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Jian He; Jing He; Panyue Zhou

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RELATIVE AUSLANDER BIJECTION IN n-EXANGULATED CATEGORIES

Jian He, Lanzhou, Jing He, Panyue Zhou, Changsha

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Abstract. The aim of this article is to study the relative Auslander bijection in n-exangulated categories. More precisely, we introduce the notion of generalized Auslander-Reiten-Serre duality and exploit a bijection triangle, which involves the generalized Auslander-Reiten-Serre duality and the restricted Auslander bijection relative to the subfunctor. As an application, this result generalizes the work by Zhao in extriangulated categories.

Keywords: n-exangulated category; generalized Auslander-Reiten-Serre duality; restricted Auslander bijection

MSC 2020: 16G70, 18G80, 18E10

1. Introduction

The notion of extriangulated categories was introduced by Nakaoka-Palu (see [19]), which can be viewed as a simultaneous generalization of exact categories and triangulated categories. The data of such a category is a triplet $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$, where $\mathscr{C}$ is an additive category, $\mathbb{E}\colon \mathscr{C}^{\rm op} \times \mathscr{C} \to \mathsf{Ab}$ is an additive bifunctor and $\mathfrak{s}$ assigns to each $\delta \in \mathbb{E}(C, A)$ a class of 3-term sequences with end terms A and C such that certain axioms hold. Recently, Herschend-Liu-Nakaoka in [11] introduced the notion of n-exangulated categories for any positive integer n. It is not only a higher dimensional analogue of extriangulated categories defined by Nakaoka-Palu (see [19]), but also gives a common generalization of $(n+2)$-angulated categories in the sense of Geiss-Keller-Oppermann (see [6]) and n-exact categories in the sense of Jasso,

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see [15]. However, there are some examples of n-exangulated categories which are neither n-exact nor $(n+2)$-angulated, see [11], [12], [13], [18].

Functors and morphisms determined by objects were introduced by Auslander, see [1]. These concepts generalize the previous work of Auslander and Reiten on almost split sequences, see [2], [3]. Later, Ringel in [20] presented a survey of these results, rearranged them as lattice isomorphisms (the Auslander bijections) and added many examples. The concept of a morphism determined by an object provides a method to construct or classify morphisms in a fixed category. Chen in [4] investigated the Auslander bijection in a k-linear Hom-finite Krull-Schmidt abelian category having Auslander-Reiten duality. Subsequently, Jiao in [16], [17] considered a generalized version on exact categories. Recently, Zhao-Tan-Huang extended Chen and Jiao's result to the extriangulated category $\mathscr{C}$. Namely, let $\mathscr{C}$ be an exangulated category, they studied the generalized Auslander-Reiten theory and Auslander bijection in [22], [23], and He-He-Zhou showed that Zhao-Tan-Huang's results have the higher counterparts in [7], [8].

As the above related work extends to further generalization, Zhao in [21] studied the Auslander bijection relative to an additive subfunctor in exangulated categories by using the generalized Auslander-Reiten theory. Specifically, suppose that $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is a k-linear Hom-finite Krull-Schmidt extriangulated category, where k is a field. Zhao constructed a bijection triangle, which involves the generalized Auslander-Reiten-Serre duality and the restricted Auslander bijection relative to the subfunctor. Our main result shows that Zhao's result has a higher counterpart.

Theorem 1.1 (see Theorem 4.13 for more detail)**.** *Assume that $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is a k-linear Hom-finite Krull-Schmidt n-exangulated category. Let $\mathbb{F}$ be an additive closed subfunctor of $\mathbb{E}$ and $X \in \mathscr{C}_{\mathbb{F},l}$. The bijection triangle*

$$\begin{array}{ccc} & \mathrm{sub}_{\mathrm{End}_{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X)^{\mathrm{op}}}\underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X, Y) & \\ \nearrow^{\eta_{\tau_{\mathbb{F}}^{-}X,Y}} & & \nwarrow^{\Upsilon_{X,Y}} \\ {}^{\tau_{\mathbb{F}}^{-}X}[\to Y\rangle_{\mathfrak{s}|_{\mathbb{F}}\text{-def}} & \xrightarrow{\quad \xi_{X,Y} \quad} & \mathrm{sub}_{\mathrm{End}_{\mathscr{C}}(X)}\mathbb{F}(Y, X) \end{array}$$

is commutative. In particular, we get the restricted Auslander bijection at Y relative to $\tau_{\mathbb{F}}^{-}X$

$$\eta_{\tau_{\mathbb{F}}^{-}X,Y} \colon {}^{\tau_{\mathbb{F}}^{-}X}[\to Y\rangle_{\mathfrak{s}|_{\mathbb{F}}\text{-def}} \to \mathrm{sub}_{\mathrm{End}_{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X)^{\mathrm{op}}}\underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X, Y),$$

which is an isomorphism of posets.

This article is organized as follows. In Section 2, we review some elementary definitions and facts on n-exangulated categories. In Section 3, we introduce the notion of generalized Auslander-Reiten-Serre duality and study its basic properties. In Section 4, we prove our main result.

Let $\mathscr{C}$ be a skeletally small additive category and n be a positive integer. Suppose that $\mathscr{C}$ is equipped with an additive bifunctor $\mathbb{E}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathsf{Ab}$, where Ab is the category of abelian groups. Next we briefly recall some definitions and basic properties of n-exangulated categories from [11]. We omit some details here, but the reader can find them in [11].

For any pair of objects $A, C \in \mathscr{C}$, an element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-*extension* or simply an *extension*. We also write such δ as ${}_A\delta_C$ when we indicate A and C. The zero element ${}_A0_C = 0 \in \mathbb{E}(C, A)$ is called the *split* $\mathbb{E}$-*extension*. For any pair of $\mathbb{E}$-extensions ${}_A\delta_C$ and ${}_{A'}\delta'_{C'}$, let $\delta \oplus \delta' \in \mathbb{E}(C \oplus C', A \oplus A')$ be the element corresponding to $(\delta, 0, 0, \delta')$ through the natural isomorphism $\mathbb{E}(C \oplus C', A \oplus A') \simeq \mathbb{E}(C, A) \oplus \mathbb{E}(C, A') \oplus \mathbb{E}(C', A) \oplus \mathbb{E}(C', A')$.

For any $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C', C)$, $\mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A')$ and $\mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A)$ are simply denoted by $a_*\delta$ and $c^*\delta$, respectively.

Let ${}_A\delta_C$ and ${}_{A'}\delta'_{C'}$ be any pair of $\mathbb{E}$-extensions. A *morphism* $(a, c)\colon \delta \to \delta'$ of extensions is a pair of morphisms $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C, C')$ in $\mathscr{C}$, satisfying the equality $a_*\delta = c^*\delta'$.

Definition 2.1 ([11], Definition 2.7)**.** Let $\mathbf{C}_{\mathscr{C}}$ be the category of complexes in $\mathscr{C}$. As its full subcategory, define $\mathbf{C}^{n+2}_{\mathscr{C}}$ to be the category of complexes in $\mathscr{C}$ whose components are zero in the degrees outside of $\{0, 1, \ldots, n+1\}$. Namely, an object in $\mathbf{C}^{n+2}_{\mathscr{C}}$ is a complex $X_\bullet = \{X_i, d_i^X\}$ of the form

$$X_0 \xrightarrow{d_0^X} X_1 \xrightarrow{d_1^X} \cdots \xrightarrow{d_{n-1}^X} X_n \xrightarrow{d_n^X} X_{n+1}.$$

We write a morphism $f_\bullet\colon X_\bullet \to Y_\bullet$ simply $f_\bullet = (f_0, f_1, \ldots, f_{n+1})$, only indicating the terms of degrees $0, \ldots, n+1$.

Definition 2.2 ([11], Definition 2.11)**.** By Yoneda lemma, any extension $\delta \in \mathbb{E}(C, A)$ induces natural transformations

$$\delta_\sharp\colon \mathscr{C}(-, C) \Rightarrow \mathbb{E}(-, A) \quad \text{and} \quad \delta^\sharp\colon \mathscr{C}(A, -) \Rightarrow \mathbb{E}(C, -).$$

For any $X \in \mathscr{C}$, these $(\delta_\sharp)_X$ and $\delta^\sharp_X$ are given as

(1) $(\delta_\sharp)_X\colon \mathscr{C}(X, C) \to \mathbb{E}(X, A)\colon f \mapsto f^*\delta$,
(2) $\delta^\sharp_X\colon \mathscr{C}(A, X) \to \mathbb{E}(C, X)\colon g \mapsto g_*\delta$.

We simply denote $(\delta_\sharp)_X(f)$ and $\delta^\sharp_X(g)$ by $\delta_\sharp(f)$ and $\delta^\sharp(g)$, respectively.

Definition 2.3 ([11], Definition 2.9). Let $\mathscr{C}, \mathbb{E}, n$ be as before. Define a category $\text{Æ} := \text{Æ}^{n+2}_{(\mathscr{C},\mathbb{E})}$ as follows.

(1) An object in $\text{Æ}^{n+2}_{(\mathscr{C},\mathbb{E})}$ is a pair $\langle X_\bullet, \delta \rangle$ of $X_\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ and $\delta \in \mathbb{E}(X_{n+1}, X_0)$ satisfying

$$(d_0^X)_* \delta = 0 \quad \text{and} \quad (d_n^X)^* \delta = 0.$$

We call such a pair an $\mathbb{E}$-attached complex of length $n+2$. We also denote it by

$$X_0 \xrightarrow{d_0^X} X_1 \xrightarrow{d_1^X} \dots \xrightarrow{d_{n-2}^X} X_{n-1} \xrightarrow{d_{n-1}^X} X_n \xrightarrow{d_n^X} X_{n+1} \overset{\delta}{\dashrightarrow} .$$

(2) For such pairs $\langle X_\bullet, \delta \rangle$ and $\langle Y_\bullet, \varrho \rangle$, a morphism $f_\bullet \colon \langle X_\bullet, \delta \rangle \to \langle Y_\bullet, \varrho \rangle$ is defined to be a morphism $f_\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X_\bullet, Y_\bullet)$ satisfying $(f_0)_* \delta = (f_{n+1})^* \varrho$.

We use the same composition and identities as in $\mathbf{C}^{n+2}_{\mathscr{C}}$.

Definition 2.4 ([11], Definition 2.13). An *n-exangle* is a pair $\langle X_\bullet, \delta \rangle$ of $X_\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ and $\delta \in \mathbb{E}(X_{n+1}, X_0)$ which satisfies the following conditions. (1) The sequence

$$\mathscr{C}(-, X_0) \xrightarrow{\mathscr{C}(-,d_0^X)} \cdots \xrightarrow{\mathscr{C}(-,d_n^X)} \mathscr{C}(-, X_{n+1}) \xrightarrow{\delta_\sharp} \mathbb{E}(-, X_0)$$

of functors $\mathscr{C}^{\text{op}} \to \mathsf{Ab}$ is exact. (2) The sequence

$$\mathscr{C}(X_{n+1}, -) \xrightarrow{\mathscr{C}(d_n^X,-)} \cdots \xrightarrow{\mathscr{C}(d_0^X,\ -)} \mathscr{C}(X_0, -) \xrightarrow{\delta^\sharp} \mathbb{E}(X_{n+1}, -)$$

of functors $\mathscr{C} \to \mathsf{Ab}$ is exact.

In particular any n-exangle is an object in Æ. A *morphism of n-exangles* simply means a morphism in Æ. Thus, n-exangles form a full subcategory of Æ.

Let $X_\bullet$ be a complex of length $n+2$ with fixed end-terms. In other words, $X_\bullet$ satisfies $X_0 = A$ and $X_{n+1} = C$. We also write it as ${}_AX_{\bullet C}$ when we emphasize A and C.

Definition 2.5 ([11], Definition 2.22). Let $\mathfrak{s}$ be a correspondence which associates a homotopic equivalence class $\mathfrak{s}(\delta) = [{}_AX_{\bullet C}]$ to each extension $\delta = {}_A\delta_C$. Such $\mathfrak{s}$ is called a *realization* of $\mathbb{E}$ in $\mathbf{C}^{n+2}_{\mathscr{C}}$ if it satisfies the following condition for any $\mathfrak{s}(\delta) = [X_\bullet]$ and any $\mathfrak{s}(\varrho) = [Y_\bullet]$.

(R0) For any morphism of extensions $(a, c) \colon \delta \to \varrho$, there exists a morphism $f_\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X_\bullet, Y_\bullet)$ of the form $f_\bullet = (a, f_1, \dots, f_n, c)$. Such $f_\bullet$ is called a *lift* of (a, c).

In such a case, we simply say that "$X_\bullet$ realizes δ" whenever they satisfy $\mathfrak{s}(\delta) = [X_\bullet]$.

Moreover, a realization $\mathfrak{s}$ of $\mathbb{E}$ is said to be *exact* if it satisfies the following conditions.

(R1) For any $\mathfrak{s}(\delta) = [X_\bullet]$, the pair $\langle X_\bullet, \delta\rangle$ is an n-exangle.

(R2) For any $A \in \mathscr{C}$, the zero element ${}_A0_0 = 0 \in \mathbb{E}(0, A)$ satisfies

$$\mathfrak{s}({}_A0_0) = [A \xrightarrow{\mathrm{id}_A} A \to 0 \to \ldots \to 0 \to 0].$$

Dually, $\mathfrak{s}({}_00_A) = [0 \to 0 \to \ldots \to 0 \to A \xrightarrow{\mathrm{id}_A} A]$ holds for any $A \in \mathscr{C}$.

