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ON THE SIGNLESS LAPLACIAN AND NORMALIZED SIGNLESS LAPLACIAN SPREADS OF GRAPHS

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Abstract. Let $G=(V,E)$, $V=\{v_1,v_2,\dots,v_n\}$, be a simple connected graph with n vertices, m edges and a sequence of vertex degrees $d_1\geqslant d_2\geqslant\dots\geqslant d_n$. Denote by A and D the adjacency matrix and diagonal vertex degree matrix of G, respectively. The signless Laplacian of G is defined as $L^+=D+A$ and the normalized signless Laplacian matrix as $\mathcal{L}^+=D^{-1/2}L^+D^{-1/2}$. The normalized signless Laplacian spreads of a connected nonbipartite graph G are defined as $r(G)=\gamma_2^+/\gamma_n^+$ and $l(G)=\gamma_2^+-\gamma_n^+$, where $\gamma_1^+\geqslant\gamma_2^+\geqslant\dots\geqslant\gamma_n^+\geqslant 0$ are eigenvalues of $\mathcal{L}^+$. We establish sharp lower and upper bounds for the normalized signless Laplacian spreads of connected graphs. In addition, we present a better lower bound on the signless Laplacian spread.

Keywords: Laplacian graph spectra; bipartite graph; spread of graph

MSC 2020: 15A18, 05C50

1. Introduction

Let $G=(V,E)$, $V=\{v_1,v_2,\dots,v_n\}$, be a simple connected graph with n vertices, m edges and a sequence of vertex degrees $\Delta=d_1\geqslant d_2\geqslant\dots\geqslant d_n=\delta>0$, $d_i=d(v_i)$. By $i\sim j$ we denote the adjacency of vertices v_i and v_j in graph G.

Let $A=(a_{ij})_{n\times n}$ and $D=\operatorname{diag}(d_1,d_2,\dots,d_n)$ be the adjacency and the diagonal degree matrix of G, respectively. Then $L=D-A$ is the Laplacian matrix of G. Because graph G is assumed to be connected, it has no isolated vertices and therefore the matrix $D^{-1/2}$ is well-defined. The normalized Laplacian is defined as $\mathcal{L}=D^{-1/2}LD^{-1/2}=I-D^{-1/2}AD^{-1/2}=I-R$. Here I is the unity matrix,

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and $R = D^{-1/2}AD^{-1/2}$ the Randić matrix, see [6]. Further, denote by $L^+ = D + A$ and $\mathcal{L}^+ = D^{-1/2}L^+D^{-1/2} = I + D^{-1/2}AD^{-1/2} = I + R$ signless Laplacian and normalized signless Laplacian matrix, respectively. For more information on these matrices one can refer to [10], [12].

Eigenvalues of matrix $\mathcal{L}$, $\gamma_1^- \geqslant \gamma_2^- \geqslant \ldots \geqslant \gamma_{n-1}^- \geqslant \gamma_n^- = 0$ are called *normalized Laplacian eigenvalues* of G. Some well known properties of these eigenvalues are (see [28])

$$\sum_{i=1}^{n-1} \gamma_i^- = n \quad \text{and} \quad \sum_{i=1}^{n-1} (\gamma_i^-)^2 = n + 2R_{-1}(G), \tag{1.1}$$

where

$$R_{-1}(G) = \sum_{i\sim j} \frac{1}{d_i d_j}$$

is the general Randić index R_{-1}, see e.g., [8], [26].

Eigenvalues of matrix L^+, $\gamma_1 \geqslant \gamma_2 \geqslant \ldots \geqslant \gamma_n \geqslant 0$, are called the *signless Laplacian eigenvalues* of G. Some well known properties of signless Laplacian eigenvalues are, see [13]

$$\sum_{i=1}^{n} \gamma_i = 2m \quad \text{and} \quad \sum_{i=1}^{n} \gamma_i^2 = M_1(G) + 2m, \tag{1.2}$$

where

$$M_1(G) = \sum_{i=1}^{n} d_i^2$$

is the first Zagreb index, see [20].

