U) = \mathbb{R} \setminus U$. Then choose $a = h$. Notice that $f \in \mathrm{C}^\infty(V)$, which allows h to be chosen in $\mathrm{C}^\infty(\mathbb{R}) \subseteq \mathrm{C}^k(\mathbb{R})$. □

The corollary also says that the set of nonzero divisors of the ring $\mathrm{C}^k(\mathbb{R})$ is $\{g \in \mathrm{C}^k(\mathbb{R})\colon \operatorname{coz} g \text{ is dense in } \mathbb{R}\}$. Indeed, if $b \in \mathrm{C}^k(\mathbb{R})$ and $\operatorname{coz} b$ is not dense, there is an open set $U \neq \emptyset$ disjoint from $\operatorname{coz} b$. Then if $a \in \mathrm{C}^k(\mathbb{R})$ with $\operatorname{coz} a = U$, it follows that $ab = \mathbf{0}$. Moreover, if U is chosen to be $\operatorname{Int} \mathrm{z}(b)$, then $U \cup \operatorname{coz} b$ is dense and then $ab = \mathbf{0}$ and $a + b$ is a nonzero divisor. This property says that the ring $\mathrm{C}^k(\mathbb{R})$ is *complemented* (see for example, [10], Introduction). Notice that a nonzero divisor in any $\mathrm{C}^k(\mathbb{R})$ is also a nonzero divisor in the larger ring $\mathrm{C}(\mathbb{R})$.

3. Applications to rings of quotients

In the situation of Theorem 2.1, and similarly with that of Theorem 2.4, there is, by restriction, a ring homomorphism $\mathrm{C}^k(\mathbb{R}) \to \mathrm{C}^k(U)$. The nature of the relationship between these two rings will be examined. It may be necessary to recall a few terms. The first is the notion of an *epimorphism* in a category which is recalled. The category here will be that of all rings (with 1), $\mathcal{R}$.

Definition 3.1. In a category $\mathcal{C}$, a morphism $e\colon A \to B$ is called an *epimorphism* if for two morphisms $f, g\colon B \to C$, $f \circ e = g \circ e$ implies $f = g$.

Proposition 3.2. *Let $\emptyset \neq U$ be an open set in $\mathbb{R}$. Then the homomorphism $\psi\colon \mathrm{C}^k(\mathbb{R}) \to \mathrm{C}^k(U)$, given by restriction, is a flat epimorphism in the category of rings. In particular, if U is dense, ψ is a monomorphism.*

P r o o f. Since classical localization gives a flat epimorphism (see [12], Proposition 10.8), it suffices to show that $\mathrm{C}^k(U)$ is a classical localization of $\mathrm{C}^k(\mathbb{R})$. Consider $M = \{a \in \mathrm{C}^k(\mathbb{R})\colon U \subseteq \operatorname{coz} a\}$. This is a multiplicatively closed set. If $a \in M$, then $a|_U$ is invertible in $\mathrm{C}^k(U)$. By the theorems of Section 2, if $f \in \mathrm{C}^k(U)$, there are $g, h \in \mathrm{C}^k(\mathbb{R})$ with $U \subseteq \operatorname{coz} g$ (i.e., $g \in M$), and $fg|_U = h|_U$, or $f = (g|_U)^{-1} h|_U$, giving the result. If U is dense, then the elements of M are nonzero divisors by Corollary 2.5. □

Notice that as in the proposition, for a dense open V, the corresponding multiplicatively closed set of nonzero divisors is $M = \{g \in \mathrm{C}^k(\mathbb{R})\colon V \subseteq \operatorname{coz} g\}$. Every nonzero divisor of $\mathrm{C}^k(\mathbb{R})$ lies in one of these sets.

See [3] for more information about epimorphisms and the rings $\mathrm{C}(X)$: earlier works on epimorphisms in the category of rings are found in the references.