Note that the above condition (R1) does not depend on representatives of the class $[X_\bullet]$.

Definition 2.6 ([11], Definition 2.23). Let $\mathfrak{s}$ be an exact realization of $\mathbb{E}$.

(1) An n-exangle $\langle X_\bullet, \delta\rangle$ is called an $\mathfrak{s}$-*distinguished* n-exangle if it satisfies $\mathfrak{s}(\delta) = [X_\bullet]$. We often simply say a *distinguished n-exangle* when $\mathfrak{s}$ is clear from the context.

(2) An object $X_\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ is called an $\mathfrak{s}$-*conflation* or simply a *conflation* if it realizes some extension $\delta \in \mathbb{E}(X_{n+1}, X_0)$.

(3) A morphism f in $\mathscr{C}$ is called an $\mathfrak{s}$-*inflation* or simply an *inflation* if it admits some conflation $X_\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ satisfying $d_0^X = f$.

(4) A morphism g in $\mathscr{C}$ is called an $\mathfrak{s}$-*deflation* or simply a *deflation* if it admits some conflation $X_\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ satisfying $d_n^X = g$.

Definition 2.7 ([11], Definition 2.27). For a morphism $f_\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X_\bullet, Y_\bullet)$ satisfying $f_0 = \mathrm{id}_A$ for some $A = X_0 = Y_0$, its *mapping cone* $M^f_\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ is defined to be the complex

$$X_1 \xrightarrow{d_0^{M_f}} X_2 \oplus Y_1 \xrightarrow{d_1^{M_f}} X_3 \oplus Y_2 \xrightarrow{d_2^{M_f}} \ldots \xrightarrow{d_{n-1}^{M_f}} X_{n+1} \oplus Y_n \xrightarrow{d_n^{M_f}} Y_{n+1}$$

where

$$d_0^{M_f} = \begin{bmatrix} -d_1^X \\ f_1 \end{bmatrix}, \quad d_i^{M_f} = \begin{bmatrix} -d_{i+1}^X & 0 \\ f_{i+1} & d_i^Y \end{bmatrix} \quad (1 \leqslant i \leqslant n-1), \quad d_n^{M_f} = [\, f_{n+1} \quad d_n^Y \,].$$

The *mapping cocone* is defined dually, for morphisms $h_\bullet$ in $\mathbf{C}^{n+2}_{\mathscr{C}}$ satisfying $h_{n+1} = \mathrm{id}$.

Definition 2.8 ([11], Definition 2.32). An *n-exangulated category* is a triplet $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ of an additive category $\mathscr{C}$, an additive bifunctor $\mathbb{E}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathsf{Ab}$, and its exact realization $\mathfrak{s}$ in $\mathbf{C}^{n+2}_{\mathscr{C}}$, satisfying the following conditions.

(EA1) Let $A \xrightarrow{f} B \xrightarrow{g} C$ be any sequence of morphisms in $\mathscr{C}$. If both f and g are inflations, then so is $g \circ f$. Dually, if f and g are deflations, then so is $g \circ f$.

(EA2) For $\varrho \in \mathbb{E}(D, A)$ and $c \in \mathscr{C}(C, D)$, let ${}_A\langle X_\bullet, c^*\varrho\rangle_C$ and ${}_A\langle Y^\cdot, \varrho\rangle_D$ be distinguished n-exangles. Then (id_A, c) has a *good lift* $f_\bullet$, in the sense that its mapping cone gives a distinguished n-exangle $\langle M^f_\bullet, (d_0^X)_*\varrho\rangle$.

(EA2$^{\mathrm{op}}$) Dual of (EA2).

Remark 2.9.

(1) Note that in the case $n = 1$, a triplet $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is a 1-exangulated category if and only if it is an extriangulated category, see [11], Proposition 4.3.

(2) From [11], Proposition 4.34 and [11], Proposition 4.5, we know that $(n + 2)$-angulated in the sense of Geiss-Keller-Oppermann (see [6]) and n-exact categories in the sense of Jasso (see [15]) are n-exangulated categories. There are some other examples of n-exangulated categories which are neither n-exact nor $(n + 2)$-angulated, see [11], [12], [13], [18].

The following are some very useful lemmas and they will be needed later on.

Lemma 2.10 ([11], Claim 2.15)**.** *Let $\mathscr{C}$ be an n-exangulated category, and*

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-2}} A_{n-1} \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1} \overset{\theta}{\dashrightarrow} \tag{2.1}$$

be a distinguished n-exangle in $\mathscr{C}$. Then the following are equivalent:

(1) *α_0 is a split monomorphism (also known as a section);*

(2) *α_n is a split epimorphism (also known as a retraction);*

(3) *$\theta = 0$.*

If a distinguished n-exangle (2.1) *satisfies one of the above equivalent conditions, it is called split.*

Definition 2.11 ([24], Definition 3.14 and [18], Definition 3.2)**.** Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category. An object $P \in \mathscr{C}$ is called *projective* if for any distinguished n-exangle

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-2}} A_{n-1} \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1} \overset{\delta}{\dashrightarrow}$$

and any morphism c in $\mathscr{C}(P, A_{n+1})$, there exists a morphism $b \in \mathscr{C}(P, A_n)$ satisfying $\alpha_n \circ b = c$. The concept of injective objects is defined dually.

Lemma 2.12 ([18], Lemma 3.4)**.** *Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category. Then the following statements are equivalent for an object $P \in \mathscr{C}$.*

(1) *$\mathbb{E}(P, A) = 0$ for any $A \in \mathscr{C}$.*

(2) *P is projective.*

(3) *Any distinguished n-exangle $A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-2}} A_{n-1} \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} P \overset{\delta}{\dashrightarrow}$ splits.*

Lemma 2.13 ([24], Lemma 3.3). *Let $\mathscr{C}$ be an n-exangulated category, and*

$$\begin{array}{ccccccccccc}
X_0 & \xrightarrow{f_0} & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} \cdots \longrightarrow & X_n & \xrightarrow{f_n} & X_{n+1} & \overset{\delta}{\dashrightarrow} \\
\downarrow{\scriptstyle a_0} & & \downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle a_2} & & \downarrow{\scriptstyle a_n} & & \downarrow{\scriptstyle a_{n+1}} & \\
Y_0 & \xrightarrow{g_0} & Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} \cdots \longrightarrow & Y_n & \xrightarrow{g_n} & Y_{n+1} & \overset{\eta}{\dashrightarrow}
\end{array}$$

any morphism of distinguished n-exangles. Then the following are equivalent:

(1) *There is a morphism $h_1\colon X_1 \to Y_0$ such that $h_1 f_0 = a_0$.*
(2) *There is a morphism $h_{n+1}\colon X_{n+1} \to Y_n$ such that $g_n h_{n+1} = a_{n+1}$.*
(3) $(a_0)_*\delta = (a_{n+1})^*\eta = 0$.

3. The generalized Auslander-Reiten-Serre duality

Unless otherwise specified, we always assume that $\mathscr{C}$ is a k-linear Hom-finite Krull-Schmidt n-exangulated category, where k is a field. We put $D := \mathrm{Hom}_k(-, k)$.

We denote by $\mathrm{rad}_{\mathscr{C}}$ the Jacobson radical of $\mathscr{C}$. Namely, $\mathrm{rad}_{\mathscr{C}}$ is an ideal of $\mathscr{C}$ such that $\mathrm{rad}_{\mathscr{C}}(A, A)$ coincides with the Jacobson radical of the endomorphism ring $\mathrm{End}(A)$ for any $A \in \mathscr{C}$.

Assume that $\mathcal{B}$ is an additive category.

(a) A morphism $\alpha_n\colon A_n \to A_{n+1}$ in $\mathcal{B}$ is called right almost split if
  (1) α_n is not a split epimorphism and
  (2) for every $f\colon Y \to A_{n+1}$ in $\mathcal{B}$ that is not a split epimorphism there exists $h\colon Y \to A_n$ such that $\alpha_n h = f$, that is, h makes the triangle

$$\begin{array}{ccc}
 & & Y \\
 & {\scriptstyle h}\swarrow & \downarrow{\scriptstyle f} \\
A_n & \xrightarrow{\alpha_n} & A_{n+1}
\end{array}$$

  commutative.

(b) A morphism $\alpha_0\colon A_0 \to A_1$ in $\mathcal{B}$ is called left almost split if
  (1) α_0 is not a split monomorphism and
  (2) for every $g\colon A_0 \to Z$ in $\mathcal{B}$ that is not a split monomorphism there exists $h\colon A_1 \to Z$ such that $g = h\alpha_0$, that is, h makes the triangle

$$\begin{array}{ccc}
A_0 & \xrightarrow{\alpha_0} & A_1 \\
{\scriptstyle g}\downarrow & \swarrow{\scriptstyle h} & \\
Z & &
\end{array}$$

  commutative.

Next, let us recall the notion of Auslander-Reiten n-exangles in an n-exangulated category.

Definition 3.1 ([9], Definition 3.1). A distinguished n-exangle

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-2}} A_{n-1} \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1} \overset{\delta}{\dashrightarrow} \tag{3.1}$$

in $\mathscr{C}$ is called an *Auslander-Reiten n-exangle* if α_0 is left almost split, α_n is right almost split and when for $n \geqslant 2$, $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ are in $\mathrm{rad}_{\mathscr{C}}$.

Lemma 3.2 ([9], Lemma 3.3). *Let*

$$A_\bullet\colon\ A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-2}} A_{n-1} \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1} \overset{\delta}{\dashrightarrow}$$

be a distinguished n-exangle in $\mathscr{C}$. Then the following statements are equivalent:

(1) *$A_\bullet$ is an Auslander-Reiten n-exangle;*
(2) $\mathrm{End}(A_0)$ *is local, $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ are in* $\mathrm{rad}_{\mathscr{C}}$ *and α_n is right almost split;*
(3) $\mathrm{End}(A_{n+1})$ *is local, $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ are in* $\mathrm{rad}_{\mathscr{C}}$ *and α_0 is left almost split.*

The following lemma shows that a distinguished n-exangle in an equivalence class can be chosen in a minimal way in a Krull-Schmidt n-exangulated category.

Lemma 3.3 ([10], Lemma 3.4). *Let A_0, A_{n+1} be two objects in $\mathscr{C}$. Then for every equivalence class associated with $\mathbb{E}$-extension $\delta = {}_{A_0}\delta_{A_{n+1}}$, there exists a representation*

$$A_\bullet\colon\ A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-2}} A_{n-1} \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1} \overset{\delta}{\dashrightarrow}$$

such that $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ are in $\mathrm{rad}_{\mathscr{C}}$. *Moreover, $A_\bullet$ is a direct summand of every other elements in this equivalent class.*

In what follows, let $\mathbb{F} \subseteq \mathbb{E}$ be an additive sub-bifunctor. Then we have $a_*\delta \in \mathbb{F}(C, A')$ and $c^*\delta \in \mathbb{F}(C', A)$ for any $a \in \mathscr{C}(A, A')$, $c \in \mathscr{C}(C', C)$ and $\delta \in \mathbb{F}(C, A)$. For a realization $\mathfrak{s}$ of $\mathbb{E}$, define $\mathfrak{s}|_{\mathbb{F}}$ to be the restriction of $\mathfrak{s}$ onto $\mathbb{F}$. Then $\mathfrak{s}|_{\mathbb{F}}$ is an exact realization of $\mathbb{F}$. Moreover, the triplet $(\mathscr{C}, \mathbb{F}, \mathfrak{s}|_{\mathbb{F}})$ satisfies the condition (EA2) and (EA2$^{\mathrm{op}}$), see [11], Claim 3.9. Thus, we may speak of $\mathfrak{s}|_{\mathbb{F}}$-conflations (or $\mathfrak{s}|_{\mathbb{F}}$-inflations, or $\mathfrak{s}|_{\mathbb{F}}$-deflations, respectively) and $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangles as in Definition 2.6. However, it is worth noting that $(\mathscr{C}, \mathbb{F}, \mathfrak{s}|_{\mathbb{F}})$ is not an n-exangulated category in general, see [11], Proposition 3.16 and [21], Example 2.12.

Definition 3.4. An $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangle

$$A_\bullet\colon\ A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-2}} A_{n-1} \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1} \overset{\delta}{\dashrightarrow}$$

in $\mathscr{C}$ is called *Auslander-Reiten $\mathfrak{s}|_{\mathbb{F}}$-$n$-exangle* if $A_\bullet$ is an *Auslander-Reiten n-exangle.*

We always assume that the following condition, analogous to the (WIC) condition in [19], Condition 5.8, holds.

Condition 3.5. Let $f \in \mathscr{C}(A, B)$, $g \in \mathscr{C}(B, C)$ be any composable pair of morphisms. Consider the following conditions.
(1) If $g \circ f$ is an $\mathfrak{s}|_{\mathbb{F}}$-deflation, then so is g.
(2) If $g \circ f$ is an $\mathfrak{s}|_{\mathbb{F}}$-inflation, then so is f.