The signless Laplacian spread of a graph G is defined as $s_{L^+}(G) = \gamma_1 - \gamma_n$, see [22]. For details and several lower and upper bounds on $s_{L^+}(G)$, see [1], [23].

The eigenvalues of matrix $\mathcal{L}^+$, $\gamma_1^+ \geqslant \gamma_2^+ \geqslant \ldots \geqslant \gamma_n^+ \geqslant 0$, are normalized signless Laplacian eigenvalues of G. The following identities are valid for them, see [9]:

$$\sum_{i=1}^{n} \gamma_i^+ = n \quad \text{and} \quad \sum_{i=1}^{n} (\gamma_i^+)^2 = n + 2R_{-1}(G). \tag{1.3}$$

The normalized Laplacian and normalized signless Laplacian eigenvalues are, respectively, of the form, see [18], [21]

$$\gamma_i^- = 1 - \varrho_{n-i+1} \quad \text{and} \quad \gamma_i^+ = 1 + \varrho_i \quad \text{for } i = 1, 2, \ldots, n, \tag{1.4}$$

where $1 = \varrho_1 \geqslant \varrho_2 \geqslant \ldots \geqslant \varrho_n$ are Randić matrix eigenvalues, see [6], [12], [21].

The normalized Laplacian spread is defined by $s_{\mathcal{L}}(G) = \gamma_1^- - \gamma_{n-1}^-$, see [7], [19]. In [16], the Randić spread is introduced as $s_R(G) = \varrho_2 - \varrho_n$. It is also observed that $s_{\mathcal{L}}(G) = s_R(G)$, see [2], [19]. For more details on normalized Laplacian (Randić) spread, see [2], [17], [24].

The normalized signless Laplacian spreads of a connected nonbipartite graph G are defined as $r(G) = \gamma_2^+/\gamma_n^+$ and $l(G) = \gamma_2^+ - \gamma_n^+$. Recall that the normalized Laplacian and normalized signless Laplacian eigenvalues of bipartite graphs coincide, see [4]. Then, for bipartite graphs, $\gamma_n^+ = \gamma_n^- = 0$, see [10]. Therefore, the normalized signless Laplacian spreads of a connected bipartite graph G are considered as $r(G) = \gamma_2^+/\gamma_{n-1}^+$ and $l(G) = \gamma_2^+ - \gamma_{n-1}^+$.

In this paper, we obtain sharp lower and upper bounds for the normalized signless Laplacian spreads of connected graphs. Moreover, we present a better lower bound on the signless Laplacian spread.

2. Preliminaries

In this section we recall some known results about graph spectra and analytical discrete inequalities, which will be used later.

Lemma 2.1 ([18]). *For any connected graph G, the largest normalized signless Laplacian eigenvalue is*

$$\gamma_1^+ = 2.$$

Lemma 2.2 ([18]). *Let G be a graph of order $n \geqslant 2$ with no isolated vertices. Then*

$$\gamma_2^+ = \gamma_3^+ = \ldots = \gamma_n^+ = \frac{n-2}{n-1}$$

if and only if $G \cong K_n$.

Lemma 2.3 ([5]). *Let G be a connected nonbipartite graph with $n \geqslant 3$ vertices. Then $\gamma_i^+ > 0$ for $i = 1, 2, \ldots, n$.*

Lemma 2.4 ([4]). *If G is a bipartite graph, then the eigenvalues of $\mathcal{L}$ and $\mathcal{L}^+$ coincide.*

Lemma 2.5 ([15]). *Let G be a connected graph with $n > 2$ vertices. Then $\gamma_2^- = \gamma_3^- = \ldots = \gamma_{n-1}^-$ if and only if $G \cong K_n$ or $G \cong K_{p,q}$.*

Lemma 2.6 ([23]). *Let G be a simple connected graph with $n \geqslant 2$ vertices. Then*

$$s_{L^+}(G) \leqslant \sqrt{\frac{2(n(M_1(G) + 2m) - 4m^2)}{n}}. \tag{2.1}$$

Let us note that (2.1) was also proven in [1]. Besides, it was proven that equality holds if and only if $G \cong K_{n/2,n/2}$, for n even. In the same paper the following was proven.

