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CHARACTERIZATIONS OF COMMUTATORS OF THE HARDY-LITTLEWOOD MAXIMAL FUNCTION ON TRIEBEL-LIZORKIN SPACES

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Abstract. We study the mapping property of the commutator of Hardy-Littlewood maximal function on Triebel-Lizorkin spaces. Also, some new characterizations of the Lipschitz spaces are given.

Keywords: characterization; commutator; Hardy-Littlewood maximal function; Lipschitz space; Triebel-Lizorkin space

MSC 2020: 42B25, 47B25, 47B47

1. Introduction

The purpose of this article is first, to obtain the boundedness of commutators of Hardy-Littlewood maximal function from Lebesgue spaces to Triebel-Lizorkin spaces, and second, to give the characterization of Lipschitz space via the boundedness results (precise definitions are given in the next section).

We briefly summarize some classical and recent works in the literature, which lead to the results presented here. A well known result of Coifman, Rochberg and Weiss (see [2]) states that the commutator

$$[b, T](f) := bT(f) - T(bf)$$

is bounded on some L^p, $1 < p < \infty$, if and only if $b \in \mathrm{BMO}$, where T is the Calderón-Zygmund operator with smooth homogeneous kernels. A particular case of the result

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of Janson (see [4]) states that for $1 < p < q < \infty$ with $1/q = 1/p - \alpha/n$, $b \in \mathrm{Lip}_\alpha$ if and only if $[b, T]$ is bounded from L^p to L^q. In addition, in [13], Zhang proved that the commutator of Hardy-Littlewood maximal function is bounded from L^p to L^q if and only if the symbol belongs to Lip_α, where $1/q = 1/p - \alpha/n$. Specially, using Sobolev-Besov embedding, Paluszyński in [8] obtained that for $1 < p < \infty$ and $0 < \alpha < 1$, $b \in \mathrm{Lip}_\alpha$ if and only if $[b, T]$ is bounded from L^p to the homogeneous Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,\infty}$. Paluszyński's idea was novel for the study about the boundedness of commutators from L^p to $\dot{F}_p^{\alpha,\infty}$ and shed new light on the characterization of the Lipschitz space via commutators.

On the other hand, it is well known that maximal operators play a key role in differentiation theory, where they are used in obtaining almost everywhere convergence for certain integral averages. An interesting question is raised: Can we extend Paluszyński's result to the commutator of Hardy-Littlewood maximal function? In this paper, we give an affirmative answer as follows. It should be pointed out that, in the study of the mapping properties of commutators on Triebel-Lizorkin spaces, some of the techniques employed in [8] cannot be applied to Hardy-Littlewood maximal function.

Theorem 1.1. *Let $0 < \alpha < n/(n+1)$ and $1/(1-\alpha) < p < n/\alpha$. Then the following statements are equivalent:*

(a1) $b \in \mathrm{Lip}_\alpha$;

(a2) *M_b is a bounded operator from L^p to $\dot{F}_p^{\alpha,\infty}$.*

Theorem 1.2. *Let $m \geqslant 2$, $0 < \alpha < n/((n+1)m)$ and $1/(1-m\alpha) < p < n/(m\alpha)$. Then the following statements are equivalent:*

(b1) $b \in \mathrm{Lip}_\alpha$;

(b2) *$M_{b,m}$ is a bounded operator from L^p to $\dot{F}_p^{m\alpha,\infty}$.*

The subsequent investigations about multilinear operators in the late 70s have added to the success of Calderón's work on commutators. A classical bilinear estimate, the so-called Kato-Ponce commutator estimate (see [6]), is crucial in the study of the Navier-Stokes equations. In the multilinear setting, commutators of Calderón-Zygmund operators and fractional integrals started to receive attention only a few years ago. Characterization of commutators of multilinear operators have just began to be studied, see [1], [11], [12]. In this paper, we will characterize the boundedness of commutators of bilinear Hardy-Littlewood maximal operator from the product of Lebesgue spaces to Triebel-Lizorkin spaces.

Theorem 1.3. *Let $\vec{b} = (b_1, b_2)$, $0 < \alpha < 1$ and $1/(1-\alpha) < p_1$, $p_2 < \infty$ such that $1 < p < n/\alpha$, where $1/p = 1/p_1 + 1/p_2$. Then the following statements are equivalent:*

(c1) $b_1, b_2 \in \mathrm{Lip}_\alpha$;

(c2) $\mathcal{M}_{\vec{b}}$ *is a bounded operator from* $L^{p_1} \times L^{p_2}$ *to* $\dot{F}_p^{\alpha,\infty}$.