Definition 3.6. (1) A morphism $f\colon A \to B$ in $\mathscr{C}$ is called $\mathbb{F}$-*projectively trivial* if for each $C \in \mathscr{C}$, the induced map $\mathbb{F}(f, C)\colon \mathbb{F}(B, C) \to \mathbb{F}(A, C)$ is zero. Dually, a morphism $g\colon A \to B$ in $\mathscr{C}$ is called $\mathbb{F}$-*injectively trivial* if for each $C \in \mathscr{C}$, the induced map $\mathbb{F}(C, g)\colon \mathbb{F}(C, A) \to \mathbb{F}(C, B)$ is zero.

(2) An object $C \in \mathscr{C}$ is called $\mathbb{F}$-*projectively trivial* if the identity morphism id_C is $\mathbb{F}$-projectively trivial. Dually, an object $C \in \mathscr{C}$ is called $\mathbb{F}$-*injectively trivial* if the identity morphism id_C is $\mathbb{F}$-injectively trivial.

For an $\mathbb{F}$-projectively trivial morphism, we have the following equivalent characterization.

Lemma 3.7. *Let $f \in \mathscr{C}(A, B)$ be a morphism. Then the following statements are equivalent.*
(1) *f is $\mathbb{F}$-projectively trivial.*
(2) *f factors through any $\mathfrak{s}|_{\mathbb{F}}$-deflation $g\colon X_n \to B$.*
(3) *For any $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangle $X_\bullet\colon X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{g} B \overset{\theta}{\dashrightarrow}$, if there exists a morphism of $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangles*

$$
\begin{array}{cccccccccccccc}
X'_\bullet\colon & X_0 & \xrightarrow{\alpha'_0} & X'_1 & \xrightarrow{\alpha'_1} & X'_2 & \xrightarrow{\alpha'_2} & \cdots & \xrightarrow{\alpha'_{n-1}} & X'_n & \xrightarrow{g'} & A & \overset{f^*\theta}{\dashrightarrow} & \\
 & \| & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle f} & & \\
X_\bullet\colon & X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{g} & B & \overset{\theta}{\dashrightarrow} & ,
\end{array}
\tag{3.2}
$$

then the top $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangle $X'_\bullet$ is split.

Proof. (1) $\Leftrightarrow$ (3) $\Rightarrow$ (2) It is straightforward to verify.
(2) $\Rightarrow$ (3) For any $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangle

$$X_\bullet\colon X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{g} B \overset{\theta}{\dashrightarrow},$$

consider the diagram (3.2). By the assumption (2), f factors through g, and so α'_0 is a split monomorphism by Lemma 2.13. Thus, $f^\star\theta = 0$, that is, the top $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangle $X'_\bullet$ is split. $\square$

Construction 3.8. Let A and B be two objects in $\mathscr{C}$. We denote by $\mathcal{P}_{\mathbb{F}}(A,B)$ (or $\mathcal{I}_{\mathbb{F}}(A,B)$) the set of $\mathbb{F}$-projectively trivial (or $\mathbb{F}$-injectively trivial, respectively) morphisms from A to B. The *stable category* $\underline{\mathscr{C}}$ (or *costable category* $\overline{\mathscr{C}}$) of $\mathscr{C}$ is defined as follows: the category whose objects are objects of $\mathscr{C}$ and whose morphisms are elements of $\underline{\mathscr{C}}(A,B) = \mathscr{C}(A,B)/\mathcal{P}_{\mathbb{F}}(A,B)$ (or $\overline{\mathscr{C}}(A,B) = \mathscr{C}(A,B)/\mathcal{I}_{\mathbb{F}}(A,B)$, respectively). Given a morphism $f\colon A \to B$ in $\mathscr{C}$, we denote by $\underline{f}$ the image of f in $\underline{\mathscr{C}}$ (or $\overline{f}$ the image of f in $\overline{\mathscr{C}}$, respectively).

Given an *Auslander-Reiten* $\mathfrak{s}|_{\mathbb{F}}$*-$n$-exangle*

$$X_\bullet\colon\ X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} X_{n+1} \overset{\gamma}{\dashrightarrow} .$$

Put $D = \operatorname{Hom}_k(-,k)$. Since $X_\bullet$ is not split, there exists some $\varphi \in D\mathbb{F}(X_{n+1}, X_0)$ such that $\varphi(\gamma) \neq 0$. Next, for each object Y in $\mathscr{C}$, we can get a *non-degenerate* k-bilinear map

$$\langle -,-\rangle_Y\colon\ \overline{\mathscr{C}}(Y,X_0) \times \mathbb{F}(X_{n+1},Y) \to k, \quad (\overline{f},\delta) \mapsto \varphi(f_*\delta).$$

In fact, for any non-split $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangle

$$Y_\bullet\colon\ Y \xrightarrow{\beta_0} Y_1 \xrightarrow{\beta_1} Y_2 \xrightarrow{\beta_2} \dots \xrightarrow{\beta_{n-1}} Y_n \xrightarrow{\beta_n} X_{n+1} \overset{\delta}{\dashrightarrow},$$

since $X_\bullet$ is an Auslander-Reiten $\mathfrak{s}|_{\mathbb{F}}$-$n$-exangle, we obtain the commutative diagram

$$\begin{array}{rccccccccccc}
Y_\bullet\colon & Y & \xrightarrow{\beta_0} & Y_1 & \xrightarrow{\beta_1} & \cdots & \xrightarrow{\beta_{n-2}} & Y_{n-1} & \xrightarrow{\beta_{n-1}} & Y_n & \xrightarrow{\beta_n} & X_{n+1} \overset{\delta}{\dashrightarrow} \\
 & {\scriptstyle f}\,\big\downarrow & & \big\downarrow & & & & \big\downarrow & & \big\downarrow{\scriptstyle f_n} & & \big\| \\
X_\bullet\colon & X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & \cdots & \xrightarrow{\alpha_{n-2}} & X_{n-1} & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & X_{n+1} \overset{\gamma}{\dashrightarrow}
\end{array}$$

by the dual of [11], Proposition 3.6. Hence, $f_*\delta = \gamma$ and $f \in \overline{\mathscr{C}}(Y,X_0)$. Then we have that $\langle \overline{f},\delta\rangle_Y = \varphi(f_*\delta) = \varphi(\gamma) \neq 0$.

On the other hand, suppose $0 \neq \overline{f} \in \overline{\mathscr{C}}(Y,X_0)$, then $f\colon Y \to X_0$ representing $\overline{f}$ is not $\mathfrak{s}|_{\mathbb{F}}$-injective, and there exist $Z \in \mathscr{C}$ and $\varepsilon \in \mathbb{F}(Z,Y)$ such that $f_*\varepsilon$ is non-split by the dual of Lemma 3.7. Since $X_\bullet$ is an Auslander-Reiten $\mathfrak{s}|_{\mathbb{F}}$-$n$-exangle, by [11], Proposition 3.6 we have the commutative diagram

$$\begin{array}{rccccccccccc}
Z_\bullet\colon & Y & \xrightarrow{\eta_0} & Z_1 & \xrightarrow{\eta_1} & \cdots & \xrightarrow{\eta_{n-2}} & Z_{n-1} & \xrightarrow{\eta_{n-1}} & Z_n & \xrightarrow{\eta_n} & Z \overset{\varepsilon}{\dashrightarrow} \\
 & {\scriptstyle f}\,\big\downarrow & & \big\downarrow & & & & \big\downarrow & & \big\downarrow{\scriptstyle f_n} & & \big\| \\
U_\bullet\colon & X_0 & \xrightarrow{\zeta_0} & U_1 & \xrightarrow{\zeta_1} & \cdots & \xrightarrow{\zeta_{n-2}} & U_{n-1} & \xrightarrow{\zeta_{n-1}} & U_n & \xrightarrow{\zeta_n} & Z \overset{f_*\varepsilon}{\dashrightarrow} \\
 & \big\| & & {\scriptstyle h_1}\big\uparrow & & & & \big\uparrow & & \big\uparrow & & {\scriptstyle h}\big\uparrow \\
X_\bullet\colon & X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & \cdots & \xrightarrow{\alpha_{n-2}} & X_{n-1} & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & X_{n+1} \overset{\gamma}{\dashrightarrow} .
\end{array}$$

Then $\gamma = h^*(f_*\varepsilon) = f_*h^*\varepsilon$, therefore, we have that $\langle \overline{f}, h^*\varepsilon\rangle_Y = \varphi(f_*(h^*\varepsilon)) = \varphi(\gamma) \neq 0$.

Thus, we have the following proposition.

Proposition 3.9. *Let* $X_\bullet\colon\ X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} X_{n+1} \overset{\gamma}{\dashrightarrow}$ *be an Auslander-Reiten* $\mathfrak{s}|_{\mathbb{F}}$*-*$n$*-exangle in* $\mathscr{C}$ *and* $\varphi \in D\mathbb{F}(X_{n+1}, X_0)$ *with* $\varphi(\gamma) \neq 0$.

(1) *For each* $Y \in \mathscr{C}$, *we have a non-degenerate* k*-bilinear map*

$$\langle -,-\rangle_Y\colon\ \overline{\mathscr{C}}(Y, X_0) \times \mathbb{F}(X_{n+1}, Y) \to k, \quad (\overline{f}, \delta) \mapsto \varphi(f_*\delta).$$

Moreover, the induced map

$$\varphi_{X_{n+1},Y}\colon\ \overline{\mathscr{C}}(Y, X_0) \to D\mathbb{F}(X_{n+1}, Y), \quad \overline{f} \mapsto \langle \overline{f}, -\rangle_Y,$$

is a natural isomorphism and functorial in $Y \in \mathscr{C}$ *with* $\varphi = \varphi_{X_{n+1},X_0}(\overline{\mathrm{Id}_{X_0}})$.

(2) *For each* $Y \in \mathscr{C}$, *we have a non-degenerate* k*-bilinear map*

$${}_Y\langle -,-\rangle\colon\ \mathbb{F}(Y, X_0) \times \underline{\mathscr{C}}(X_{n+1}, Y) \to k, \quad (\delta, \underline{g}) \mapsto \varphi(g^*\delta).$$

Moreover, the induced map

$$\psi_{Y,X_0}\colon\ \underline{\mathscr{C}}(X_{n+1}, Y) \to D\mathbb{F}(Y, X_0), \quad \underline{g} \mapsto_Y \langle -, \underline{g}\rangle,$$

is a natural isomorphism and functorial in $Y \in \mathscr{C}$ *with* $\varphi = \psi_{X_{n+1},X_0}(\underline{\mathrm{Id}_{X_{n+1}}})$.

P r o o f. (1) The functoriality of $\varphi_{X_{n+1}}\colon\ \overline{\mathscr{C}}(-, X_0) \to D\mathbb{F}(X_{n+1}, -)$ follows from a direct verifition.

(2) It is similar to (1). □

Proposition 3.10. *Let* X_{n+1} (*or* Y_0) *be a non-*$\mathfrak{s}|_{\mathbb{F}}$*-projective* (*or non-*$\mathfrak{s}|_{\mathbb{F}}$*-injective, respectively*) *indecomposable object in* $\mathscr{C}$.

(1) *Assume that* $\varphi_{X_{n+1},-}\colon\ \overline{\mathscr{C}}(-, X') \to D\mathbb{F}(X_{n+1}, -)$ *is an isomorphism of functors for some* $X' \in \mathscr{C}$, *which has a non-*$\mathfrak{s}|_{\mathbb{F}}$*-injective indecomposable direct summand, then there exists an Auslander-Reiten* $\mathfrak{s}|_{\mathbb{F}}$*-*$n$*-exangle ending at* X_{n+1} *in* $\mathscr{C}$.

(2) *Assume that* $\psi_{-,Y_0}\colon\ \underline{\mathscr{C}}(Y', -) \to D\mathbb{F}(-, Y_0)$ *is an isomorphism of functors for some* $Y' \in \mathscr{C}$, *which has a non-*$\mathfrak{s}|_{\mathbb{F}}$*-projective indecomposable direct summand, then there exists an Auslander-Reiten* $\mathfrak{s}|_{\mathbb{F}}$*-*$n$*-exangle starting at* Y_0 *in* $\mathscr{C}$.

P r o o f. (1) For each object and each morphism $f\colon\ U \to X'$, by the naturality of $\varphi_{X_{n+1},-}$, we obtain the commutative diagram

$$\begin{array}{ccc}
\overline{\mathscr{C}}(X', X') & \xrightarrow{\varphi_{X_{n+1},X'}} & D\mathbb{F}(X_{n+1}, X') \\
{\scriptstyle \overline{\mathscr{C}}(f,X')}\big\downarrow & & \big\downarrow{\scriptstyle D\mathbb{F}(X_{n+1},f)} \\
\overline{\mathscr{C}}(U, X') & \xrightarrow{\varphi_{X_{n+1},U}} & D\mathbb{F}(X_{n+1}, U).
\end{array}$$

Set $\varphi = \varphi_{X_{n+1},X'}(\overline{\mathrm{Id}_{X'}})$, then we have

$$\varphi_{X_{n+1},U}(\overline{f}) = D\mathbb{F}(X_{n+1}, f)(\varphi) = \varphi \circ \mathbb{F}(X_{n+1}, f).$$

It follows that $\varphi_{X_{n+1},U}(\overline{f})(\theta) = \varphi(f_*\theta)$ for each $\theta \in \mathbb{F}(X_{n+1}, U)$.