Lemma 2.7 ([1]). *Let G be a simple connected graph with $n \geqslant 2$ vertices. Then*

$$s_{L^+}(G) \geqslant \frac{2}{n}\sqrt{n(M_1(G)+2m)-4m^2}. \tag{2.2}$$

Denote by $t(G)$ the total number of spanning trees of G, and by $G_1 \times G_2$ the Cartesian product of graphs G_1 and G_2. In [14] the following quantity was introduced:

$$t_1(G) = \frac{2t(G \times K_2)}{t(G)}.$$

Lemma 2.8 ([12]). *Let G be a connected graph with n vertices, m edges and $t(G)$ spanning trees. Then*

$$\det \mathcal{L}^- = \prod_{i=1}^{n-1} \gamma_i^- = \frac{2mt(G)}{\det D}.$$

Lemma 2.9 ([4]). *If G is a connected bipartite graph with n vertices, m edges and $t(G)$ spanning trees, then*

$$\det \mathcal{L}^+ = \det \mathcal{L}^- = \frac{2mt(G)}{\det D}.$$

If G is a connected nonbipartite graph with n vertices, then

$$\det \mathcal{L}^+ = \prod_{i=1}^{n} \gamma_i^+ = \frac{t_1(G)}{\det D}.$$

Lemma 2.10 ([10]). *Let G be a bipartite graph of order n. Then $\gamma_i^- = 2-\gamma_{n-i+1}^-$ for $i = 1, 2, \ldots, n$.*

For the real number sequences $a = (a_i)$ and $b = (b_i)$, $i = 1, 2, \ldots, n$, in [3] (see also [25]) the following result was proven.

Lemma 2.11 ([3]). *Let $a = (a_i)$ and $b = (b_i)$, $i = 1, 2, \ldots, n$ be two real number sequences with the properties $a \leqslant a_i \leqslant A$ and $b \leqslant b_i \leqslant B$. Then*

$$\left|\sum_{i=1}^{n} a_i b_i - \frac{1}{n}\sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i\right| \leqslant (A-a)(B-b)\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right). \tag{2.3}$$

The expression (2.3) can be observed in an equivalent, more appropriate form. Namely, when n is even, the expression

$$\alpha(n) = \frac{1}{n}\Big[\frac{n}{2}\Big]\Big(1 - \frac{1}{n}\Big[\frac{n}{2}\Big]\Big)$$

is equal to

$$\alpha(n) = \frac{1}{4},$$

while for n odd we have

$$\alpha(n) = \frac{(n-1)(n+1)}{4n^2}.$$

Thus, we obtain

$$\alpha(n) = \frac{1}{4}\Big(1 - \frac{(-1)^{n+1}+1}{2n^2}\Big) = \begin{cases} \dfrac{1}{4} & \text{if } n \text{ is even,} \\ \dfrac{(n-1)(n+1)}{4n^2} & \text{if } n \text{ is odd.} \end{cases} \tag{2.4}$$

Having this in mind, inequality (2.3) can be considered as

$$\Big| n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \Big| \leqslant n^2\alpha(n)(A-a)(B-b). \tag{2.5}$$

Lemma 2.12 ([11])**.** *Let $a = (a_i)$, $i = 1, 2, \ldots, n$, be a positive real number sequence with the property $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n > 0$. Then*

$$\frac{\sum_{i=1}^{n} a_i}{n\big(\prod_{i=1}^{n} a_i\big)^{1/n}} \geqslant \bigg(\frac{1}{2}\bigg(\sqrt{\frac{a_1}{a_n}} + \sqrt{\frac{a_n}{a_1}}\bigg)\bigg)^{2/n} \tag{2.6}$$

with equality if $a_2 = \ldots = a_{n-1} = \frac{1}{2}(a_1 + a_n)$.