2. Definitions and preliminaries

Definition 2.1. For a locally integrable function f, the Hardy-Littlewood maximal function M is defined by

$$M(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \, \mathrm{d}y.$$

We recall the definitions of commutators of the Hardy-Littlewood maximal function.

Definition 2.2. Let $m \in \mathbb{N}$. For a locally integrable function b, the commutator of the Hardy-Littlewood maximal function is defined by

$$M_b(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |b(x) - b(y)||f(y)| \, \mathrm{d}y;$$

the high-order commutator of the Hardy-Littlewood maximal function is defined by

$$M_{b,m}(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |b(x) - b(y)|^m |f(y)| \, \mathrm{d}y.$$

In 2009, Lerner, Ombrosi, Pérez, Torres and Trujillo-González in [7] introduced the following multilinear maximal function that adapts to the multilinear Calderón-Zygmund theory. In this paper, we only consider the bilinear case. A similar argument also works for the multilinear cases.

Definition 2.3. For a collection of locally integrable functions $\vec{f} = (f_1, f_2)$, the bilinear maximal function $\mathcal{M}$ is defined by

$$\mathcal{M}(\vec{f})(x) = \sup_{B \ni x} \prod_{i=1}^{2} \frac{1}{|B|} \int_B |f_i(y_i)| \, \mathrm{d}y_i.$$

We now give the definition of the commutator related to the bilinear maximal function.

Definition 2.4. For two collections of locally integrable functions $\vec{f} = (f_1, f_2)$ and $\vec{b} = (b_1, b_2)$, the commutator $\mathcal{M}_{\vec{b}}$ is defined by

$$\mathcal{M}_{\vec{b}}(\vec{f})(x) = \sum_{i=1}^{2} \mathcal{M}_{b_i,i}(\vec{f})(x),$$

where

$$\mathcal{M}_{b_i,i}(\vec{f})(x) = \sup_{B\ni x} \frac{1}{|B|^2} \int_B \int_B |b_i(x) - b_i(y_i)| \prod_{j=1}^{2} |f_j(y_j)| \, \mathrm{d}y_1 \, \mathrm{d}y_2.$$

Definition 2.5. Let $0 < \alpha < 1$ and $0 < p < \infty$. A locally integrable function f is said to belong to the Campanato space $\mathcal{C}_{\alpha,p}$ if

$$\sup_Q \frac{1}{|Q|^{\alpha/n}} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p \, \mathrm{d}x \right)^{1/p},$$

where $f_Q = |Q|^{-1} \int_Q f(x) \, \mathrm{d}x$.

Definition 2.6. Let $0 < \alpha < 1$. The Lipschitz space Lip_α is a set of functions f such that

$$\|f\|_{\mathrm{Lip}_\alpha} := \sup_{x,h\in\mathbb{R}^n, h\neq 0} \frac{|f(x+h) - f(x)|}{|h|^\alpha} < \infty.$$

Campanato spaces are a useful tool in the regularity theory of PDEs due to their better structures, which allows us to give an integral characterization of the spaces of Lipschitz continuous functions. This leads to a generalization of the classical Sobolev embedding theorem. We refer the reader to Triebel's books (see [9], [10]) for more concrete definition and properties of Triebel-Lizorkin spaces on $\mathbb{R}^n$. This kind of function spaces is closely related to some other function spaces, such as Sobolev and Besov spaces. All these spaces are basic for many branches of mathematics such as harmonic analysis, PDE, functional analysis and approximation theory.

Next, we will use the following characterizations of relevant function spaces.

Lemma 2.1.

(a) *For* $0 < \alpha < 1$ *and* $0 < p \leqslant \infty$ *we have*

$$\|b\|_{\mathrm{Lip}_\alpha} \approx \sup_B \frac{1}{|B|^{1+\alpha/n}} \int_B |b(x) - b_B| \, \mathrm{d}x \approx \sup_B \frac{1}{|B|^{\alpha/n}} \left(\frac{1}{|B|} \int_B |b(x) - b_B|^p \, \mathrm{d}x \right)^{1/p}$$

with the obvious changes when $p = \infty$.

(b) *For* $0 < \alpha < 1$ *and* $1 < p < \infty$ *we have*

$$\|f\|_{\dot{F}_p^{\alpha,\infty}} \approx \left\| \sup_{B\ni\cdot} \frac{1}{|B|^{1+\alpha/n}} \int_B |f - f_B| \right\|_{L^p}.$$

Proof. The first equivalence in (a) can be found in [3], pages 14 and 38, the second equivalence can be found in [5] for $1 \leqslant p < \infty$ and in [11] for $0 < p < 1$, and the proof of (b) in [5]. $\square$

3. Proofs of Theorems 1.1, 1.2 and 1.3

We now proceed with the proofs of Theorems 1.1, 1.2 and 1.3.