Let X_0 *be* a non-$\mathfrak{s}|_{\mathbb{F}}$-injective indecomposable direct summand of X'. Then the isomorphism φ_{X_{n+1},X_0} induces a *non-degenerate* k-bilinear map

$$\langle -, -\rangle_{X_0} \colon \overline{\mathscr{C}}(X_0, X') \times \mathbb{F}(X_{n+1}, X_0) \to k, \quad (\overline{f}, \delta) \mapsto \varphi(f_*\delta).$$

Take

$$\Xi = \{f \in \mathscr{C}(X_0, X') \colon f \text{ is a non-split monomorphism}\}.$$

Since X_0 is non-$\mathfrak{s}|_{\mathbb{F}}$-injective, we have $\mathcal{I}_{\mathbb{F}}(X_0, X') \subseteq \Xi$. Hence $\overline{\Xi} := \Xi/\mathcal{I}_{\mathbb{F}}(X_0, X')$ is properly contained in $\overline{\mathscr{C}}(X_0, X')$. Then there exists a non-split $\mathbb{F}$-extension $\delta \in \mathbb{F}(X_{n+1}, X_0)$ of the form

$$X_\bullet \colon X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} X_{n+1} \overset{\delta}{\dashrightarrow}$$

such that $\langle \overline{h}, \delta\rangle_{X_0} = \varphi(h_*\delta) = 0$ for each non-split monomorphisms $h \colon X_0 \to X'$ in Ξ. Here, we may assume that $\alpha_i \in \mathrm{rad}_{\mathscr{C}}$ for $i \in \{1, 2, \dots, n-1\}$ by Lemma 3.3.

Next, we claim that the morphism α_0 is left almost split. Suppose that $s \colon X_0 \to V$ is not a split monomorphism, then for each $t \colon V \to X'$, the morphism $t \circ s$ lies in Ξ. Hence, we have $\langle \overline{t \circ s}, \delta\rangle_{X_0} = 0$. Consider the *non-degenerate* k-bilinear map

$$\langle -, -\rangle_V \colon \overline{\mathscr{C}}(V, X') \times \mathbb{F}(X_{n+1}, V) \to k, \quad (\overline{f}, \beta) \mapsto \varphi(f_*\beta),$$

which is induced by $\varphi_{X_{n+1},V}$. Hence, we have

$$\langle \overline{t}, s_*\delta\rangle_V = \varphi(t_*(s_*\delta)) = \langle \overline{t \circ s}, \delta\rangle_{X_0} = 0.$$

This implies that the $\mathbb{F}$-extension $s_*\delta$ splits by the non-degeneracy of $\langle -, -\rangle_V$. By Lemma 2.13, the morphism s factors through α_0. This shows the morphism α_0 is left almost split. Therefore, $X_\bullet$ is an Auslander-Reiten $\mathfrak{s}|_{\mathbb{F}}$-$n$-exangle from Lemma 3.2 since $\mathrm{End}(X_{n+1})$ is local.

(2) It is similar to (1). □

We define two full subcategories of $\mathscr{C}$ as

$$\mathscr{C}_{\mathbb{F},r} = \{X \in \mathscr{C} \colon \text{the functor } D\mathbb{F}(X, -) \colon \overline{\mathscr{C}} \to \mathrm{mod}\, k \text{ is representable}\},$$
$$\mathscr{C}_{\mathbb{F},l} = \{X \in \mathscr{C} \colon \text{the functor } D\mathbb{F}(-, X) \colon \underline{\mathscr{C}} \to \mathrm{mod}\, k \text{ is representable}\}.$$

Then we have the following result.

Proposition 3.11. *Let X and Y be indecomposable objects in $\mathscr{C}$.*

(1) *If X is non-$\mathfrak{s}|_{\mathbb{F}}$-projective, then $X \in \mathscr{C}_{\mathbb{F},r}$ if and only if there exists an Auslander-Reiten $\mathfrak{s}|_{\mathbb{F}}$-$n$-exangle ending at X.*

(2) *If Y is non-$\mathfrak{s}|_{\mathbb{F}}$-injective, then $Y \in \mathscr{C}_{\mathbb{F},l}$ if and only if there exists an Auslander-Reiten $\mathfrak{s}|_{\mathbb{F}}$-$n$-exangle starting at Y.*

Proof. It follows from Propositions 3.9 and 3.10. □

Based on these two full subcategories $\mathscr{C}_{\mathbb{F},r}$ and $\mathscr{C}_{\mathbb{F},l}$, next we will construct two functors $\tau_{\mathbb{F}}\colon \underline{\mathscr{C}_{\mathbb{F},r}} \to \overline{\mathscr{C}_{\mathbb{F},l}}$ and $\tau_{\mathbb{F}}^{-}\colon \overline{\mathscr{C}_{\mathbb{F},l}} \to \underline{\mathscr{C}_{\mathbb{F},r}}$.

(1) For $X \in \mathscr{C}_{\mathbb{F},r}$, we define $\tau_{\mathbb{F}}X$ to be an object in $\mathscr{C}$ that contains no injective summands such that there exists an isomorphism

$$\varphi_{X,-}\colon \overline{\mathscr{C}}(-,\tau_{\mathbb{F}}X) \to D\mathbb{F}(X,-).$$

Then $\tau_{\mathbb{F}}$ gives a map from $\mathscr{C}_{\mathbb{F},r}$ to $\mathscr{C}$.

(2) For each Y in $\mathscr{C}_{\mathbb{F},l}$, we define $\tau_{\mathbb{F}}^{-}Y$ to be an object in $\mathscr{C}$ that contains no projective summands such that there exists an isomorphism of functors

$$\psi_{-,Y}\colon \underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}Y,-) \to D\mathbb{F}(-,Y).$$

Then $\tau_{\mathbb{F}}^{-}$ gives a map from $\mathscr{C}_{\mathbb{F},l}$ to $\mathscr{C}$.

Let $\underline{\mathscr{C}_{\mathbb{F},r}}$ be the image of $\mathscr{C}_{\mathbb{F},r}$ under the canonical functor $\mathscr{C} \to \underline{\mathscr{C}}$ and $\overline{\mathscr{C}_{\mathbb{F},l}}$ be the image of $\mathscr{C}_{\mathbb{F},l}$ under the canonical functor $\mathscr{C} \to \overline{\mathscr{C}}$. One can check that the above procedures induce two functors, which we still denote by $\tau_{\mathbb{F}}$ and $\tau_{\mathbb{F}}^{-}$. That is, we have

$$\tau_{\mathbb{F}}\colon \underline{\mathscr{C}_{\mathbb{F},r}} \to \overline{\mathscr{C}_{\mathbb{F},l}} \quad \text{and} \quad \tau_{\mathbb{F}}^{-}\colon \overline{\mathscr{C}_{\mathbb{F},l}} \to \underline{\mathscr{C}_{\mathbb{F},r}}.$$

Remark 3.12.

(1) If $X, Y \in \mathscr{C}_{\mathbb{F},r}$ and $X \cong Y$ in $\underline{\mathscr{C}}$, then $\tau_{\mathbb{F}}X \cong \tau_{\mathbb{F}}Y$ in $\overline{\mathscr{C}}$. If $X \in \mathscr{C}_{\mathbb{F},l}$ and $X \cong Y$ in $\overline{\mathscr{C}}$, then $\tau_{\mathbb{F}}^{-}X \cong \tau_{\mathbb{F}}^{-}Y$ in $\underline{\mathscr{C}}$.

(2) If $X_{n+1} \in \mathscr{C}_{\mathbb{F},r}$, then $X_{n+1} \cong \tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}X_{n+1}$ in $\underline{\mathscr{C}_{\mathbb{F},r}}$. If $Y_0 \in \mathscr{C}_{\mathbb{F},l}$, then $Y_0 \cong \tau_{\mathbb{F}}\tau_{\mathbb{F}}^{-}Y_0$ in $\overline{\mathscr{C}_{\mathbb{F},l}}$.

Theorem 3.13. *The functors*

$$\tau_{\mathbb{F}}\colon \underline{\mathscr{C}_{\mathbb{F},r}} \to \overline{\mathscr{C}_{\mathbb{F},l}} \quad \text{and} \quad \tau_{\mathbb{F}}^{-}\colon \overline{\mathscr{C}_{\mathbb{F},l}} \to \underline{\mathscr{C}_{\mathbb{F},r}}$$

are quasi-inverse to each other.

Proof. We only prove that $\underline{\nu}\colon \tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}} \to \mathrm{Id}_{\underline{\mathscr{C}_{\mathbb{F},r}}}$ is a natural isomorphism. Firstly, we prove that $\underline{\nu}$ is a natural transformation. For each $\underline{f}\colon X_{n+1} \to U_{n+1}$ in $\underline{\mathscr{C}_{\mathbb{F},r}}$, consider the following two diagrams,

$$\begin{array}{ccccc}
\overline{\mathscr{C}}(\tau_{\mathbb{F}}X_{n+1}, \tau_{\mathbb{F}}X_{n+1}) & \xrightarrow{\varphi_{X_{n+1},\tau_{\mathbb{F}}X_{n+1}}} & D\mathbb{F}(X_{n+1}, \tau_{\mathbb{F}}X_{n+1}) & \xleftarrow{\psi_{X_{n+1},\tau_{\mathbb{F}}X_{n+1}}} & \underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}X_{n+1}, X_{n+1}) \\
\Big\downarrow{\scriptstyle \overline{\mathscr{C}}(\tau_{\mathbb{F}}X_{n+1},\tau_{\mathbb{F}}(\underline{f}))} & (1) & \Big\downarrow{\scriptstyle D\mathbb{F}(f,\tau_n X_{n+1})} & (2) & \Big\downarrow{\scriptstyle \underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}X_{n+1},\underline{f})} \\
\overline{\mathscr{C}}(\tau_{\mathbb{F}}X_{n+1}, \tau_{\mathbb{F}}U_{n+1}) & \xrightarrow{\varphi_{U_{n+1},\tau_{\mathbb{F}}X_{n+1}}} & D\mathbb{F}(U_{n+1}, \tau_{\mathbb{F}}X_{n+1}) & \xleftarrow{\psi_{U_{n+1},\tau_{\mathbb{F}}X_{n+1}}} & \underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}X_{n+1}, U_{n+1})
\end{array}$$

and

$$\begin{array}{ccccc}
\overline{\mathscr{C}}(\tau_{\mathbb{F}}U_{n+1}, \tau_{\mathbb{F}}U_{n+1}) & \xrightarrow{\varphi_{U_{n+1},\tau_{\mathbb{F}}U_{n+1}}} & D\mathbb{F}(U_{n+1}, \tau_{\mathbb{F}}U_{n+1}) & \xleftarrow{\psi_{U_{n+1},\tau_{\mathbb{F}}U_{n+1}}} & \underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}U_{n+1}, U_{n+1}) \\
\Big\downarrow{\scriptstyle \overline{\mathscr{C}}(\tau_n(\underline{f}),\tau_{\mathbb{F}}U_{n+1})} & (3) & \Big\downarrow{\scriptstyle D\mathbb{F}(U_{n+1},\tau_{\mathbb{F}}(\underline{f}))} & (4) & \Big\downarrow{\scriptstyle \underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}(\underline{f}),U_{n+1})} \\
\overline{\mathscr{C}}(\tau_{\mathbb{F}}X_{n+1}, \tau_{\mathbb{F}}U_{n+1}) & \xrightarrow{\varphi_{U_{n+1},\tau_{\mathbb{F}}X_{n+1}}} & D\mathbb{F}(U_{n+1}, \tau_{\mathbb{F}}X_{n+1}) & \xleftarrow{\psi_{U_{n+1},\tau_{\mathbb{F}}X_{n+1}}} & \underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}X_{n+1}, U_{n+1}).
\end{array}$$

The square (1) commutes by the definition of $\tau_{\mathbb{F}}(\underline{f})$ and the square (2) commutes since the isomorphism $\psi_{-,\tau_{\mathbb{F}}X_{n+1}}$ is natural. Similarly, the square (3) commutes since the isomorphism $\varphi_{-,\tau_{\mathbb{F}}U_{n+1}}$ is natural and the square (4) commutes by the definition of $\tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}(\underline{f})$.

By a diagram chasing, we have

$$\tau_{\mathbb{F}}(\underline{f}) = \varphi^{-1}_{U_{n+1},\tau_{\mathbb{F}}X_{n+1}}(\psi_{U_{n+1},\tau_{\mathbb{F}}X_{n+1}}(\underline{f} \circ \underline{\nu_{X_{n+1}}}))$$

and

$$\tau_{\mathbb{F}}(\underline{f}) = \varphi^{-1}_{U_{n+1},\tau_{\mathbb{F}}X_{n+1}}(\psi_{U_{n+1},\tau_{\mathbb{F}}X_{n+1}}(\underline{\nu_{U_{n+1}}} \circ \tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}(\underline{f}))).$$

Thus, $\underline{f} \circ \underline{\nu_{X_{n+1}}} = \underline{\nu_{U_{n+1}}} \circ \tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}(\underline{f})$. It follows that $\underline{\nu}$ is a natural transformation.