3. Main results

In the next theorem we determine a lower bound for the normalized signless Laplacian ratio spread $r(G) = \gamma_2^+/\gamma_n^+$ in terms of $R_{-1}(G)$ and n.

Theorem 3.1. *Let G be a connected nonbipartite graph with $n \geqslant 3$ vertices. Then*

$$r(G) = \frac{\gamma_2^+}{\gamma_n^+} \geqslant \bigg(\frac{\sqrt{2(n-1)R_{-1}(G) - n} + \sqrt{(n-1)(n + 2R_{-1}(G) - 4)}}{n-2}\bigg)^2. \tag{3.1}$$

Equality holds if and only if $G \cong K_n$.

Proof. For every i, $i = 2, 3, \ldots, n$,

$$(\gamma_2^+ - \gamma_i^+)(\gamma_n^+ - \gamma_i^+) \leqslant 0,$$

that is

$$(\gamma_i^+)^2 + \gamma_2^+\gamma_n^+ \leqslant (\gamma_2^+ + \gamma_n^+)\gamma_i^+. \tag{3.2}$$

Summing up the above inequality over i, $i = 2, 3, \ldots, n$, gives

$$\sum_{i=2}^{n}(\gamma_i^+)^2 + \gamma_2^+\gamma_n^+\sum_{i=2}^{n} 1 \leqslant (\gamma_2^+ + \gamma_n^+)\sum_{i=2}^{n}\gamma_i^+.$$

Then, by Lemma 2.1 and equation (1.3)

$$(n + 2R_{-1}(G) - 4) + (n-1)\gamma_2^+\gamma_n^+ \leqslant (n-2)(\gamma_2^+ + \gamma_n^+).$$

By arithmetic-geometric mean inequality, AM-GM (see e.g. [25]), we have

$$2\sqrt{(n-1)(n+2R_{-1}(G)-4)\gamma_2^+\gamma_n^+} \leqslant (n-2)(\gamma_2^+ + \gamma_n^+),$$

that is

$$\left(\sqrt{\frac{\gamma_2^+}{\gamma_n^+}} + \sqrt{\frac{\gamma_n^+}{\gamma_2^+}}\right)^2 \geqslant \frac{4(n-1)(n+2R_{-1}(G)-4)}{(n-2)^2}. \tag{3.3}$$

Since

$$\left(\sqrt{\frac{\gamma_2^+}{\gamma_n^+}} - \sqrt{\frac{\gamma_n^+}{\gamma_2^+}}\right)^2 = \left(\sqrt{\frac{\gamma_2^+}{\gamma_n^+}} + \sqrt{\frac{\gamma_n^+}{\gamma_2^+}}\right)^2 - 4, \tag{3.4}$$

from (3.3) we have that

$$\left(\sqrt{\frac{\gamma_2^+}{\gamma_n^+}} - \sqrt{\frac{\gamma_n^+}{\gamma_2^+}}\right)^2 \geqslant \frac{4(2(n-1)R_{-1}(G)-n)}{(n-2)^2}. \tag{3.5}$$

Now, from (3.3) and (3.5) we obtain

$$\sqrt{\frac{\gamma_2^+}{\gamma_n^+}} \geqslant \frac{\sqrt{2(n-1)R_{-1}(G)-n}}{n-2} + \frac{\sqrt{(n-1)(n+2R_{-1}(G)-4)}}{n-2},$$

from which we arrive at (3.1).

Equality in (3.2) holds if and only if $\gamma_i^+ \in \{\gamma_2^+, \gamma_n^+\}$ for $i = 3, \ldots, n-1$. Equality in (3.3), that is in (3.5), holds if and only if $\gamma_2^+ = \ldots = \gamma_n^+$. This implies that equality in (3.1) holds if and only if $\gamma_2^+ = \ldots = \gamma_n^+$, that is, by Lemma 2.2, if and only if $G \cong K_n$. □

Corollary 3.2. *Let G be a connected nonbipartite graph with $n \geqslant 3$ vertices. Then*

$$r(G) = \frac{\gamma_2^+}{\gamma_n^+} \geqslant \frac{\big(\sqrt{n(n-1-\Delta)} + \sqrt{(n-1)(n+(n-4)\Delta)}\big)^2}{(n-2)^2\Delta}.$$

Equality holds if and only if $G \cong K_n$.