Proof of Theorem 1.1. (a1) $\Rightarrow$ (a2): It is easy to see that M_b is bounded from L^p to L^q with $1 < p < \infty$ and $1/q = 1/p - \alpha/n$ when $b \in \mathrm{Lip}_\alpha$. Indeed, noting that for any $x \in \mathbb{R}^n$,

$$M_b(f)(x) \leqslant \|b\|_{\mathrm{Lip}_\alpha} M_\alpha(f)(x),$$

where

$$M_\alpha(f)(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/n}} \int_B |f(y)| \, \mathrm{d}y,$$

we obtain that for $f \in L^p$,

$$M_b(f)(x) < \infty \quad \text{a.e.}$$

Let B be a fixed ball. Without loss of generality for any $x, x' \in B$ we may assume that

$$x \neq x', \quad M_b(f)(x') \leqslant M_b(f)(x) \quad \text{and} \quad M_b(f)(x) < \infty.$$

Thus, for any $\varepsilon > 0$ there is a ball $B_1 := B(x_0, r) \ni x$ such that

$$M_b(f)(x) - \frac{1}{|B_1|} \int_{B_1} |b(x) - b(y)||f(y)| \, \mathrm{d}y < \varepsilon. \tag{3.1}$$

Since $x' \in B(x_0, r + |x - x'|) =: B_2$, we deduce that

$$\frac{1}{|B_2|} \int_{B_2} |b(x') - b(y)||f(y)| \, \mathrm{d}y \leqslant M_b(f)(x'). \tag{3.2}$$

By (3.1) and (3.2), we have

$$\begin{aligned}
&M_b(f)(x) - M_b(f)(x') \\
&\quad \leqslant \frac{1}{|B_1|} \int_{B_1} |b(x) - b(y)||f(y)| \, \mathrm{d}y - \frac{1}{|B_2|} \int_{B_2} |b(x') - b(y)||f(y)| \, \mathrm{d}y + \varepsilon \\
&\quad \leqslant |b(x) - b(x')| \frac{1}{|B_1|} \int_{B_1} |f(y)| \, \mathrm{d}y + \frac{1}{|B_1|} \int_{B_1} |b(x') - b(y)||f(y)| \, \mathrm{d}y \\
&\quad\quad - \frac{1}{|B_2|} \int_{B_2} |b(x') - b(y)||f(y)| \, \mathrm{d}y + \varepsilon.
\end{aligned}$$

To estimate for $M_b(f)(x) - M_b(f)(x')$, we need the following result.

Claim 3.1. *Let $0 < \alpha < 1$, $1/(1-\alpha) \leqslant s < p$ and $x, x' \in \mathbb{R}^n$ with $x \neq x'$. For any ball $B_1 := B(x_0, r) \ni x$ and $B_2 := B(x_0, r + |x - x'|)$, there exists a constant C such that*

$$\left| \frac{1}{|B_1|} \int_{B_1} |b(x') - b(y)||f(y)| \,\mathrm{d}y - \frac{1}{|B_2|} \int_{B_2} |b(x') - b(y)||f(y)| \,\mathrm{d}y \right| \leqslant C|x - x'|^\alpha M_s(f)(x),$$

where $M_s(f)(x) = (M(|f|^s)(x))^{1/s}$.

Proof. There are two possibilities as follows, and then we use different techniques to analyze each part.

Case 1: $r \leqslant |x - x'|$. Then $r + |x - x'| \leqslant 2|x - x'|$, which implies that

$$\begin{aligned} &\left| \frac{1}{|B_1|} \int_{B_1} |b(x') - b(y)||f(y)| \,\mathrm{d}y - \frac{1}{|B_2|} \int_{B_2} |b(x') - b(y)||f(y)| \,\mathrm{d}y \right| \\ &\qquad \leqslant \left| \frac{1}{|B_1|} \int_{B_1} |b(x') - b(y)||f(y)| \,\mathrm{d}y \right| \\ &\qquad\quad + \left| \frac{1}{|B_2|} \int_{B_2} |b(x') - b(y)||f(y)| \,\mathrm{d}y \right| \\ &\qquad \leqslant C|B_2|^{\alpha/n} M(f)(x) \leqslant C|x - x'|^\alpha M(f)(x). \end{aligned}$$