With this in hand, it is possible to show that for $\mathrm{C}^k(\mathbb{R})$, $k \in \mathbb{N} \cup \{\infty\}$, the classical ring of quotients $\mathrm{Q}_{\mathrm{cl}}(\mathrm{C}^k(\mathbb{R}))$ and the complete ring of quotients $\mathrm{Q}(\mathrm{C}^k(\mathbb{R}))$ coincide and are self-injective (von Neuman) regular rings. (For the maximal flat epimorphic ring of quotients, $\mathrm{Q}_{\mathrm{tot}}(\mathrm{C}^k(\mathbb{R}))$, see [12], Chapter XI, Section 4.)

Theorem 3.3. *Let $k \in \mathbb{N} \cup \{\infty\}$. Then the classical ring of quotients $\mathrm{Q}_{\mathrm{cl}}(\mathrm{C}^k(\mathbb{R}))$, the complete ring of quotients $\mathrm{Q}(\mathrm{C}^k(\mathbb{R}))$, and the maximal flat epimorphism extension $\mathrm{Q}_{\mathrm{tot}}(\mathrm{C}^k(\mathbb{R}))$ coincide and is a self-injective (von Neumann) regular ring.*

Proof. To simplify notation, put $R = \mathrm{C}^k(\mathbb{R})$ and for dense open V, $R_V = \mathrm{C}^k(V)$. By restrictions, there is a directed diagram, where for $V \subseteq V'$, $\psi_{V',V}\colon R_{V'} \to R_V$ and $\psi_V\colon R \to R_V$. Call the direct limit T with maps $\tau_V\colon R_V \to T$. The ring R embeds in T. The diagram can also be thought of as a diagram of R-modules, via the embeddings of R in each R_V and from this, of R in T. Hence, T_R is flat because a direct limit of flat modules is flat.

The next step is to show that $T = \mathrm{Q}_{\mathrm{cl}}(R)$. An element $g \in R$ is a nonzero divisor if and only if its cozero set is dense, see Corollary 2.5 again. If $g \in R$ with $\operatorname{coz} g = V$ is dense, then $g|_V$ is invertible in R_V, and so its image is invertible in L. Moreover, if $t \in T$ with representative $f \in R_V$, by Theorem 2.1, there exist $g, h \in R$ with $\operatorname{coz} g = V$ and $fg|_V = h|_V$. Then $f = h|_V (g|_V)^{-1}$, showing that $t = hg^{-1} \in T$.

The next step is to show that $T = \mathrm{Q}_{\mathrm{cl}}(R)$ is the complete ring of quotients. Recall that the complete ring of quotients $\mathrm{Q}(R)$ is the direct limit $\varinjlim \operatorname{Hom}_R(D, R)$, where D ranges over the dense ideals of R, i.e., with $\operatorname{ann} D = \mathbf{0}$. For D a dense ideal, put $V_D = \bigcup\limits_{a \in D} \operatorname{coz} a$. By Corollary 2.5, this is a dense open set. On the other hand, for V dense open, $D_V = \{a \in R\colon \operatorname{coz} a \subseteq V\}$ is a dense ideal because it contains nonzero divisors. In addition, for any commutative ring S, $\mathrm{Q}_{\mathrm{cl}}(S) \subseteq \mathrm{Q}(S)$. The reasoning of [6], page 15 for $\mathrm{C}(\mathbb{R})$ applies here as well, and will only be sketched. It will be seen that each $\operatorname{Hom}(D, R)$ can be embedded as an R-module in $\mathrm{C}^k(V_D)$. For $\varphi \in \operatorname{Hom}(D, R)$, define for $x \in V_D$, $P(\varphi)(x) = \varphi(d)(x)/d(x)$, whenever $x \in \operatorname{coz} d$. If $x \in \operatorname{coz} d \cap \operatorname{coz} d'$, then $\varphi(d)(x)/d(x) = \varphi(d)(x)d'(x)/(d(x)d'(x)) = \varphi(dd')(x)/dd'(x) = \varphi(d')(x)d(x)/(d(x)d'(x)) = \varphi(d')(x)/d'(x)$. Moreover, for $d \in D$, $P(\varphi)$ is C^k on $\operatorname{coz} d$ and by the above calculation, $P(\varphi)$ is C^k on all of V_D. Then $P\colon \operatorname{Hom}(D, R) \to \mathrm{C}^k(V_D)$ is an embedding of R-modules; indeed, for $r \in R$, $P(\varphi r) = P(\varphi)r$ because φ is an R-module map. Using these embeddings, the directed diagram for $\mathrm{Q}(R)$ embeds in that for $\mathrm{Q}_{\mathrm{cl}}(R)$, giving that $\mathrm{Q}(R) \subseteq \mathrm{Q}_{\mathrm{cl}}(R)$. This with the opposite inclusion yields $\mathrm{Q}(R) = \mathrm{Q}_{\mathrm{cl}}(R)$.