Proof. In [27] it was proven that

$$\frac{n}{2\Delta} \leqslant R_{-1}(G) \leqslant \frac{n}{2\delta}. \tag{3.6}$$

From the left-hand part of (3.6) and (3.1) the required result is obtained. □

Considering the proof techniques in Theorem 3.1 with Lemmas 2.4 and 2.5, we get the following result.

Theorem 3.3. *Let G be a connected bipartite graph with $n \geqslant 3$ vertices. Then*

$$r(G) = \frac{\gamma_2^+}{\gamma_{n-1}^+} = \frac{\gamma_2^-}{\gamma_{n-1}^-} \geqslant \left(\sqrt{\frac{n+2R_{-1}(G)-4}{n-2}} + \sqrt{\frac{2(R_{-1}(G)-1)}{n-2}}\right)^2. \tag{3.7}$$

Equality holds if and only if $G \cong K_{p,q}$, $p+q=n$.

By Lemma 2.10 and Theorem 3.3, we have:

Corollary 3.4. *Let G be a connected bipartite graph with $n \geqslant 3$ vertices. Then*

$$\gamma_{n-1}^+ = \gamma_{n-1}^- \leqslant \frac{2}{1 + \big(\sqrt{(n+2R_{-1}(G)-4)/(n-2)} + \sqrt{(2(R_{-1}(G)-1))/(n-2)}\big)^2}.$$

Equality holds if and only if $G \cong K_{p,q}$, $p+q=n$.

In the next theorem we determine an upper bound for the normalized signless Laplacian ratio spread $r(G) = \gamma_2^+/\gamma_n^+$ in terms of $\det \mathcal{L}^+ = t_1(G)/\det D$ and parameter n.

Theorem 3.5. *Let G be a simple connected nonbipartite graph with $n \geqslant 3$ vertices. Then*

$$r(G) = \frac{\gamma_2^+}{\gamma_n^+} \leqslant \frac{\det D}{t_1(G)}\left(\sqrt{2\Big(\frac{n-2}{n-1}\Big)^{n-1}} + \sqrt{2\Big(\frac{n-2}{n-1}\Big)^{n-1} - \frac{t_1(G)}{\det D}}\right)^2. \tag{3.8}$$

Equality holds if and only if $G \cong K_n$.

P r o o f. Note that by Lemma 2.3, $\gamma_i^+ > 0$, $i = 1, 2, \ldots, n$ for any connected nonbipartite graph G. Further, inequality (2.6) can be observed in the following form:

$$\frac{\sum_{i=2}^{n} a_i}{(n-1)\bigl(\prod_{i=2}^{n} a_i\bigr)^{1/(n-1)}} \geqslant \Bigl(\frac{1}{2}\Bigl(\sqrt{\frac{a_2}{a_n}} + \sqrt{\frac{a_n}{a_2}}\Bigr)\Bigr)^{2/(n-1)}.$$

For $a_i = \gamma_i^+$, $i = 2, \ldots, n$, the above inequality becomes

$$\frac{\sum_{i=2}^{n} \gamma_i^+}{(n-1)\bigl(\prod_{i=2}^{n} \gamma_i^+\bigr)^{1/(n-1)}} \geqslant \Bigl(\frac{1}{2}\Bigl(\sqrt{\frac{\gamma_2^+}{\gamma_n^+}} + \sqrt{\frac{\gamma_n^+}{\gamma_2^+}}\Bigr)\Bigr)^{2/(n-1)}.$$

From Lemma 2.1 and equation (1.3) we have

$$\frac{4(n-2)^{n-1}}{(n-1)^{n-1}\bigl(\frac{1}{2}\prod_{i=1}^{n} \gamma_i^+\bigr)} \geqslant \Bigl(\sqrt{\frac{\gamma_2^+}{\gamma_n^+}} + \sqrt{\frac{\gamma_n^+}{\gamma_2^+}}\Bigr)^2.$$