Case 2: $r > |x - x'|$. We can compute

$$\begin{aligned} &\left| \frac{1}{|B_1|} \int_{B_1} |b(x') - b(y)||f(y)| \,\mathrm{d}y - \frac{1}{|B_2|} \int_{B_2} |b(x') - b(y)||f(y)| \,\mathrm{d}y \right| \\ &\quad \leqslant \left| \frac{1}{|B_1|} \int_{B_1} |b(x') - b(y)||f(y)| \,\mathrm{d}y - \frac{1}{|B_2|} \int_{B_1} |b(x') - b(y)||f(y)| \,\mathrm{d}y \right| \\ &\quad\quad + \left| \frac{1}{|B_2|} \int_{B_1} |b(x') - b(y)||f(y)| \,\mathrm{d}y - \frac{1}{|B_2|} \int_{B_2} |b(x') - b(y)||f(y)| \,\mathrm{d}y \right| \\ &\quad = \mathrm{I}_1 + \mathrm{I}_2. \end{aligned}$$

By the fact that $x, y \in B_1$ and $|x - x'| < r$, we obtain that

$$|x' - y| \leqslant |x - x'| + |x - y| \leqslant 3r.$$

In addition, we can conclude that $|B_2| - |B_1| \leqslant Cr^{n-1}|x - x'|$. In fact, if $n = 1$, we get $|B_2| - |B_1| = 2|x - x'|$ immediately. If $n \geqslant 2$, it follows from $|B_2| - |B_1| \approx (r + |x - x'|)^n - r^n$ and the differential mean value theorem that

$$(r + |x - x'|)^n - r^n = n\xi^{n-1}|x - x'| \leqslant Cr^{n-1}|x - x'|$$

for some $\xi \in (r, r+|x-x'|)$. Therefore,

$$\begin{aligned}\mathrm{I}_1 &= \frac{|B_2|-|B_1|}{|B_2|}\cdot\frac{1}{|B_1|}\int_{B_1}|b(x')-b(y)||f(y)|\,\mathrm{d}y \leqslant C\frac{|x-x'|}{r}|B_1|^{\alpha/n}M(f)(x)\\ &\leqslant C|x-x'|^{\alpha}M(f)(x).\end{aligned}$$

For any $1/(1-\alpha)\leqslant s<p$ we have $\alpha\leqslant 1/s'$, then

$$\begin{aligned}\mathrm{I}_2 &\leqslant \frac{1}{|B_2|}\int_{B_2\setminus B_1}|b(x')-b(y)||f(y)|\,\mathrm{d}y \leqslant \frac{C}{|B_2|^{1-\alpha/n}}\int_{B_2\setminus B_1}|f(y)|\,\mathrm{d}y\\ &\leqslant C\frac{(|B_2|-|B_1|)^{1/s'}}{|B_2|^{1/s'-\alpha/n}}M_s(f)(x)\leqslant C|x-x'|^{\alpha}M_s(f)(x).\end{aligned}$$

Thus, we obtain the desired result. $\square$

Set $\varepsilon = |x-x'|^{\alpha}M_s(f)(x)$. Then Claim 3.1 gives us that

$$|M_b(f)(x)-M_b(f)(x')|\leqslant C|x-x'|^{\alpha}M_s(f)(x).$$

Therefore, it follows from Lemma 2.1 that

$$\begin{aligned}\|M_b(f)\|_{\dot F_p^{\alpha,\infty}} &\approx \Big\|\sup_{B\ni\cdot}\frac{1}{|B|^{1+\alpha/n}}\int_B|M_b(f)(x)-(M_b(f))_B|\,\mathrm{d}x\Big\|_{L^p}\\ &\leqslant \Big\|\sup_{B\ni\cdot}\frac{1}{|B|^{2+\alpha/n}}\int_B\int_B|M_b(f)(x)-M_b(f)(x')|\,\mathrm{d}x'\,\mathrm{d}x\Big\|_{L^p}\\ &\leqslant C\Big\|\sup_{B\ni\cdot}\frac{1}{|B|}\int_B M_s(f)(x)\,\mathrm{d}x\Big\|_{L^p}\leqslant C\|M(M_s(f))\|_{L^p}\leqslant C\|f\|_{L^p}.\end{aligned}$$

Thus, the proof of (a1) $\Rightarrow$ (a2) is completed.