Now we prove that $\underline{\nu_{X_{n+1}}}$ is an isomorphism for each $X_{n+1} \in \mathscr{C}_{\mathbb{F},r}$. We may assume that X_{n+1} is indecomposable and non-$\mathfrak{s}|_{\mathbb{F}}$-projective in $\mathscr{C}$. Put

$$\alpha = \psi_{\tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}X_{n+1},\tau_{\mathbb{F}}X_{n+1}}(\underline{\mathrm{Id}_{\tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}X_{n+1}}}) \in D\mathbb{F}(\tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}X_{n+1}, \tau_{\mathbb{F}}X_{n+1})$$

and

$$\beta = \varphi_{X_{n+1},\tau_{\mathbb{F}}X_{n+1}}(\overline{\mathrm{Id}_{\tau_{\mathbb{F}}X_{n+1}}}) \in D\mathbb{F}(X_{n+1}, \tau_{\mathbb{F}}X_{n+1}).$$

Thus, we have $\beta = \psi_{X_{n+1},\tau_{\mathbb{F}}X_{n+1}}(\underline{\nu_{X_{n+1}}})$ by the definition of $\underline{\nu_{X_{n+1}}}$. Consider the commutative diagram

$$\begin{array}{ccc} \underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}X_{n+1}, \tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}X_{n+1}) & \xrightarrow{\psi_{\tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}X_{n+1},\tau_{\mathbb{F}}X_{n+1}}} & D\mathbb{F}(\tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}X_{n+1}, \tau_{\mathbb{F}}X_{n+1}) \\ {\scriptstyle \underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}X_{n+1},\underline{\vartheta_{X_{n+1}}})}\downarrow & & \downarrow{\scriptstyle D\mathbb{F}(\vartheta_{X_{n+1}},\tau_{\mathbb{F}}X_{n+1})} \\ \underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}X_{n+1}, X_{n+1}) & \xrightarrow{\psi_{X_{n+1},\tau_{\mathbb{F}}X_{n+1}}} & D\mathbb{F}(X_{n+1}, \tau_{\mathbb{F}}X_{n+1}), \end{array}$$

and note that

$$\psi_{\tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}X_{n+1},\tau_{\mathbb{F}}X_{n+1}}(\underline{\mathrm{Id}_{\tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}X_{n+1}}}) = \alpha \text{ and } \underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}X_{n+1}, \underline{\vartheta_{X_{n+1}}})(\underline{\mathrm{Id}_{\tau_{\mathbb{F}}^{-}\tau_{\mathbb{F}}X_{n+1}}}) = \underline{\nu_{X_{n+1}}}.$$

Then we have

$$\beta = D\mathbb{F}(\nu_{X_{n+1}}, \tau_{\mathbb{F}}X_{n+1})(\alpha) = \alpha \circ \mathbb{F}(\nu_{X_{n+1}}, \tau_{\mathbb{F}}X_{n+1}).$$

Since X_{n+1} is non-$\mathfrak{s}|_{\mathbb{F}}$-projective in $\mathscr{C}$, $X_0 \cong \tau_{\mathbb{F}}X_{n+1}$ in $\overline{\mathscr{C}}$ is nonzero and then non-$\mathfrak{s}|_{\mathbb{F}}$-injective in $\mathscr{C}$. Thus, there is an isomorphism $\varphi_{X_{n+1},-}\colon \overline{\mathscr{C}}(-, X_0) \to D\mathbb{F}(X_{n+1}, -)$. By Proposition 3.10, there exists an Auslander-Reiten $\mathfrak{s}|_{\mathbb{F}}$-$n$-exangle

$$X_\bullet\colon X_0 \to X_1 \to X_2 \to \ldots \to X_n \to X_{n+1} \overset{\eta}{\dashrightarrow}.$$

By Proposition 3.9, we have a natural isomorphism

$$\varphi'_{X_{n+1},-}\colon \overline{\mathscr{C}}(-, X_0) \to D\mathbb{F}(X_{n+1}, -)$$

such that $\varphi'_{X_{n+1},X_0}(\overline{\mathrm{Id}_{X_0}})(\eta) \neq 0$. Setting $\beta' := \varphi'_{X_{n+1},X_0}(\overline{\mathrm{Id}_{X_0}})$, we have $\beta'(\eta) \neq 0$. By Yoneda's lemma, there exists some $k\colon X_0 \to \tau_{\mathbb{F}}X_{n+1}$ such that $\overline{\mathscr{C}}(-, k) = \varphi^{-1}_{X_{n+1},-} \circ \varphi'_{X_{n+1},-}$. We thus, obtain

$$\beta' = \varphi'_{X_{n+1},X_0}(\overline{\mathrm{Id}_{X_0}}) = (\varphi_{X_{n+1},X_0} \circ \overline{\mathscr{C}}(-, s))(\overline{\mathrm{Id}_{X_0}}) = \varphi_{X_{n+1},X_0}(\overline{k}).$$

Consider the commutative diagram

$$\begin{array}{ccc} \overline{\mathscr{C}}(\tau_{\mathbb{F}}X_{n+1}, \tau_{\mathbb{F}}X_{n+1}) & \xrightarrow{\varphi_{X_{n+1},\tau_{\mathbb{F}}X_{n+1}}} & D\mathbb{F}(X_{n+1}, \tau_{\mathbb{F}}X_{n+1}) \\ {\scriptstyle \overline{\mathscr{C}}(k,\tau_{\mathbb{F}}X_{n+1})}\downarrow & & \downarrow{\scriptstyle D\mathbb{F}(X_{n+1},k)} \\ \overline{\mathscr{C}}(X_0, \tau_{\mathbb{F}}X_{n+1}) & \xrightarrow{\varphi_{X_{n+1},X_0}} & D\mathbb{F}(X_{n+1}, X_0). \end{array}$$

Since $\varphi_{X_{n+1},\tau_{\mathbb{F}}X_{n+1}}(\overline{\mathrm{Id}_{\tau_{\mathbb{F}}X_{n+1}}}) = \beta$ and $\overline{\mathscr{C}}(s,\tau_{\mathbb{F}}X_{n+1})(\overline{\mathrm{Id}_{\tau_{\mathbb{F}}X_{n+1}}}) = \overline{k}$, we have

$$\beta' = D\mathbb{F}(X_{n+1},k)(\beta) = \beta \circ \mathbb{F}(X_{n+1},k) = \alpha \circ \mathbb{F}(\nu_{X_{n+1}},\tau_{\mathbb{F}}X_{n+1}) \circ \mathbb{F}(X_{n+1},k).$$

Thus,

$$0 \neq \beta'(\eta) = \alpha(\nu^*_{X_{n+1}}(k_*\eta)) = \alpha(k_*(\nu^*_{X_{n+1}}\eta)),$$

which implies that the distinguished $\mathfrak{s}|_{\mathbb{F}}$-$n$-exangle

$$U_\bullet\colon\ X_0 \to U_1 \to U_2 \to \ldots \to U_n \to \tau_{\mathbb{F}}^-\tau_{\mathbb{F}}X_{n+1} \overset{\nu^*_{X_{n+1}}\eta}{\dashrightarrow}$$

is non-split. We claim that $\nu_{X_{n+1}}\colon \tau_{\mathbb{F}}^-\tau_{\mathbb{F}}X_{n+1} \to X_{n+1}$ is a split epimorphism in $\mathscr{C}$. Otherwise, suppose that $\nu_{X_{n+1}}\colon \tau_{\mathbb{F}}^-\tau_{\mathbb{F}}X_{n+1} \to X_{n+1}$ is not a split epimorphism in $\mathscr{C}$. Since $X_\bullet$ is an Auslander-Reiten $\mathfrak{s}|_{\mathbb{F}}$-$n$-exangle, we have the commutative diagram

$$\begin{array}{lccccccccccl}
U_\bullet\colon & X_0 & \longrightarrow & U_1 & \longrightarrow \cdots \longrightarrow & U_{n-1} & \longrightarrow & U_n & \longrightarrow & \tau_n^-\tau_n X_{n+1} & \xrightarrow{\nu^*_{X_{n+1}}\eta} & \\
 & \| & & \downarrow & & \downarrow & & \downarrow & \swarrow & \downarrow{\scriptstyle \nu_{X_{n+1}}} & & \\
X_\bullet\colon & X_0 & \longrightarrow & X_1 & \longrightarrow \cdots \longrightarrow & X_{n-1} & \longrightarrow & X_n & \longrightarrow & X_{n+1} & \overset{\eta}{\dashrightarrow} & .
\end{array}$$

By Lemma 2.13, the top distinguished n-exangle is split, which is a contradiction. Thus, $\underline{\nu_{X_{n+1}}}$ is an isomorphism in $\underline{\mathscr{C}_{\mathbb{F},r}}$ since $\tau_{\mathbb{F}}^-\tau_{\mathbb{F}}X_{n+1} \cong X_{n+1}$ in $\underline{\mathscr{C}_{\mathbb{F},r}}$ by Remark 3.12. □

Definition 3.14. This sextuple $\{\mathscr{C}_{\mathbb{F},l}, \mathscr{C}_{\mathbb{F},r}, \varphi, \psi, \tau_{\mathbb{F}}, \tau_{\mathbb{F}}^-\}$ is called the generalized Auslander-Reiten-Serre duality on $\mathscr{C}$.

Remark 3.15.

(1) If $\mathbb{E} = \mathbb{F}$, then we put $\mathscr{C}_l = \mathscr{C}_{\mathbb{F},l}$, $\mathscr{C}_r = \mathscr{C}_{\mathbb{F},r}$, $\tau = \tau_{\mathbb{E}}$, $\tau^- = \tau_{\mathbb{E}}^-$.

(2) If $\mathbb{E} = \mathbb{F}$ and $\mathscr{C} = \mathscr{C}_l = \mathscr{C}_r$, then the generalized Auslander-Reiten-Serre duality is exactly the Auslander-Reiten-Serre duality in the sense of [7].

(3) If $\mathscr{C}$ is an extriangulated category, then Definition 3.14 coincides with the definition of generalized Auslander-Reiten-Serre duality of extriangulated category, cf. [21]. Moreover, if $\mathbb{E} = \mathbb{F}$ and $\mathscr{C} = \mathscr{C}_l = \mathscr{C}_r$, then the generalized Auslander-Reiten-Serre duality is exactly the Auslander-Reiten-Serre duality in the sense of [14].

Set

$$\begin{aligned}
\lambda_X &:= \varphi_{X,\tau_{\mathbb{F}}X}(\overline{\mathrm{Id}_{\tau_{\mathbb{F}}X}}) \in D\mathbb{F}(X,\tau_{\mathbb{F}}X), & \underline{\mu_X} &:= \psi^{-1}_{X,\tau_{\mathbb{F}}X}(\lambda_X) \in \underline{\mathscr{C}}(\tau_{\mathbb{F}}^-\tau_{\mathbb{F}}X, X),\\
\kappa_X &:= \psi_{\tau_{\mathbb{F}}^-X,X}(\underline{\mathrm{Id}_{\tau_{\mathbb{F}}^-X}}) \in D\mathbb{F}(\tau_{\mathbb{F}}^-X, X), & \overline{\nu_X} &:= \varphi^{-1}_{\tau_{\mathbb{F}}^-X,X}(\kappa_X) \in \overline{\mathscr{C}}(X, \tau_{\mathbb{F}}\tau_{\mathbb{F}}^-X).
\end{aligned}$$

Let us end this section with the following key lemma.

Lemma 3.16. *Let* $X_0 \to X_1 \to X_2 \to \ldots \to X_{n-1} \to X_n \to Y \overset{\delta}{\dashrightarrow}$ *be an* $\mathfrak{s}|_{\mathbb{F}}$*-distinguished* n*-exangle in* $\mathscr{C}$*.*

(1) *For any* $X \in \mathscr{C}_{\mathbb{F},r}$*, we have the commutative diagram*

$$\begin{array}{ccc} D\mathbb{F}(X,X_0) & \xrightarrow{D(\delta_\sharp)_X} & D\underline{\mathscr{C}}(X,Y) \\ \uparrow{\scriptstyle \varphi_{X,X_0}} & & \uparrow{\scriptstyle D(\psi_{Y,\tau_{\mathbb{F}}X}\underline{\mathscr{C}}(\underline{\mu_X},Y))} \\ \overline{\mathscr{C}}(X_0,\tau_{\mathbb{F}}X) & \xrightarrow{\delta^\sharp{}_{\tau_{\mathbb{F}}X}} & \mathbb{F}(Y,\tau_{\mathbb{F}}X), \end{array}$$

which is natural in both δ *and* X*.*

(2) *For any* $X \in \mathscr{C}_{\mathbb{F},l}$*, we have the commutative diagram*

$$\begin{array}{ccc} D\mathbb{F}(Y,X) & \xrightarrow{D\delta^\sharp{}_X} & D\overline{\mathscr{C}}(X_0,X) \\ \uparrow{\scriptstyle \psi_{Y,X}} & & \uparrow{\scriptstyle D(\varphi_{\tau_{\mathbb{F}}^- X,X_0}\overline{\mathscr{C}}(X_0,\overline{\nu_X}))} \\ \underline{\mathscr{C}}(\tau_{\mathbb{F}}^- X,Y) & \xrightarrow{(\delta_\sharp)_{\tau_{\mathbb{F}}^- X}} & \mathbb{F}(\tau_{\mathbb{F}}^- X,X_0), \end{array}$$

which is natural in both δ *and* X*.*

P r o o f. Since the proof is very similar to [21], Lemma 3.9, we omit it. For more details, one also can see [23]. □

4. A bijection triangle

In this section, we will show that there is a bijective triangle which involves the generalized Auslander-Reiten-Serre duality and the restricted Auslander bijection relative to the subfunctor $\mathbb{F}$. Firstly, we recall the concept of morphisms being determined by objects.