According to the above we have that

$$\Bigl(\sqrt{\frac{\gamma_2^+}{\gamma_n^+}} + \sqrt{\frac{\gamma_n^+}{\gamma_2^+}}\Bigr)^2 \leqslant \frac{8((n-2)/(n-1))^{n-1}}{\det \mathcal{L}^+}. \tag{3.9}$$

From the above and (3.4) we get

$$\Bigl(\sqrt{\frac{\gamma_2^+}{\gamma_n^+}} - \sqrt{\frac{\gamma_n^+}{\gamma_2^+}}\Bigr)^2 \leqslant 4\frac{2((n-2)/(n-1))^{n-1} - \det \mathcal{L}^+}{\det \mathcal{L}^+}. \tag{3.10}$$

Now, from (3.9) and (3.10) we obtain

$$\sqrt{\frac{\gamma_2^+}{\gamma_n^+}} \leqslant \sqrt{\frac{2((n-2)/(n-1))^{n-1}}{\det \mathcal{L}^+}} + \sqrt{\frac{2((n-2)/(n-1))^{n-1} - \det \mathcal{L}^+}{\det \mathcal{L}^+}}.$$

Inequality (3.8) is obtained from the above inequality and Lemma 2.9.

Equality in (3.9) holds if $\gamma_3^+ = \ldots = \gamma_{n-1}^+ = \frac{1}{2}(\gamma_2^+ + \gamma_n^+)$, which implies that equality in (3.8) holds if $\gamma_2^+ = \ldots = \gamma_n^+$. From Lemma 2.2 it follows that the equality in (3.8) holds if and only if $G \cong K_n$. □

Considering the techniques in Theorem 3.5 with Lemmas 2.4, 2.5, 2.8 and 2.9, we obtain the following result for bipartite graphs:

Theorem 3.6. *Let G be a simple connected bipartite graph with $n \geqslant 2$ vertices and m edges. Then*

$$r(G) = \frac{\gamma_2^+}{\gamma_{n-1}^+} = \frac{\gamma_2^-}{\gamma_{n-1}^-} \leqslant \frac{\det D}{2mt(G)}\Bigl(\sqrt{2} + \sqrt{2 - \frac{2mt(G)}{\det D}}\Bigr)^2.$$

Equality holds if $G \cong K_{p,q}$, $p + q = n$.

By Lemma 2.10 and Theorem 3.6, we get:

Corollary 3.7. *Let G be a connected bipartite graph with $n \geqslant 2$ vertices. Then*

$$\gamma_{n-1}^{+} = \gamma_{n-1}^{-} \geqslant \frac{2}{1 + (\det D/2mt(G))\big(\sqrt{2} + \sqrt{2 - 2mt(G)/\det D}\big)^{2}}.$$

Equality holds if and only if $G \cong K_{p,q}$, $p + q = n$.

We now consider the normalized signless Laplacian linear spread $l(G)$ of a connected graph G. At first, we state the following remark.

Remark 3.8. From equation (1.4) we conclude that normalized signless Laplacian linear spread coincides with normalized Laplacian (Randić) spread. Then, the results derived for normalized signless Laplacian linear spread can be immediately re-stated for normalized Laplacian (Randić) spread and vice versa.

By Theorem 4 of [19] and Remark 3.8, we directly have the following result.

Theorem 3.9. *Let G be a connected nonbipartite graph with $n \geqslant 3$ vertices. Then*

$$l(G) = \gamma_2^{+} - \gamma_n^{+} \leqslant \sqrt{\frac{2(2(n-1)R_{-1}(G) - n)}{n-1}}. \tag{3.11}$$

Equality holds if and only if $G \cong K_n$.