(a2) $\Rightarrow$ (a1): Let B be any fixed ball. For any $x\in B$,

$$|b(x)-b_Q|\leqslant\frac{1}{|B|}\int_B|b(x)-b(y)|\chi_B(y)\,\mathrm{d}y\leqslant M_b(\chi_B)(x). \tag{3.3}$$

We set q such that $1/p-1/q=\alpha/n$. By Sobolev-Besov embedding,

$$\|\cdot\|_{L^q}=\|\cdot\|_{\dot F_q^{0,2}}\leqslant C\|\cdot\|_{\dot F_p^{\alpha,\infty}},$$

which shows that M_b is bounded from L^p to L^q. Then inequality (3.3) shows that

$$\Big(\int_B|b(x)-b_B|^q\,\mathrm{d}x\Big)^{1/q}\leqslant\|M_b(\chi_B)\|_{L^q}\leqslant C\|\chi_B\|_{L^p}\leqslant C|B|^{1/p}.$$

From Lemma 2.1 we get $b\in\mathrm{Lip}_{\alpha}$. $\square$

P r o o f of Theorem 1.2. (b1) $\Rightarrow$ (b2) is similar as the proof of (a1) $\Rightarrow$ (a2).

(b2) $\Rightarrow$ (b1): From the fact that

$$\begin{aligned}|b(x)-b_B| &\leqslant \frac{1}{|B|}\int_B |b(x)-b(y)|\,\mathrm{d}y\\ &\leqslant \Big(\frac{1}{|B|}\int_B |b(x)-b(y)|^m\chi_B(y)\,\mathrm{d}y\Big)^{1/m}\\ &\leqslant (M_{b,m}(\chi_B)(x))^{1/m}.\end{aligned}$$

We set q such that $1/p-1/q=\alpha/n$. By Sobolev-Besov embedding,

$$\|\cdot\|_{L^q}=\|\cdot\|_{\dot F_q^{0,2}}\leqslant C\|\cdot\|_{\dot F_p^{m\alpha,\infty}},$$

which shows that $M_{b,m}$ is bounded from L^p to L^q. Then

$$\Big(\int_B |b(x)-b_B|^{mq}\,\mathrm{d}x\Big)^{1/mq}\leqslant \|M_{b,m}(\chi_B)\|_{L^q}^{1/m}\leqslant C\|\chi_B\|_{L^p}^{1/m}\leqslant C|B|^{1/mp}.$$

This shows that $b\in \mathrm{Lip}_\alpha$. $\square$

P r o o f of Theorem 1.3. (c1) $\Rightarrow$ (c2): From the fact that

$$\mathcal{M}_{b_1,1}(f_1,f_2)(x)\leqslant \|b_1\|_{\mathrm{Lip}_\alpha}\mathcal{M}_\alpha(f_1,f_2)(x),$$

where $\mathcal{M}_\alpha$ is the bilinear fractional maximal operator

$$\mathcal{M}_\alpha(\vec f)(x)=\sup_{B\ni x}|B|^{\alpha/n}\prod_{i=1}^2\frac{1}{|B|}\int_B |f_i(y_i)|\,\mathrm{d}y_i.$$

Then we obtain that for $f_1\in L^{p_1}$ and $f_2\in L^{p_2}$,

$$\mathcal{M}_{b_1,1}(f_1,f_2)(x)<\infty\quad \text{a.e.}$$

Let B be a fixed ball. Without loss of generality, for any $x,x'\in B$ we may assume that

$$x\neq x',\quad \mathcal{M}_{b_1,1}(f_1,f_2)(x')\leqslant \mathcal{M}_{b_1,1}(f_1,f_2)(x)\quad\text{and}\quad \mathcal{M}_{b_1,1}(f_1,f_2)(x)<\infty.$$

Thus, for any $\varepsilon>0$ there is a ball $B_1:=B(x_0,r)\ni x$ such that

$$\mathcal{M}_{b_1,1}(f_1,f_2)(x)-\frac{1}{|B_1|^2}\int_{B_1}\int_{B_1}|b_1(x)-b_1(y_1)||f_1(y_1)||f_2(y_2)|\,\mathrm{d}y_1\,\mathrm{d}y_2\leqslant\varepsilon. \tag{3.4}$$

Since $x' \in B(x_0, r + |x - x'|) =: B_2$, we deduce that

$$\frac{1}{|B_2|^2} \int_{B_2} \int_{B_2} |b_1(x) - b_1(y_1)||f_1(y_1)||f_2(y_2)| \,\mathrm{d}y_1 \,\mathrm{d}y_2 \leqslant \mathcal{M}_{b_1,1}(f_1, f_2)(x'). \tag{3.5}$$