Definition 4.1 ([1])**.** Let $\mathscr{C}$ be an additive category. Let $f \in \mathscr{C}(X,Y)$ and $C \in \mathscr{C}$. The morphism f is called *right* C*-determined* and C is called a *right determiner* of f, if the following condition is satisfied: each $g \in \mathscr{C}(L,Y)$ factors through f, provided that for each $h \in \mathscr{C}(C,L)$ the morphism $g \circ h$ factors through f.

Definition 4.2 ([20])**.** Two morphisms $f\colon X \to Y$ and $f'\colon X' \to Y$ are called *right equivalent* if f factors through f' and f' factors through f, i.e., we have the commutative diagram

$$\begin{array}{ccccc} & \swarrow & X & \nwarrow & \\ & & \downarrow{\scriptstyle f} & & \\ X' & \xrightarrow[f']{} & Y & \xleftarrow[f']{} & X'. \end{array}$$

One can make some easy observations.

Remark 4.3.

(a) A right equivalence relation is an equivalence relation on the set of all morphisms ending in some object $Y \in \mathscr{C}$. Put

$$[f\rangle := \{\text{the right equivalence class of a morphism } f \in \mathscr{C}(X, Y)\}.$$

(b) Assume that f and f' are right equivalent. Then f is right C-determined if and only if so is f'. We say that $[f\rangle$ is right C-determined if a representative element f is right C-determined.

(c) Assume that f and f' are right equivalent. Then $\operatorname{Im}\mathscr{C}(C, f)= \operatorname{Im}\mathscr{C}(C, f')$.

(d) If f and f' are right C-determined, then f and f' are right equivalent if and only if $\operatorname{Im}\mathscr{C}(C, f)= \operatorname{Im}\mathscr{C}(C, f')$.

Definition 4.4 ([20])**.** Suppose $f_1 \in \mathscr{C}(X_1, Y)$ and $f_2 \in \mathscr{C}(X_2, Y)$. Then put $[f_1\rangle \leqslant [f_2\rangle$ if and only if f_1 factors through f_2.

We define two sets as follows:

(1) $[\,\to Y\rangle := \{$the set of right equivalence classes of morphisms to $Y\}$. Then $\leqslant$ induces a poset relation on $[\,\to Y\rangle$.

(2) ${}^{C}[\,\to Y\rangle := \{$the subset of $[\,\to Y\rangle$ consisting of all right equivalence classes that are right C-determined$\}$.

We denote by $\operatorname{Sub}_{\operatorname{End}_{\mathscr{C}}(C)^{\mathrm{op}}}\mathscr{C}(C, Y)$ the poset formed by $\operatorname{End}_{\mathscr{C}}(C)^{\mathrm{op}}$-submodules of $\mathscr{C}(C, Y)$, ordered by the inclusion. Then the map

$$\eta_{C,Y}\colon\ [\,\to Y\rangle \to \operatorname{Sub}_{\operatorname{End}_{\mathscr{C}}(C)^{\mathrm{op}}}\mathscr{C}(C, Y), \quad [f\rangle \mapsto \operatorname{Im}\mathscr{C}(C, f)$$

is well-defined by Remark 4.3 (c).

The restriction of $\eta_{C,Y}$ on ${}^{C}[\,\to Y\rangle$ is injective and reflects the orders, that is, for two classes $[f_1\rangle, [f_2\rangle \in {}^{C}[\,\to Y\rangle$, $[f_1\rangle \leqslant [f_2\rangle$ if and only if $\eta_{C,Y}([f_1\rangle) \subseteq \eta_{C,Y}([f_2\rangle)$.

Remark 4.5. Since each $\operatorname{End}_{\mathscr{C}}(C)^{\mathrm{op}}$-submodule of $\underline{\mathscr{C}}(C, Y)$ corresponds to a unique $\operatorname{End}_{\mathscr{C}}(C)^{\mathrm{op}}$-submodule of the set $\mathscr{C}(C, Y)$ containing $\mathcal{P}(C, Y)$, the poset $\operatorname{Sub}_{\operatorname{End}_{\mathscr{C}}(C)^{\mathrm{op}}}\underline{\mathscr{C}}(C, Y)$ is viewed as a subset of $\operatorname{Sub}_{\operatorname{End}_{\mathscr{C}}(C)^{\mathrm{op}}}\mathscr{C}(C, Y)$.

In the following, we are going to consider n-exangulated categories. Under Condition 3.5, put

$$[\,\to Y\rangle_{\mathfrak{s}|_{\mathbb{F}}\text{-def}} := \{[f\rangle \in [\,\to Y\rangle\colon\ f \text{ is a } \mathfrak{s}|_{\mathbb{F}}\text{-deflation}\}.$$

Note that $\mathcal{P}_{\mathbb{F}}(C, Y) \subseteq \operatorname{Im}\mathscr{C}(C, f)$ for any $[f\rangle \in [\,\to Y\rangle_{\mathrm{def}}$. Then we have the map

$$\eta_{C,Y}\colon\ [\,\to Y\rangle_{\mathfrak{s}|_{\mathbb{F}}\text{-def}} \to \operatorname{Sub}_{\operatorname{End}_{\mathscr{C}}(C)^{\mathrm{op}}}\underline{\mathscr{C}}(C, Y), \quad [f\rangle \mapsto \operatorname{Im}\mathscr{C}(C, f)/\mathcal{P}_{\mathbb{F}}(C, Y).$$

Put

$${}^{C}[\,\to Y\rangle_{\mathfrak{s}|_{\mathbb{F}}\text{-def}} := [\,\to Y\rangle_{\mathfrak{s}|_{\mathbb{F}}\text{-def}} \cap {}^{C}[\,\to Y\rangle.$$

Then we have the map

$$\eta_{C,Y}\colon\ {}^{C}[\to Y\rangle_{\mathfrak{s}|_{\mathbb{F}}\text{-def}} \to \mathrm{Sub}_{\mathrm{End}_{\mathscr{C}}(C)^{\mathrm{op}}}\underline{\mathscr{C}}(C,Y), \quad [f\rangle \mapsto \mathrm{Im}\,\mathscr{C}(C,f)/\mathcal{P}_{\mathbb{F}}(C,Y).$$

Definition 4.6. If the map $\eta_{C,Y}\colon\ {}^{C}[\to Y\rangle_{\mathfrak{s}|_{\mathbb{F}}\text{-def}} \to \mathrm{Sub}_{\mathrm{End}_{\mathscr{C}}(C)^{\mathrm{op}}}\underline{\mathscr{C}}(C,Y)$ above is surjective, then we say that the restricted Auslander bijection at Y relative to C holds.

Lemma 4.7. *The correspondence*

$$\xi_{X,Y}\colon\ [\to Y\rangle_{\mathfrak{s}|_{\mathbb{F}}\text{-def}} \to \mathrm{Sub}_{\mathrm{End}_{\mathscr{C}}(X)}\mathbb{F}(Y,X), \quad [f\rangle \mapsto \mathrm{Im}\,\delta^{\sharp}_{f\,X}$$

is a well-defined map.

Proof. We show that $\xi_{X,Y}([f\rangle)$ is independent of the choice of the representative elements. In fact, let $f_1 \in \mathscr{C}(Z_1,Y)$ and $f_2 \in \mathscr{C}(Z_2,Y)$ be two $\mathfrak{s}|_{\mathbb{F}}$-deflations, which are right equivalent. Then there are two $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangles

$$A_0 \to A_1 \to A_2 \to \ldots \to A_{n-1} \to Z_1 \xrightarrow{f_1} Y \overset{\delta_1}{\dashrightarrow}$$

and

$$B_0 \to B_1 \to B_2 \to \ldots \to B_{n-1} \to Z_2 \xrightarrow{f_2} Y \overset{\delta_2}{\dashrightarrow}.$$

Thus, we obtain the commutative diagram

$$\begin{array}{ccccccccccccc}
A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots & \longrightarrow & A_{n-1} & \longrightarrow & Z_1 & \xrightarrow{f_1} & Y \overset{\delta_1}{\dashrightarrow} \\
{\scriptstyle k_0}\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \| \\
B_0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots & \longrightarrow & B_{n-1} & \longrightarrow & Z_2 & \xrightarrow{f_2} & Y \overset{\delta_2}{\dashrightarrow} \\
{\scriptstyle l_0}\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \| \\
A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots & \longrightarrow & A_{n-1} & \longrightarrow & Z_1 & \xrightarrow{f_1} & Y \overset{\delta_1}{\dashrightarrow}
\end{array}$$

by the dual of [11], Proposition 3.6. Applying $\mathscr{C}(-X)$ to the commutative diagram above, we have the commutative diagram

$$\begin{array}{ccc}
\mathscr{C}(A_0,X) & \xrightarrow{\delta^{\sharp}_{1X}} & \mathbb{F}(Y,X) \\
{\scriptstyle \mathscr{C}(k_0,X)}\uparrow & \circlearrowleft & \| \\
\mathscr{C}(B_0,X) & \xrightarrow{\delta^{\sharp}_{2X}} & \mathbb{F}(Y,X) \\
{\scriptstyle \mathscr{C}(l_0,X)}\uparrow & \circlearrowleft & \| \\
\mathscr{C}(A_0,X) & \xrightarrow{\delta^{\sharp}_{1X}} & \mathbb{F}(Y,X).
\end{array}$$

Hence, we have $\mathrm{Im}\,\delta^{\sharp}_{1X} = \mathrm{Im}\,\delta^{\sharp}_{2X}$. $\square$

We denote by ${}_X[\to Y\rangle_{\mathfrak{s}|_{\mathbb{F}}\text{-def}}$ the subset of $[\to Y\rangle_{\mathfrak{s}|_{\mathbb{F}}\text{-def}}$ consisting of those classes $[f\rangle$ that have a representative element f such that there exists an $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangle

$$X_0 \to X_1 \to X_2 \to \ldots \to X_{n-1} \to W \to fY \overset{\delta_f}{\dashrightarrow}$$

with $X_0 \in \operatorname{add} X$. In this case, $\mathscr{C}(X_0, X)$ is a finitely generated projective $\operatorname{End}_{\mathscr{C}}(X)$-module, and hence, $\xi_{X,Y}([f\rangle) = \operatorname{Im} \delta^{\sharp}_{f\,X}$ is a finitely generated $\operatorname{End}_{\mathscr{C}}(X)$-module.

Put

$$\operatorname{sub}_{\operatorname{End}_{\mathscr{C}}(X)}\mathbb{F}(Y,X) := \{\text{the subset of } \operatorname{Sub}_{\operatorname{End}_{\mathscr{C}}(X)}\mathbb{F}(Y,X) \text{ consisting of finitely generated } \operatorname{End}_{\mathscr{C}}(X)\text{-modules}\}.$$

Before we begin the following proposition, let us recall the definition of anti-isomorphism. A map between posets is called *anti-isomorphism* if it is a bijection and reverses the orders of the two posets.

Proposition 4.8. *The correspondence*

$$\xi_{X,Y}\colon\ {}_X[\to Y\rangle_{\mathfrak{s}|_{\mathbb{F}}\text{-def}} \to \operatorname{sub}_{\operatorname{End}_{\mathscr{C}}(X)}\mathbb{F}(Y,X), \quad [f\rangle \mapsto \operatorname{Im}\delta^{\sharp}_{f\,X}$$

is a well-defined bijection. Moreover, it is an anti-isomorphism of posets.

Proof. We know that the $\xi_{X,Y}$ is a well-defined map by Lemma 4.7.