Since the complete ring of quotients of a semiprime commutative ring is always regular (see [12], Chapter XII, Proposition 2.2), it follows that $\mathrm{Q}_{\mathrm{cl}}(\mathrm{C}^k(\mathbb{R}))$ is regular.

As final steps, it is only necessary to quote (see [12], page 235, Example 2), which states that if $\mathrm{Q}_{\mathrm{cl}}(R)$ is regular, then $\mathrm{Q}_{\mathrm{cl}}(R) = \mathrm{Q}_{\mathrm{tot}}(R)$ and moreover, this ring is self-injective by Lambek, see [11], Section 4.5, Corollary to Proposition 2. $\square$

For metric spaces X, $\mathrm{Q}_{\mathrm{cl}}(\mathrm{C}(X)) = \mathrm{Q}(\mathrm{C}(X))$ by [6], page 20, in particular for $X = \mathbb{R}$. Since $\mathrm{C}^k(\mathbb{R})$ is complemented (the remark after Corollary 2.5), it follows that $\mathrm{Q}_{\mathrm{cl}}(\mathrm{C}^k(\mathbb{R}))$ is regular (in fact the two properties are equivalent, see [1], Theorem 2.3 and also [10], Theorem 2.5), but the theorem supplies more information than the regularity, namely that $\mathrm{Q}_{\mathrm{cl}}(\mathrm{C}^k(\mathbb{R})) = \mathrm{Q}(\mathrm{C}^k(\mathbb{R}))$.

Corollary 3.4. *Let $k \in \mathbb{N} \cup \{\infty\}$ and $\emptyset \neq U$ be an open set in $\mathbb{R}$. Then $\mathrm{Q}_{\mathrm{cl}}(\mathrm{C}^k(U)) = Q(\mathrm{C}^k(U)) = \mathrm{Q}_{\mathrm{tot}}(\mathrm{C}^k(U))$ is a regular ring.*

P r o o f. Since $U = \bigcup\limits_{n\in\mathbb{N}} I_n$ is a disjoint union of intervals, it follows that $\mathrm{C}^k(U) = \prod\limits_{n\in N} \mathrm{C}^k(I_n)$. Both types of rings of quotients commute with products and hence, it suffices to consider each $\mathrm{C}^k(I_n)$. However, for each open interval I_n there is a C^∞ bijection between I_n and $\mathbb{R}$. Theorem 3.3 then gives the result. $\square$

Another word about the rings of quotients of $\mathrm{C}^k(\mathbb{R})$: they are all distinct.