By (3.4) and (3.5), we have

$$\begin{aligned}
&\mathcal{M}_{b_1,1}(f_1, f_2)(x) - \mathcal{M}_{b_1,1}(f_1, f_2)(x') \\
&\qquad \leqslant \frac{1}{|B_1|^2} \int_{B_1} \int_{B_1} |b_1(x) - b_1(y_1)||f_1(y_1)||f_2(y_2)| \,\mathrm{d}y_1 \,\mathrm{d}y_2 \\
&\qquad\quad - \frac{1}{|B_2|^2} \int_{B_2} \int_{B_2} |b_1(x') - b_1(y_1)||f_1(y_1)||f_2(y_2)| \,\mathrm{d}y_1 \,\mathrm{d}y_2 + \varepsilon \\
&\qquad \leqslant |b_1(x) - b_1(x')| \frac{1}{|B_1|^2} \int_{B_1} \int_{B_1} |f_1(y_1)||f_2(y_2)| \,\mathrm{d}y_1 \,\mathrm{d}y_2 \\
&\qquad\quad + \frac{1}{|B_1|^2} \int_{B_1} \int_{B_1} |b_1(x') - b_1(y_1)||f_1(y_1)||f_2(y_2)| \,\mathrm{d}y_1 \,\mathrm{d}y_2 \\
&\qquad\quad - \frac{1}{|B_2|^2} \int_{B_2} \int_{B_2} |b_1(x') - b_1(y_1)||f_1(y_1)||f_2(y_2)| \,\mathrm{d}y_1 \,\mathrm{d}y_2 + \varepsilon.
\end{aligned}$$

We claim that there exists a constant C such that for $1/(1-\alpha) \leqslant s < \min\{p_1, p_2\}$,

$$\begin{aligned}
&\int_{B_2} \int_{B_2} \Big| \frac{\chi_{B_1}(y_1)\chi_{B_1}(y_2)}{|B_1|^2} - \frac{\chi_{B_2}(y_1)\chi_{B_2}(y_2)}{|B_2|^2} \Big| |f_1(y_1)||f_2(y_2)| \,\mathrm{d}y_1 \,\mathrm{d}y_2 \\
&\qquad \leqslant \frac{C|x - x'|^\alpha}{|B_2|^{\alpha/n}} M_s(f_1)(x) M_s(f_2)(x).
\end{aligned} \tag{3.6}$$

Then, we conclude that

$$\begin{aligned}
&|\mathcal{M}_{b_1,1}(f_1, f_2)(x) - \mathcal{M}_{b_1,1}(f_1, f_2)(x')| \\
&\qquad \leqslant C|x - x'|^\alpha M(f_1)(x) M(f_2)(x) + \varepsilon \\
&\qquad\quad + \int_{B_2} \int_{B_2} \Big| \frac{\chi_{B_1}(y_1)\chi_{B_1}(y_2)}{|B_1|^2{}_2} - \frac{\chi_{B_2}(y_1)\chi_{B_2}(y_2)}{|B_2|^2} \Big| \\
&\qquad\qquad \times |b_1(x') - b_1(y_1)| \prod_{i=1} |f_i(y_i)| \,\mathrm{d}y_1 \,\mathrm{d}y_2 \\
&\qquad \leqslant C|x - x'|^\alpha M_s(f_1)(x) M_s(f_2)(x).
\end{aligned}$$

Using the same argument as above, we conclude that

$$|\mathcal{M}_{b_2,2}(f_1, f_2)(x) - \mathcal{M}_{b_2,2}(f_1, f_2)(x')| \leqslant C|x - x'|^\alpha M_s(f_1)(x) M_s(f_2)(x).$$

This shows that

$$|\mathcal{M}_{\vec{b}}(f_1,f_2)(x)-\mathcal{M}_{\vec{b}}(f_1,f_2)(x')|\leqslant C|x-x'|^{\alpha}M_s(f_1)(x)M_s(f_2)(x),$$

which implies that

$$\begin{aligned}\|\mathcal{M}_{\vec{b}}(f_1,f_2)\|_{\dot F_p^{\alpha,\infty}} &\approx \Big\|\sup_{B\ni\cdot}\frac{1}{|B|^{1+\alpha/n}}\int_B|\mathcal{M}_{\vec{b}}(f_1,f_2)(x)-(\mathcal{M}_{\vec{b}}(f_1),f_2)_B|\,\mathrm{d}x\Big\|_{L^p}\\ &\leqslant \Big\|\sup_{B\ni\cdot}\frac{1}{|B|^{2+\alpha/n}}\int_B\int_B|\mathcal{M}_{\vec{b}}(f_1,f_2)(x)-\mathcal{M}_{\vec{b}}(f_1,f_2)(x')|\,\mathrm{d}x'\,\mathrm{d}x\Big\|_{L^p}\\ &\leqslant C\Big\|\sup_{Q\ni\cdot}\frac{1}{|B|}\int_B M_s(f_1)(x)M_s(f_2)(x)\,\mathrm{d}x\Big\|_{L^p}\\ &\leqslant C\|M(M_s(f_1)M_s(f_2))\|_{L^p}\leqslant C\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.\end{aligned}$$

Thus, the proof of (c1) ⇒ (c2) is completed. □

Now we give the proof of inequality (3.6).