Step 1: We will prove that $\xi_{X,Y}$ is injective. Let

$$A_0 \to A_1 \to A_2 \to \ldots \to A_{n-1} \to Z_1 \xrightarrow{f_1} Y \overset{\delta_1}{\dashrightarrow}$$

and

$$B_0 \to B_1 \to B_2 \to \ldots \to B_{n-1} \to Z_2 \xrightarrow{f_2} Y \overset{\delta_2}{\dashrightarrow}$$

be two $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangles satisfying $A_0, B_0 \in \operatorname{add} X$. Assume that $\operatorname{Im}\delta^{\sharp}_{1X} = \operatorname{Im}\delta^{\sharp}_{2X}$. Since $B_0 \in \operatorname{add} X$, $\mathscr{C}(B_0, X) \in \operatorname{End}_{\mathscr{C}}(X)$-proj, and hence, we have the commutative diagram of exact rows

$$\begin{array}{ccc} \mathscr{C}(A_0,X) & \xrightarrow{\delta^{\sharp}_{1X}} & \operatorname{Im}\delta^{\sharp}_{1X} \\ {\scriptstyle s}\,\uparrow & \circlearrowright & \| \\ \mathscr{C}(B_0,X) & \xrightarrow{\delta^{\sharp}_{2X}} & \operatorname{Im}\delta^{\sharp}_{2X}. \end{array}$$

By the Yoneda lemma, there exists $\omega \in \mathscr{C}(A_0, B_0)$ such that $\mathscr{C}(\omega\text{‘}, X) = s$. So $\delta^{\sharp}_{2X} = \delta^{\sharp}_{1X}\mathscr{C}(\omega, X)$. Thus, for any $f \in \mathscr{C}(B_0, X)$, we have

$$f_*\delta_2 = \delta^{\sharp}_{2X}(f) = (\delta^{\sharp}_{1X}\mathscr{C}(\omega, X))(f) = \delta^{\sharp}_{1X}(f\omega) = (f\omega)_*\delta_1 = f_*\omega_*\delta_1.$$

Moreover, since $B_0 \in \operatorname{add} X$, we have $pi = \mathrm{id}_{B_0}$, where $p\colon X \to B_0$ is the natural projection and $i\colon B_0 \to X$ is the natural injection. Thus, we get

$$\begin{aligned}\delta_2 &= (\mathrm{id}_{B_0})_*\delta_2 = (pi)_*\delta_2 = p_*(i_*\delta_2) = p_*(i_*\omega_*\delta_1)\\ &= (p_*i_*)(\omega_*\delta_1) = (\mathrm{id}_{B_0})_*(\omega_*\delta_1) = \omega_*\delta_1.\end{aligned}$$

By (R0), we can obtain that (ω, id_Y) has a lift $\omega_\bullet = (\omega, \omega_1, \omega_2, \ldots, \omega_n, \mathrm{id}_Y)$, that is, there exists the commutative diagram of $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangles

$$\begin{array}{ccccccccccccc}
A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow \cdots \longrightarrow & A_{n-1} & \longrightarrow & Z_1 & \xrightarrow{f_1} & Y & \overset{\delta_1}{\dashrightarrow} & \\
\downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle\omega_1} & & \downarrow{\scriptstyle\omega_2} & & \downarrow{\scriptstyle\omega_{n-1}} & & \downarrow{\scriptstyle\omega_n} & & \| & & \\
B_0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow \cdots \longrightarrow & B_{n-1} & \longrightarrow & Z_2 & \xrightarrow{f_2} & Y & \overset{\delta_2}{\dashrightarrow} & .
\end{array}$$

In particular, f_1 factors through f_2. Dually one can prove f_2 factors through f_1. This shows that f_2 and f_1 are right equivalent and hence $[f_1\rangle = [f_2\rangle$.

Step 2: We will prove that $\xi_{X,Y}$ is surjective. Let F be any finitely generated $\mathrm{End}_{\mathscr{C}}(X)$-submodule of $\mathbb{F}(Y, X)$. Then there exists a morphism $h\colon \mathscr{C}(A_0, X) \to \mathbb{F}(Y, X)$ with $A_0 \in \operatorname{add} X$ and $\operatorname{Im} h = F$. By Yoneda's lemma, we obtain a natural isomorphism

$$\mathbb{F}(Y, A_0) \to \mathrm{Hom}_{\mathrm{End}_{\mathscr{C}}(X)}(\mathscr{C}(A_0, X), \mathbb{F}(Y, X)), \quad \delta \mapsto \delta^\sharp_X.$$

It follows that there exists an $\mathbb{F}$-extension $\delta \in \mathbb{F}(Y, A_0)$ such that $\delta^\sharp_X = h$. Let

$$A_0 \to A_1 \to A_2 \to \ldots \to A_{n-1} \to Z_1 \xrightarrow{f} Y \overset{\delta}{\dashrightarrow}$$

be an $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangle. Then $\xi_{X,Y}([f\rangle) = \operatorname{Im} \delta^\sharp_X = \operatorname{Im} h = F$.

Moreover, $\xi_{X,Y}$ is an anti-isomorphism of posets. Indeed, consider two $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangles

$$A_0 \to A_1 \to A_2 \to \ldots \to A_{n-1} \to Z_1 \xrightarrow{f_1} Y \overset{\delta_1}{\dashrightarrow}$$

and

$$B_0 \to B_1 \to B_2 \to \ldots \to B_{n-1} \to Z_2 \xrightarrow{f_2} Y \overset{\delta_2}{\dashrightarrow},$$

where $A_0, B_0 \in \operatorname{add} X$. If $[f_1\rangle \leqslant [f_2\rangle$, then there exists a morphism $g\colon Z_1 \to Z_2$ such that $f_1 = f_2 g$. Thus, we obtain the commutative diagram

$$\begin{array}{ccccccccccccc}
A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow \cdots \longrightarrow & A_{n-1} & \longrightarrow & Z_1 & \xrightarrow{f_1} & Y & \overset{\delta_1}{\dashrightarrow} \\
\downarrow{\scriptstyle g_0} & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle g} & & \| & \\
B_0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow \cdots \longrightarrow & B_{n-1} & \longrightarrow & Z_2 & \xrightarrow{f_1} & Y & \overset{\delta_2}{\dashrightarrow}
\end{array}$$

by the dual of [11], Proposition 3.6. Then $\delta_2 = (g_0)_*\delta_1$, hence we have $\operatorname{Im} \delta^\sharp_{2X} \subseteq \operatorname{Im} \delta^\sharp_{1X}$. □

Assume $\mathscr{C}$ has a generalized Auslander-Reiten-Serre duality.

Lemma 4.9. Let $X \in \mathscr{C}_{\mathbb{F},l}$. There is a bijection

$$\Upsilon_{X,Y}\colon \operatorname{sub}_{\operatorname{End}_{\mathscr{C}}(X)}\mathbb{F}(Y,X) \to \operatorname{sub}_{\operatorname{End}_{\mathscr{C}}(X)^{\mathrm{op}}}\underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X,Y)$$

such that for any finitely generated $\operatorname{End}_{\mathscr{C}}(X)$-submodule F of $\mathbb{F}(Y,X)$, $\Upsilon_{X,Y}(F)=H$ is defined by an exact sequence

$$0 \longrightarrow H \longrightarrow \underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X,Y) \xrightarrow{D(i)\psi_{Y,X}} DF \longrightarrow 0,$$

where $i\colon F \to \mathbb{F}(Y,X)$ is the inclusion. The bijection $\Upsilon_{X,Y}$ is an anti-isomorphism of posets.

Proof. Since the proof is very similar to [22], Lemma 5.1, we omit it. Moreover, one also can see [4], Lemma 4.2. □

For any $X \in \mathscr{C}_{\mathbb{F},l}$, since $\tau_{\mathbb{F}}^{-}$ is an equivalence, we can identify via $\tau_{\mathbb{F}}^{-}$ the $\operatorname{End}_{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X)^{\mathrm{op}}$-module structure on $\underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X,Y)$ with the corresponding $\operatorname{End}_{\mathscr{C}}(X)^{\mathrm{op}}$-module structure. Hence, we can identify the poset $\operatorname{Sub}_{\operatorname{End}_{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X)^{\mathrm{op}}}\underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X,Y)$ with $\operatorname{Sub}_{\operatorname{End}_{\mathscr{C}}(X)^{\mathrm{op}}}\underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X,Y)$. Under the identification, we have the bijection

$$\Upsilon_{X,Y}\colon \operatorname{sub}_{\operatorname{End}_{\mathscr{C}}(X)}\mathbb{F}(Y,X) \to \operatorname{sub}_{\operatorname{End}_{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X)^{\mathrm{op}}}\underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X,Y).$$

Proposition 4.10. Let $X \in \mathscr{C}_{\mathbb{F},l}$. Then we have the commutative triangle

$$\begin{array}{ccc} & \operatorname{sub}_{\operatorname{End}_{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X)^{\mathrm{op}}}\underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X,Y) & \\ {\scriptstyle \eta_{\tau_{\mathbb{F}}^{-}X,Y}}\nearrow & & \nwarrow{\scriptstyle \Upsilon_{X,Y}} \\ [\to Y\rangle_{\mathfrak{s}|_{\mathbb{F}}\text{-def}} & \xrightarrow{\qquad\xi_{X,Y}\qquad} & \operatorname{sub}_{\operatorname{End}_{\mathscr{C}}(X)}\mathbb{F}(Y,X). \end{array}$$

Proof. For any $[f\rangle \in [\to Y\rangle_{\mathfrak{s}|_{\mathbb{F}}\text{-def}}$, there is an $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangle

$$X_0 \to X_1 \to X_2 \to \ldots \to X_{n-1} \to X_n \xrightarrow{f} Y \overset{\delta}{\dashrightarrow}.$$

We obtain an exact sequence

$$\underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X,X_n) \xrightarrow{\underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X,f)} \underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X,Y) \xrightarrow{(\delta_{\sharp})_{\tau_{\mathbb{F}}^{-}X}} \mathbb{E}(\tau_{\mathbb{F}}^{-}X,X_0).$$

By definition, the following two equations hold:

$$\eta_{\tau_{\mathbb{F}}^{-}X,Y}([f\rangle) = \operatorname{Im}\underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X, f) = \operatorname{Ker}(\delta_{\sharp})_{\tau_{\mathbb{F}}^{-}X} \quad \text{and} \quad \xi_{X,Y}([f\rangle) = \operatorname{Im}\delta^{\sharp}{}_{X}.$$

By Lemma 3.16, we have the exact sequence

$$0 \longrightarrow \operatorname{Ker}(\delta_{\sharp})_{\tau_{\mathbb{F}}^{-}X} \longrightarrow \underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X, Y) \xrightarrow{D(i)\psi_{Y,X}} D\operatorname{Im}\delta^{\sharp}_{X} \longrightarrow 0,$$

where $i\colon \operatorname{Im}\delta^{\sharp}_{X} \to \mathbb{F}(Y,X)$ is the inclusion. Hence $\Upsilon_{X,Y}(\operatorname{Im}\delta^{\sharp}{}_{X}) = \operatorname{Ker}(\delta_{\sharp})_{\tau_{\mathbb{F}}^{-}X}$ by Lemma 4.9. Thus, we have $\eta_{\tau_{\mathbb{F}}^{-}X,Y} = \Upsilon_{X,Y}\xi_{X,Y}$. $\square$

Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category. Let $\mathbb{F}$ be an additive subfunctor of $\mathbb{F}$ and

$$X_{\bullet}\colon\ X_0 \xrightarrow{\lambda_0} X_1 \to \lambda_1 \longrightarrow X_2 \to \lambda_2 \longrightarrow \ldots \to \lambda_{n-1} \longrightarrow X_n \to \lambda_n \longrightarrow X_{n+1} \dashrightarrow$$

an arbitrary $\mathfrak{s}|_{\mathbb{F}}$-conflation. Recall from [5] that $\mathbb{F}$ is closed if the two sequences

$$\mathbb{F}(-, X_0) \xrightarrow{(\lambda_0)_*} \mathbb{F}(-, X_1) \xrightarrow{(\lambda_1)_*} \mathbb{F}(-, X_2)$$

and

$$\mathbb{F}(X_{n+1}, -) \xrightarrow{(\lambda_n)^*} \mathbb{F}(X_n, -) \xrightarrow{(\lambda_{n-1})^*} \mathbb{F}(X_{n-1}, -)$$

are exact.

Moreover, we have the following equivalent statements.

Lemma 4.11 ([11], Proposition 3.16)**.** *For any additive subfunctor $\mathbb{F} \subseteq \mathbb{E}$, the following statements are equivalent.*

(1) *$(\mathscr{C}, \mathbb{F}, \mathfrak{s}|_{\mathbb{F}})$ is an n-exangulated category.*
(2) *$\mathfrak{s}|_{\mathbb{F}}$-inflations are closed under composition.*
(3) *$\mathfrak{s}|_{\mathbb{F}}$-deflations are closed under composition.*
(4) *$\mathbb{F} \subseteq \mathbb{E}$ is closed.*

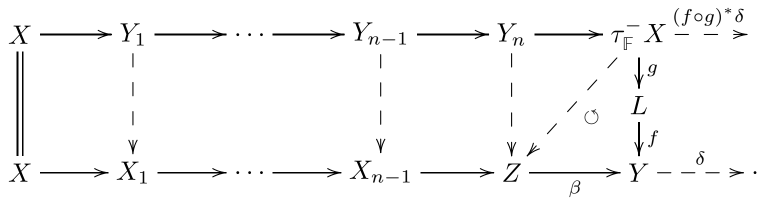
Proposition 4.12. *Let*

$$X \xrightarrow{\alpha} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_{n-2}} X_{n-1} \xrightarrow{\alpha_{n-1}} Z \xrightarrow{\beta} Y \overset{\delta}{\dashrightarrow}$$

be an $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangle with $X \in \mathscr{C}_{\mathbb{F},l}$. Then

(1) *β is right $\tau_{\mathbb{F}}^{-}X$-determined.*
(2) *Let $\mathbb{F}$ be an additive closed subfunctor of $\mathbb{E}$. If α is in $\operatorname{rad}_{\mathscr{C}}$, then β is right C-determined for some $C \in \mathscr{C}$ if and only if $\tau_{\mathbb{F}}^{-}X \in \operatorname{add} C$. Consequently, we have ${}_{X}[\to Y\rangle_{\mathfrak{s}|_{\mathbb{F}}\text{-def}} = {}^{\tau_{\mathbb{F}}^{-}X}[\to Y\rangle_{\mathfrak{s}|_{\mathbb{F}}\text{-def}}$.*

P r o o f. (1) Let $f \in \mathscr{C}(L, Y)$ be such that for each $g \in \mathscr{C}(\tau_{\mathbb{F}}^{-} X, L)$, the morphism $f \circ g$ factors through β. We need to show that the morphism f factors through β. Indeed, by (EA2), we have the commutative diagram of an $\mathfrak{s}|_{\mathbb{F}}$-distinguished n-exangle