Corollary 3.10. *Let G be a connected nonbipartite graph with $n \geqslant 3$ vertices. Then*

$$l(G) = \gamma_2^{+} - \gamma_n^{+} \leqslant \sqrt{\frac{2n(n-1-\delta)}{\delta(n-1)}}.$$

Equality holds if and only if $G \cong K_n$.

Proof. The required result is obtained from (3.11) and the right-hand side of (3.6). □

Remark 3.11. The inequality analogous to (3.11), but for signless Laplacian eigenvalues, $\gamma_1 \geqslant \gamma_2 \geqslant \ldots \geqslant \gamma_n > 0$, was proven in [23].

Theorem 3.12. *Let G be a connected bipartite graph with $n \geqslant 3$ vertices. Then*

$$l(G) = \gamma_2^{+} - \gamma_{n-1}^{+} = \gamma_2^{-} - \gamma_{n-1}^{-} \leqslant 2\sqrt{R_{-1}(G) - 1}. \tag{3.12}$$

Equality holds if and only if $G \cong K_{p,q}$, $p + q = n$.

P r o o f. By Lagrange's identity (see for example [25]), we have that

$$(n-2)\sum_{i=2}^{n-1}(\gamma_i^+)^2-\left(\sum_{i=2}^{n-1}\gamma_i^+\right)^2=\sum_{2\leqslant i<j\leqslant n-1}(\gamma_i^+-\gamma_j^+)^2.$$

From the above we get

$$(n-2)\sum_{i=2}^{n-1}(\gamma_i^+)^2-\left(\sum_{i=2}^{n-1}\gamma_i^+\right)^2\geqslant(\gamma_2^+-\gamma_{n-1}^+)^2+\sum_{i=3}^{n-2}((\gamma_2^+-\gamma_i^+)^2+(\gamma_i^+-\gamma_{n-1}^+)^2). \tag{3.13}$$

On the other hand, we have that

$$(\gamma_2^+-\gamma_i^+)^2+(\gamma_i^+-\gamma_{n-1}^+)^2\geqslant\frac{1}{2}(\gamma_2^+-\gamma_{n-1}^+)^2. \tag{3.14}$$

Now, from (3.13) and (3.14) we obtain

$$(n-2)\sum_{i=2}^{n-1}(\gamma_i^+)^2-\left(\sum_{i=2}^{n-1}\gamma_i^+\right)^2\geqslant\left(\frac{n-2}{2}\right)(\gamma_2^+-\gamma_{n-1}^+)^2.$$

Thus, from Lemma 2.1 and equation (1.3)

$$(n-2)(n+2R_{-1}(G)-4)-(n-2)^2\geqslant\frac{n-2}{2}(\gamma_2^+-\gamma_{n-1}^+)^2\,,$$

wherefrom (3.12) is obtained.

Equality in (3.13) holds if and only if $\gamma_i^+\in\{\gamma_2^+,\gamma_{n-1}^+\}$ for $i=3,\ldots,n-2$. On the other hand, equality in (3.14) holds if and only if $\gamma_i^+=\frac{1}{2}(\gamma_2^++\gamma_{n-1}^+)$ for $i=3,\ldots,n-2$. This implies that equality in (3.12) holds if and only if $\gamma_2^+=\ldots=\gamma_{n-1}^+$, which means, by Lemmas 2.4 and 2.5, that equality in (3.12) holds if and only if $G\cong K_{p,q}$. □

By Lemma 2.10 and Theorem 3.12, we have:

Corollary 3.13. *Let G be a connected bipartite graph with $n\geqslant 3$ vertices. Then*

$$\gamma_2^+=\gamma_2^-\leqslant 1+\sqrt{R_{-1}(G)-1}\quad\text{and}\quad\gamma_{n-1}^+=\gamma_{n-1}^-\geqslant 1-\sqrt{R_{-1}(G)-1}$$

with equalities if and only if $G\cong K_{p,q}$, $p+q=n$.

