Czechoslovak Mathematical Journal

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*Czechoslovak Mathematical Journal*, Vol. 73 (2023), No. 2, 553–564

Persistent URL: `http://dml.cz/dmlcz/151673`

Czechoslovak Mathematical Journal, 73 (148) (2023), 553–564

# COMMUTATIVE GRADED-$S$-COHERENT RINGS

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Received March 29, 2022. Published online February 2, 2023.

*Abstract.* Recently, motivated by Anderson, Dumitrescu's $S$-finiteness, D. Bennis, M. El Hajoui (2018) introduced the notion of $S$-coherent rings, which is the $S$-version of coherent rings. Let $R = \bigoplus_{\alpha\in G} R_\alpha$ be a commutative ring with unity graded by an arbitrary commutative monoid $G$, and $S$ a multiplicatively closed subset of nonzero homogeneous elements of $R$. We define $R$ to be graded-$S$-coherent ring if every finitely generated homogeneous ideal of $R$ is $S$-finitely presented. The purpose of this paper is to give the graded version of several results proved in D. Bennis, M. El Hajoui (2018). We show the nontriviality of our generalization by giving an example of a graded-$S$-coherent ring which is not $S$-coherent and as a special case of our study, we give the graded version of the Chase's characterization of $S$-coherent rings.

*Keywords*: $S$-finite; graded-$S$-coherent module; graded-$S$-coherent ring

*MSC 2020*: 13A02, 13A15, 16W50, 13D03

## 1. Introduction

This section is devoted to some conventions and a recall of some standard background terminology. Throughout this paper all rings are commutative with unity, and all modules are unital. The symbol $G$ will denote a commutative monoid (that is, a commutative monoid, written additively, with an identity element denoted by 0), and all the graded rings and modules are graded by $G$. The symbol $S$ will be a multiplicatively closed subset of nonzero homogeneous elements of $R$.

If $R$ is a ring and $M$ is an $R$-module, $M$ is called *$S$-finite* if there exists a finitely generated submodule $N$ of $M$ such that $sM \subseteq N$ for some $s \in S$; this notion was introduced by Anderson and Dumitrescu, see [3]. According to [7], $E$ is called an *$S$-finitely presented $R$-module* if there exists an exact sequence $0 \to F_1 \to F_0 \to E \to 0$ of $R$-modules such that $F_0$ is a finitely generated free $R$-module and $F_1$ is

DOI: 10.21136/CMJ.2023.0130-22

$S$-finite. Any finitely presented $R$-module is an $S$-finitely presented $R$-module, while the converse is false in general; for more results and details, the reader can refer to [7], Section 2. A finitely generated $R$-module $M$ is said to be an $S$-coherent $R$-module if every finitely generated $R$-submodule of $M$ is an $S$-finitely presented $R$-module; and a ring $R$ is said to be an $S$-coherent ring if $R$ is $S$-coherent as an $R$-module.

The concept of coherent rings is one of the most significant notions in homological algebra. Because of its importance, there have been many generalizations to the notion of coherent rings. Some of them are to the context of graded rings, see [4]–[6].

According to [5], a graded ring $R$ is said to be a graded-coherent ring if every finitely generated homogeneous ideal of $R$ is a finitely presented ideal of $R$. Clearly, every coherent graded ring is a graded-coherent ring, but the converse is false, see [5], Example 3.2. In this paper we are interested in the graded version of $S$-coherent modules and rings, which are called, respectively, graded-$S$-coherent modules and graded-$S$-coherent rings, see Definition 3.1. In Section 3, we introduce and study the notion of graded-$S$-coherent rings and the more general notion of graded-$S$-coherent modules (over an arbitrary graded ring). Our main aims in this section are to characterize graded-$S$-coherent modules and graded-$S$-coherent rings and to establish some of their basic properties. For rings, we have $S$-coherent $\Rightarrow$ graded-$S$-coherent and graded-coherent $\Rightarrow$ graded-$S$-coherent with neither implication being reversible.

## 2. Preliminaries

This section will be devoted to a standard recall of different basic notions and properties that are related to graded ring theory. For more details, we refer the reader to [8], Chapter II, Section 11, pages 163–176.

Let $G$ be a grading commutative monoid written additively with an identity element denoted by 0. By a graded ring $R$, we mean a ring graded by $G$, that is, a direct sum of subgroups $R_\alpha$ of $R$ such that $R_\alpha R_\beta \subseteq R_{\alpha+\beta}$ for every $\alpha$, $\beta \in G$. The set $h(R) = \bigcup\limits_{\alpha\in G} R_\alpha$ is the set of homogeneous elements of $R$. A nonzero element $x \in R$ is called *homogeneous* if it belongs to one of the $R_\alpha$'s, homogeneous of degree $\alpha$ if $x \in R_\alpha$. Every $z \in R$ may be written uniquely as a sum $z = z_{\alpha_i} + \ldots + z_{\alpha_n}$ of homogeneous elements $z_{\alpha_i} \in R_{\alpha_i}$, where $\alpha_1, \ldots, \alpha_n$ are distincts; $z_{\alpha_i}$ is called the *homogeneous component of degree* $\alpha_i$ of $z$. If $G$ is cancellative, then $R_0$ is a subring of $R$ (clearly $1 \in R_0$) and every $R_\alpha$ is an $R_0$-module.