Then we obtain $(f \circ g)^{*}\delta = 0$ by Lemma 2.13. Since $X \in \mathscr{C}_{\mathbb{F},l}$, there exists a natural isomorphism

$$\psi_{-,X} \colon \underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-} X, -) \to D\mathbb{F}(-, X).$$

Take $\varepsilon := \psi_{\tau_{\mathbb{F}}^{-} X, X}(\underline{\mathrm{Id}_{\tau_{\mathbb{F}}^{-} X}})$. By the naturality of $\psi_{-,X}$, we have the commutative diagram

$$\begin{array}{ccc}
\underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-} X, \tau_{\mathbb{F}}^{-} X) & \xrightarrow{\psi_{\tau_{\mathbb{F}}^{-} X, X}} & D\mathbb{F}(\tau_{\mathbb{F}}^{-} X, X) \\
{\scriptstyle \underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-} X, \underline{g})}\downarrow & & \downarrow{\scriptstyle D\mathbb{F}(g, X)} \\
\underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-} X, L) & \xrightarrow{\psi_{L,X}} & D\mathbb{F}(L, X).
\end{array}$$

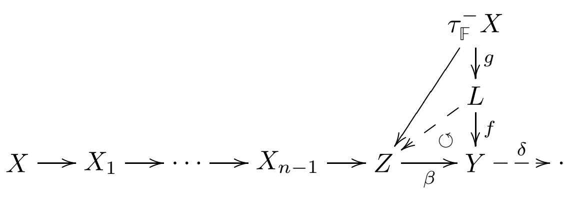
So

$$\psi_{L,X}(\underline{g}) = D\mathbb{F}(g, X)(\varepsilon) = \varepsilon \circ \mathbb{F}(g, X)$$

and hence,

$$\psi_{L,X_0}(\underline{g})(f^{*}\delta) = \varepsilon(g^{*} f^{*}\delta) = \varepsilon((f \circ g)^{*}\delta) = 0.$$

Note that $\psi_{L,X}(\underline{g})$ runs over all maps in $D\mathbb{F}(L, X)$, when $\underline{g}$ runs over all morphisms in $\underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-} X, L)$. It follows that $f^{*}\delta = 0$, thus, the morphism f factors through β by Lemma 2.13, that is, we have the commutative diagram



Therefore, α is right $\tau_{\mathbb{F}}^{-} X$-determined.

(2) The sufficiency follows from (1). It suffices to prove the necessity. We will show that each indecomposable direct summand X' of X satisfies $\tau_{\mathbb{F}}^{-} X' \in \operatorname{add} C$. Firstly, we claim that the composition of $\mathfrak{s}|_{\mathbb{F}}$-inflations $X' \xrightarrow{\iota} X \xrightarrow{\alpha} X_1$ is not

a split monomorphism, where ι is the natural inclusion. Otherwise, since $\mathbb{F}$ is closed, $\alpha\iota$ is an $\mathfrak{s}|_{\mathbb{F}}$-inflation by Lemma 4.11. If $\alpha\iota$ is a split monomorphism, then there exists a morphism $t\colon X_1 \to X'$, such that $t\alpha\iota = 1$. We have $t\alpha\iota \in \operatorname{rad}_{\mathscr{C}}$ since α is in $\operatorname{rad}_{\mathscr{C}}$. This shows $1 - t\alpha\iota$ is invertible, which is a contradiction since $1 - t\alpha\iota = 0$. Moreover, X' is not an injective object by the dual of [18], Lemma 3.4. Hence, by Lemma 3.11 there is an Auslander-Reiten $\mathfrak{s}|_{\mathbb{F}}$-$n$-exangle

$$X' \xrightarrow{\alpha'} W_1 \xrightarrow{\alpha'_1} W_2 \to \dots \xrightarrow{\alpha'_{n-1}} W_n \xrightarrow{\beta'} \tau_{\mathbb{F}}^{-} X' \overset{\sigma}{\dashrightarrow} .$$

We have the commutative diagram by [11], Proposition 3.6

$$\begin{array}{ccccccccccccc}
X' & \xrightarrow{\alpha'} & W_1 & \xrightarrow{\alpha'_1} & \cdots & \xrightarrow{\alpha'_{n-2}} & W_{n-1} & \xrightarrow{\alpha'_{n-1}} & W_n & \xrightarrow{\beta'} & \tau_{\mathbb{F}}^{-} X' & \overset{\sigma}{\dashrightarrow} & \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle i_1} & & & & \vdots{\scriptstyle i_{n-1}} & & \vdots{\scriptstyle i_n} & & \vdots{\scriptstyle i_{n+1}} & & \\
X & \xrightarrow{\alpha} & X_1 & \xrightarrow{\alpha_1} & \cdots & \xrightarrow{\alpha_{n-2}} & X_{n-1} & \xrightarrow{\alpha_{n-1}} & Z & \xrightarrow{\beta} & Y & \overset{\delta}{\dashrightarrow} &
\end{array}$$

with $\iota_*\sigma = i_{n+1}^*\delta$.

Suppose $\tau_{\mathbb{F}}^{-} X' \notin \operatorname{add} C$. Then any $f \in \mathscr{C}(C, \tau_{\mathbb{F}}^{-} X')$ is not a split epimorphism and hence factors through β', that is, $\beta' g = f$. Thus, we have

$$i_{n+1} f = i_{n+1}(\beta' g) = \beta(i_n g).$$

Moreover, since β is right C-determined, there exists $h \in \mathscr{C}(\tau_{\mathbb{F}}^{-} X', Z)$ such that $i_{n+1} = \beta h$. Consider the commutative diagram by (EA2)

$$\begin{array}{ccccccccccccc}
X & \xrightarrow{\gamma_0} & W'_1 & \xrightarrow{\gamma_1} & \cdots & \xrightarrow{\gamma_{n-2}} & W'_{n-1} & \xrightarrow{\gamma_{n-1}} & W'_n & \xrightarrow{\gamma_n} & \tau_{\mathbb{F}}^{-} X' & \overset{i_{n+1}^*\delta}{\dashrightarrow} & \\
\| & & \vdots{\scriptstyle i_1} & & & & \vdots{\scriptstyle i_{n-1}} & & \vdots{\scriptstyle i_n} & \overset{h}{\swarrow} \circlearrowleft & \downarrow{\scriptstyle i_{n+1}} & & \\
X & \xrightarrow{\alpha} & X_1 & \xrightarrow{\alpha_1} & \cdots & \xrightarrow{\alpha_{n-2}} & X_{n-1} & \xrightarrow{\alpha_{n-1}} & Z & \xrightarrow{\beta} & Y & \overset{\delta}{\dashrightarrow} & .
\end{array}$$

By Lemma 2.13, we have that id_X factors through γ_0 and hence, γ_0 is a split monomorphism. In particular, $\iota_*\sigma = i_{n+1}^*\delta = 0$. Consider the commutative diagram by ($\mathrm{EA2^{op}}$)

$$\begin{array}{ccccccccccccc}
X' & \xrightarrow{\alpha'} & W_1 & \xrightarrow{\alpha'_1} & \cdots & \xrightarrow{\alpha'_{n-2}} & W_{n-1} & \xrightarrow{\alpha'_{n-1}} & W_n & \xrightarrow{\beta'} & \tau_{\mathbb{F}}^{-} X' & \overset{\sigma}{\dashrightarrow} & \\
\downarrow{\scriptstyle \iota} & & \vdots & & & & \vdots{\scriptstyle i_{n-1}} & & \vdots{\scriptstyle i_n} & & \| & & \\
X & \xrightarrow{\alpha} & W''_1 & \xrightarrow{\alpha_1} & \cdots & \xrightarrow{\alpha_{n-2}} & W''_{n-1} & \xrightarrow{\alpha_{n-1}} & W''_n & \xrightarrow{\beta} & \tau_{\mathbb{F}}^{-} X' & \overset{\iota_*\sigma}{\dashrightarrow} & .
\end{array}$$

By Lemma 2.13, the condition $\iota_*\sigma = 0$ implies that there exists a morphism $\omega \in \mathscr{C}(W_1, X)$ satisfying $\iota = \omega\alpha'$. Since ι is a split monomorphism, α' is also a split monomorphism, which is a contradiction. Thus we have $\tau_{\mathbb{F}}^{-} X' \in \operatorname{add} C$. $\square$

We are ready to state and prove our main result.

Theorem 4.13. *Let $\mathbb{F}$ be an additive closed subfunctor of $\mathbb{E}$ and let $X \in \mathscr{C}_{\mathbb{F},l}$. The bijection triangle*

$$\begin{array}{ccc} & \mathrm{sub}_{\mathrm{End}_{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X)^{\mathrm{op}}}\underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X, Y) & \\ {}^{\eta_{\tau_{\mathbb{F}}^{-}X,Y}}\nearrow & & \nwarrow^{\Upsilon_{X,Y}} \\ {}^{\tau_{\mathbb{F}}^{-}X}[\to Y\rangle_{\mathfrak{s}|_{\mathbb{F}}\text{-def}} & \xrightarrow{\quad \xi_{X,Y} \quad} & \mathrm{sub}_{\mathrm{End}_{\mathscr{C}}(X)}\mathbb{F}(Y, X) \end{array}$$

is commutative. In particular, we get the restricted Auslander bijection at Y relative to $\tau_{\mathbb{F}}^{-}X$,

$$\eta_{\tau_{\mathbb{F}}^{-}X,Y} \colon {}^{\tau_{\mathbb{F}}^{-}X}[\to Y\rangle_{\mathfrak{s}|_{\mathbb{F}}\text{-def}} \to \mathrm{sub}_{\mathrm{End}_{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X)^{\mathrm{op}}}\underline{\mathscr{C}}(\tau_{\mathbb{F}}^{-}X, Y),$$

which is an isomorphism of posets.

Proof. It follows from Propositions 4.10 and 4.12. □

Remark 4.14. Theorem 4.13, when $\mathscr{C}$ is an extriangulated category, is just Theorem 4.11 in [21].

Let $(\mathscr{C}, \Sigma, \Theta)$ be an $(n+2)$-angulated category. Put $\mathbb{E}_{\Sigma} = \mathscr{C}(-, \Sigma-) \colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathsf{Ab}$ and, for any $\delta \in \mathbb{E}(Y, X) = \mathscr{C}(Y, \Sigma X)$, take an $(n+2)$-angle

$$X \to X_1 \to X_2 \to \ldots \to X_n \to Y \xrightarrow{\delta} \Sigma X$$

and set

$$\mathfrak{s}(\delta) = [X \to X_1 \to X_2 \to \ldots \to X_n \to Y],$$

then $(\mathscr{C}, \mathbb{E}_{\Sigma}, \mathfrak{s})$ is an n-exangulated category, see [11], Proposition 4.5. In this case, each morphism in $\mathscr{C}$ is an $\mathfrak{s}$-deflation, hence ${}^{\tau^{-}X}[\to Y\rangle_{\mathrm{def}} = {}^{\tau^{-}X}[\to Y\rangle$. Note that $\mathcal{P}_{\mathbb{E}_{\Sigma}} = 0$ in $\mathscr{C}$, thus $\underline{\mathscr{C}}(X, Y) = \mathscr{C}(X, Y)$ for any $X, Y \in \mathscr{C}$.

In particular,

$$\mathscr{C}_{\mathbb{E}_{\Sigma},l} = \{X \in \mathscr{C} \colon \text{ the functor } D\mathscr{C}(-, \Sigma X) \colon \mathscr{C} \to \mathrm{mod}\ k \text{ is representable}\}.$$

Corollary 4.15. *Let $\mathscr{C}$ is a k-linear Hom-finite Krull-Schmidt $(n+2)$-angulated category and let $X \in \mathscr{C}_{\mathscr{C}(-,\Sigma-),l}$. The bijection triangle*

$$\begin{array}{ccc} & \mathrm{sub}_{\mathrm{End}_{\mathscr{C}}(\tau^{-}X)^{\mathrm{op}}}\mathscr{C}(\tau^{-}X, Y) & \\ {}^{\eta_{\tau^{-}X,Y}}\nearrow & & \nwarrow^{\Upsilon_{X,Y}} \\ {}_{X}[\to Y\rangle = {}^{\tau^{-}X}[\to Y\rangle & \xrightarrow{\quad \xi_{X,Y} \quad} & \mathrm{sub}_{\mathrm{End}_{\mathscr{C}}(X)}\mathscr{C}(Y, \Sigma X) \end{array}$$

is commutative. In particular, we get the restricted Auslander bijection at Y relative to $\tau^{-}X$,

$$\eta_{\tau^{-}X,Y} \colon {}^{\tau^{-}X}[\to Y\rangle \to \mathrm{sub}_{\mathrm{End}_{\mathscr{C}}(\tau^{-}X)^{\mathrm{op}}}\mathscr{C}(\tau^{-}X, Y),$$

which is an isomorphism of posets.

Remark 4.16. Corollary 4.15, when $n = 1$, is just Corollary 4.12 in [21].

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Authors' addresses: Jian He, Department of Applied Mathematics, Lanzhou University of Technology, 730050 Lanzhou, Gansu, P. R. China, e-mail: `jianhe30@163.com`; Jing He (corresponding author), College of Science, Hunan University of Technology and Business, 410205 Changsha, Hunan, P. R. China; e-mail: `jinghe1003@163.com`; Panyue Zhou, School of Mathematics and Statistics, Changsha University of Science and Technology, 410114 Changsha, Hunan, P. R. China, e-mail: `panyuezhou@163.com`.