P r o o f of (3.6).

Case 1: $r\leqslant|x-x'|$. Then $r+|x-x'|\leqslant 2|x-x'|$, which implies that

$$\begin{aligned}&\int_{B_2}\int_{B_2}\Big|\frac{\chi_{B_1}(y_1)\chi_{B_1}(y_2)}{|B_1|^2}-\frac{\chi_{B_2}(y_1)\chi_{B_2}(y_2)}{|B_2|^2}\Big|\prod_{i=1}^2|f_i(y_i)|\,\mathrm{d}y_1\,\mathrm{d}y_2\\ &\qquad\leqslant\int_{B_2}\int_{B_2}\Big(\frac{\chi_{B_1}(y_1)\chi_{B_1}(y_2)}{|B_1|^2}+\frac{\chi_{B_2}(y_1)\chi_{B_2}(y_2)}{|B_2|^2}\Big)\prod_{i=1}^2|f_i(y_i)|\,\mathrm{d}y_1\,\mathrm{d}y_2\\ &\qquad\leqslant CM(f_1)(x)M(f_2)(x)\\ &\qquad\leqslant C\frac{|x-x'|^{\alpha}}{|B_2|^{\alpha/n}}M(f_1)(x)M(f_2)(x).\end{aligned}$$

Case 2: $r>|x-x'|$. Notice that by adding and subtracting

$$\int_{B_2}\int_{B_2}\frac{\chi_{B_1}(y_1)\chi_{B_1}(y_2)}{|B_2|^2}\prod_{i=1}^2|f_i(y_i)|\,\mathrm{d}y_1\,\mathrm{d}y_2$$

and

$$\int_{B_2}\int_{B_2}\frac{\chi_{B_1}(y_1)\chi_{B_2}(y_2)}{|B_2|^2}\prod_{i=1}^2|f_i(y_i)|\,\mathrm{d}y_1\,\mathrm{d}y_2,$$

we can compute

$$\begin{aligned}
&\int_{B_2}\int_{B_2}\Big|\frac{\chi_{B_1}(y_1)\chi_{B_1}(y_2)}{|B_1|^2}-\frac{\chi_{B_2}(y_1)\chi_{B_2}(y_2)}{|B_2|^2}\Big|\prod_{i=1}^{2}|f_i(y_i)|\,\mathrm{d}y_1\,\mathrm{d}y_2\\
&\quad\leqslant\int_{B_2}\int_{B_2}\Big|\frac{\chi_{B_1}(y_1)\chi_{B_1}(y_2)}{|B_1|^2}-\frac{\chi_{B_1}(y_1)\chi_{B_1}(y_2)}{|B_2|^2}\Big|\prod_{i=1}^{2}|f_i(y_i)|\,\mathrm{d}y_1\,\mathrm{d}y_2\\
&\qquad+\int_{B_2}\int_{B_2}\Big|\frac{\chi_{B_1}(y_1)\chi_{B_1}(y_2)}{|B_2|^2}-\frac{\chi_{B_1}(y_1)\chi_{B_2}(y_2)}{|B_2|^2}\Big|\prod_{i=1}^{2}|f_i(y_i)|\,\mathrm{d}y_1\,\mathrm{d}y_2\\
&\qquad+\int_{B_2}\int_{B_2}\Big|\frac{\chi_{B_1}(y_1)\chi_{B_2}(y_2)}{|B_2|^2}-\frac{\chi_{B_2}(y_1)\chi_{B_2}(y_2)}{|B_2|^2}\Big|\prod_{i=1}^{2}|f_i(y_i)|\,\mathrm{d}y_1\,\mathrm{d}y_2\\
&\quad=:\mathrm{II}_1+\mathrm{II}_2+\mathrm{II}_3.
\end{aligned}$$

For II_1 we simply have

$$\begin{aligned}
\mathrm{II}_1&=\frac{|B_2|^2-|B_1|^2}{|B_2|^2}\cdot\frac{1}{|B_1|^2}\int_{B_1}\int_{B_1}|f_1(y_1)||f_2(y_2)|\,\mathrm{d}y_1\,\mathrm{d}y_2\\
&\leqslant C\frac{|x-x'|}{r}M(f_1)(x)M(f_2)(x)\leqslant C\frac{|x-x'|^\alpha}{|B_2|^{\alpha/n}}M(f_1)(x)M(f_2)(x).
\end{aligned}$$