By Remark 3.9 of [24] and Remark 3.8, we directly have:

Theorem 3.14. *Let G be a connected nonbipartite graph with $n \geqslant 3$ vertices. Then*

$$l(G) = \gamma_2^+ - \gamma_n^+ \geqslant \frac{1}{n-1}\sqrt{\frac{2(n-1)R_{-1}(G) - n}{\alpha(n-1)}}, \tag{3.15}$$

where

$$\alpha(n-1) = \frac{1}{4}\Big(1 - \frac{(-1)^n + 1}{2(n-1)^2}\Big).$$

Equality holds if and only if $G \cong K_n$.

Theorem 3.15. *Let G be a connected bipartite graph with $n \geqslant 3$ vertices. Then*

$$l(G) = \gamma_2^+ - \gamma_{n-1}^+ = \gamma_2^- - \gamma_{n-1}^- \geqslant \sqrt{\frac{2(R_{-1}(G) - 1)}{(n-2)\alpha(n-2)}}. \tag{3.16}$$

Equality holds if and only if $G \cong K_{p,q}$, $p + q = n$.

P r o o f. For $a_i = b_i$, $i = 2, \ldots, n-1$, inequality (2.5) becomes

$$\Big|(n-2)\sum_{i=2}^{n-1} a_i^2 - \Big(\sum_{i=2}^{n-1} a_i\Big)^2\Big| \leqslant (n-2)^2\alpha(n-2)(A-a)^2.$$

For $a_i = \gamma_i^+$, $i = 2, \ldots, n-1$, $a = \gamma_{n-1}^+$, $A = \gamma_2^+$, the above inequality transforms into

$$(n-2)\sum_{i=2}^{n-1}(\gamma_i^+)^2 - \Big(\sum_{i=2}^{n-1}\gamma_i^+\Big)^2 \leqslant (n-2)^2\alpha(n-2)(\gamma_2^+ - \gamma_{n-1}^+)^2. \tag{3.17}$$

Then by Lemma 2.1 and equation (1.3), we get

$$(n-2)(n + 2R_{-1}(G) - 4) - (n-2)^2 \leqslant (n-2)^2\alpha(n-2)(\gamma_2^+ - \gamma_{n-1}^+)^2.$$

After rearrangement of the above, we obtain

$$2(n-2)(R_{-1}(G) - 1) \leqslant (n-2)^2\alpha(n-2)(\gamma_2^+ - \gamma_{n-1}^+)^2,$$

from which (3.16) is obtained.

Equality in (3.17) holds if and only if $\gamma_2^+ = \ldots = \gamma_{n-1}^+$. Then, by Lemmas 2.4 and 2.5, it follows that equality in (3.16) holds if and only if $G \cong K_{p,q}$. $\square$

By Lemma 2.10 and Theorem 3.15, we have:

Corollary 3.16. *Let G be a connected bipartite graph with $n \geqslant 3$ vertices. Then*

$$\gamma_2^+ = \gamma_2^- \geqslant 1 + \sqrt{\frac{R_{-1}(G) - 1}{2(n-2)\alpha(n-2)}} \quad \text{and} \quad \gamma_{n-1}^+ = \gamma_{n-1}^- \leqslant 1 - \sqrt{\frac{R_{-1}(G) - 1}{2(n-2)\alpha(n-2)}},$$

with equalities if and only if $G \cong K_{p,q}$, $p + q = n$.

Considering the proof techniques in Theorem 3.15 with equation (1.2), we arrive at:

Theorem 3.17. *Let G be a connected graph with $n \geqslant 2$ vertices. Then*

$$s_{L^+}(G) \geqslant \frac{1}{n}\sqrt{\frac{n(M_1(G) + 2m) - 4m^2}{\alpha(n)}}. \tag{3.18}$$

Remark 3.18. Since $\alpha(n) \leqslant \frac{1}{4}$ for any n, we have that

$$s_{L^+}(G) \geqslant \frac{1}{n}\sqrt{\frac{n(M_1(G) + 2m) - 4m^2}{\alpha(n)}} \geqslant \frac{2}{n}\sqrt{n(M_1(G) + 2m) - 4m^2},$$

which means that inequality (3.18) is stronger than (2.2) when n is odd.

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