Proposition 3.5. *For any $k, l \in \mathbb{N} \cup \{\infty\}$, $k < l$, $\mathrm{Q}_{\mathrm{cl}}(\mathrm{C}^k(\mathbb{R})) \neq \mathrm{Q}_{\mathrm{cl}}(\mathrm{C}^l(\mathbb{R}))$. Moreover, for $k < l \leqslant \infty$, $\mathrm{Q}_{\mathrm{cl}}(\mathrm{C}^k(\mathbb{R})) \supset \mathrm{Q}_{\mathrm{cl}}(\mathrm{C}^l(\mathbb{R}))$.*

P r o o f. Consider $k \in \mathbb{N}$. Let $f \in \mathrm{C}^k(\mathbb{R})$ be such that in an open set $U \neq \emptyset$, it does not have a continuous $(k+1)$st derivative anywhere in U. (Integrating the Weierstrass continuous nowhere differentiable function sufficiently many times would give an example with $U = \mathbb{R}$.) If $f \in \mathrm{Q}_{\mathrm{cl}}(\mathrm{C}^l(\mathbb{R}))$ for some $k < l \leqslant \infty$, then there would be $g, h \in \mathrm{C}^l(\mathbb{R})$ with $V = \operatorname{coz} g$ dense in $\mathbb{R}$ and $f|_V g|_V = h|_V$ or $f|_V = (g|_V)^{-1} h|_V$. However, this would show that f has a $(k+1)$st derivative in $U \cap V$, a contradiction.

The second statement follows because a nonzero divisor in $\mathrm{C}^l(\mathbb{R})$ is also one in $\mathrm{C}^k(\mathbb{R})$. $\square$

In fact, the situation of the proof of Proposition 3.5 is the only case, where $\mathrm{Q}_{\mathrm{cl}}(\mathrm{C}^k(\mathbb{R}))$ can differ from $\mathrm{Q}_{\mathrm{cl}}(\mathrm{C}^{k+1}(\mathbb{R}))$, i.e., where there is an element of $\mathrm{C}^k(\mathbb{R})$ not having a continuous $(k+1)$st derivative anywhere on a nonempty open set.

Proposition 3.6.

(1) *For each $k \in \mathbb{N} \cup \{\infty\}$, let $L_k = \{f \in \mathrm{C}(\mathbb{R})\colon$ there is $V \subseteq \mathbb{R}$, dense open, such that $f|_V \in \mathrm{C}^k(V)\}$, then the ring $L_k \subseteq \mathrm{Q}_{\mathrm{cl}}(\mathrm{C}^k(\mathbb{R}))$. Moreover, $\mathrm{Q}_{\mathrm{cl}}(L_k) = \mathrm{Q}_{\mathrm{cl}}(\mathrm{C}^k(\mathbb{R}))$.*

(2) *Suppose $f \in \mathrm{C}^k(\mathbb{R}) \setminus C^{k+1}(\mathbb{R})$ but there is a dense open set V such that $f|_V \in \mathrm{C}^{k+1}(V)$. Then $f \in \mathrm{Q}_{\mathrm{cl}}(C^{k+1}(\mathbb{R}))$.*

Proof. (1) Notice that L_k is a ring, $\mathrm{C}^k(\mathbb{R}) \subset L_k \subset \mathrm{C}(\mathbb{R})$. Suppose $f \in L_k$, i.e., f is C^k on the dense open set V. There exist $g, h \in \mathrm{C}^k(\mathbb{R})$ with $\operatorname{coz} g = V$ and $f|_V g|_V = h|_V$. Hence, $f|_V = (g|_V)^{-1} h|_V \in \mathrm{C}^k(V)$ and $\mathrm{C}^k(V)$ embeds in $\mathrm{Q}_{\mathrm{cl}}(\mathrm{C}^k(\mathbb{R}))$. This process is independent of the choice of V and clearly yields a ring homomorphism $\zeta\colon L_k \to \mathrm{Q}_{\mathrm{cl}}(\mathrm{C}^k(\mathbb{R}))$. A function in $\mathrm{C}(\mathbb{R})$ that is zero on a dense open set is zero. Hence, ζ is an embedding. However, $\mathrm{C}^k(\mathbb{R}) \subseteq L_k$, showing that $\mathrm{Q}_{\mathrm{cl}}(L_k) = \mathrm{Q}_{\mathrm{cl}}(\mathrm{C}^k(\mathbb{R}))$.

(2) The element f in the statement is in L_{k+1} and so part (1) gives the result. $\square$

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