We now move on to the control of II_2. For $1/(1-\alpha)\leqslant s<\min\{p_1,p_2\}$ we arrive at

$$\begin{aligned}
\mathrm{II}_2&\leqslant\frac{1}{|B_2|^2}\int_{B_1}\int_{B_2\setminus B_1}|f_1(y_1)||f_2(y_2)|\,\mathrm{d}y_1\,\mathrm{d}y_2\leqslant\frac{1}{|B_2|}\int_{B_2\setminus B_1}|f_2(y_2)|\,\mathrm{d}y_2M(f_1)(x)\\
&\leqslant C\frac{|x-x'|^\alpha}{|B_2|^{\alpha/n}}M_s(f_1)(x)M(f_2)(x).
\end{aligned}$$

By applying the same argument as for II_2, we have that

$$\mathrm{II}_3\leqslant C\frac{|x-x'|^\alpha}{|B_2|^{\alpha/n}}M(f_1)(x)M_s(f_2)(x).$$

Thus, we obtain the desired result.

(c2) ⇒ (c1): Let B be any fixed ball. For any $x\in B$,

$$|b_1(x)-(b_1)_B|\leqslant\frac{1}{|B|^2}\int_B\int_B|b_1(x)-b_1(y_1)|\chi_B(y_1)\chi_B(y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2\leqslant\mathcal{M}_{b_1,1}(\chi_B,\chi_B)(x).$$

Meanwhile, we can obtain

$$\sum_{i=1}^{2}|b_i(x)-(b_i)_B|\leqslant\mathcal{M}_{\vec{b}}(\chi_B,\chi_B)(x).$$

By Sobolev-Besov embedding, we have $\mathcal{M}_{\vec{b}}$ is bounded from $L^{p_1}\times L^{p_2}$ to L^q with $1/p_1+1/p_2-1/q=\alpha/n$. Then

$$\biggl(\int_B\sum_{i=1}^2|b_i(x)-(b_i)_B|^q\,{\rm d}x\biggr)^{1/q}\leqslant C\|\chi_B\|_{L^{p_1}}\|\chi_B\|_{L^{p_2}}\leqslant C|B|^{1/p}.$$

This shows that $b_1,b_2\in\mathrm{Lip}_\alpha$. □

References

[1] *L. Chaffee, R. H. Torres*: Characterization of compactness of the commutators of bilinear fractional integral operators. Potential Anal. *43* (2015), 481–494. zbl MR doi

[2] *R. R. Coifman, R. Rochberg, G. Weiss*: Factorization theorems for Hardy spaces in several variables. Ann. Math. (2) *103* (1976), 611–635. zbl MR doi

[3] *R. A. DeVore, R. C. Sharpley*: Maximal Functions Measuring Smoothness. Memoirs of the American Mathematical Society 293. AMS, Providence, 1984. zbl MR doi

[4] *S. Janson*: Mean oscillation and commutators of singular integral operators. Ark. Mat. *16* (1978), 263–270. zbl MR doi

[5] *S. Janson, M. Taibleson, G. Weiss*: Elementary characterizations of the Morrey-Campanato spaces. Harmonic analysis. Lecture Notes in Mathematics 992. Springer, Berlin, 1983, pp. 101–114. zbl MR doi

[6] *T. Kato, G. Ponce*: Commutator estimates and the Euler and Navier-Stokes equations. Commun. Pure Appl. Math. *41* (1988), 891–907. zbl MR doi

[7] *A. K. Lerner, S. Ombrosi, C. Pérez, R. H. Torres, R. Trujillo-González*: New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory. Adv. Math. *220* (2009), 1222–1264. zbl MR doi

[8] *M. Paluszyński*: Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss. Indiana Univ. Math. J. *44* (1995), 1–17. zbl MR doi

[9] *H. Triebel*: Theory of Function Spaces. Monographs in Mathematics 78. Birkhäuser, Basel, 1983. zbl MR doi

[10] *H. Triebel*: Theory of Function Spaces II. Monographs in Mathematics 84. Birkhäuser, Basel, 1992. zbl MR doi

[11] *D. H. Wang, J. Zhou, Z. D. Teng*: A note on Campanato spaces and their applications. Math. Notes *103* (2018), 483–489. zbl MR doi

[12] *P. Zhang*: Multiple weighted estimates for commutators of multilinear maximal function. Acta Math. Sin., Engl. Ser. *31* (2015), 973–994. zbl MR doi

[13] *P. Zhang*: Characterization of Lipschitz spaces via commutators of the Hardy-Littlewood maximal function. C. R., Math., Acad. Sci. Paris *355* (2017), 336–344. zbl MR doi

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