By a graded $R$-module $M$, we mean an $R$-module graded by $G$, that is, a direct sum of subgroups $M_\alpha$ of $M$ such that $R_\alpha M_\beta \subseteq M_{\alpha+\beta}$ for every $\alpha, \beta \in G$. The set $h(M) = \bigcup\limits_{\alpha\in G} M_\alpha$ is the set of homogeneous elements of $M$.

A graded $R$-module $M$ is called a *graded-free $R$-module* (gr-free) if there exists a basis $(m_i)_{i\in I}$ of $M$ consisting of homogeneous elements. Note that any graded-free $R$-module is a free $R$-module; the converse is false, see [15], page 21. When $G$ is cancellative, then $M_\alpha$ are $R_0$-modules. Obviously, $R$ is a graded $R$-module.

Let $R$ and $R'$ be two graded rings, a ring homomorphism $f\colon R \to R'$ is called *graded* if $f(R_\alpha) \subseteq R'_\alpha$ for all $\alpha \in G$. A graded ring isomorphism is a bijective graded ring homomorphism. Let $M$ and $M'$ be two graded $R$-modules and let $v\colon M \to M'$ be an $R$-module homomorphism and $\beta \in G$; $v$ is called *graded of degree $\beta$* if $v(M_\alpha) \subseteq M_{\alpha+\beta}$ for all $\alpha \in G$. An $R$-module homomorphism $v\colon M \to M'$ is called graded if there exists $\beta \in G$ such that $v$ is graded of degree $\beta$. A graded $R$-module isomorphism is a bijective graded $R$-module homomorphism of degree 0. If $v \neq 0$ and $G$ is cancellative, the degree of $v$ is then determined uniquely. An exact sequence of graded $R$-modules is an exact sequence, where the $R$-modules and the $R$-module homomorphisms in question are graded.

A submodule $N$ of $M$ is called *homogeneous* if $N = \bigoplus_{\alpha\in G} (N \cap M_\alpha)$. It is well known that the following are equivalent for a submodule $N$ of $M$:

(1) $N$ is homogeneous;

(2) the homogeneous components of every element of $N$ belong to $N$;

(3) $N$ is generated by homogeneous elements.

A homogeneous submodule of $R$ is called a *homogeneous ideal* of $R$. If $N$ is a homogeneous submodule of a graded $R$-module $M$, then $M/N$ is a graded $R$-module, where $(M/N)_\alpha := (M_\alpha + N)/N$. If $I$ is a homogeneous ideal of a graded ring $R$, then $R/I$ is a graded ring, where $(R/I)_\alpha := (R_\alpha + I)/I$.

Let $R_1$ and $R_2$ be two graded rings. Then $R = R_1 \times R_2$ is a graded ring with homogeneous elements $h(R) = \bigcup_{\alpha\in G} R_\alpha$, where $R_\alpha = (R_1)_\alpha \times (R_2)_\alpha$ for all $\alpha \in G$. It is well known that an ideal of $R_1 \times R_2$ is of the form $I_1 \times I_2$ for some ideals $I_1$ of $R_1$ and $I_2$ of $R_2$. Also, it is easily seen that $I_1 \times I_2$ is a homogeneous ideal of $R_1 \times R_2$ if and only if $I_1$, $I_2$ are homogeneous ideals of $R_1$ and $R_2$, respectively.

Let $R$ be a graded ring and let $M$ be a graded $R$-module. If the grading monoid $G$ is a group and if $S$ is a multiplicatively closed set of homogeneous elements of $R$, then $S^{-1}R$ is a graded ring and $S^{-1}M$ is a graded $S^{-1}R$-module, where $(S^{-1}R)_i = \{r/s\colon r \in R_j,\ s \in R_k \cap S \text{ and } j-k=i\}$ and $(S^{-1}M)_i = \{m/s\colon m \in M_j,\ s \in R_k \cap S$ and $j-k=i\}$.

Assume that the grading monoid is a cancellative torsion-free monoid. Let $R$ be a graded ring. Then $R$ is called a graded-Noetherian ring (gr-Noetherian ring) if it satisfies the ascending chain condition (a.c.c.) on homogeneous ideals; equivalently, if each homogeneous prime ideal of $R$ is finitely generated, see [16], Lemma 2.3. Obviously, a Noetherian graded ring is a gr-Noetherian ring, while gr-Noetherian rings need not be Noetherian. It is known that the monoid ring $A[X;G]$ over a ring $A$ is a Noetherian ring (gr-Noetherian ring) if and only if $A$ is a Noetherian ring and $G$ (each ideal of $G$) is finitely generated, see [11], Theorem 7.7, page 75, [16], Theorem 2.4. Hence, if $\mathbb{Q}$ is the additive group of rational numbers and $D$ is a Noetherian ring, the group ring, $A = D[X;\mathbb{Q}]$, is a gr-Noetherian ring but not a Noetherian ring.

## 3. Graded-$S$-coherent rings

This section initiates the study of graded-$S$-coherent modules and rings. We begin, following the classical case, by giving the definition of graded-$S$-coherent modules.

**Definition 3.1.** A graded $R$-module $M$ is said to be graded-$S$-coherent if it is finitely generated and every finitely generated homogeneous submodule of $M$ is $S$-finitely presented. And a graded ring $R$ is said to be graded-$S$-coherent, if it is graded-$S$-coherent as a graded $R$-module; that is, if every finitely generated homogeneous ideal of $R$ is $S$-finitely presented.

We next collect some immediate classes of graded-$S$-coherent modules and rings.

**Remark 3.2.** Let $R$ be a graded ring. Then the following statements hold:

(1) Every finitely generated homogeneous $R$-submodule of a graded-$S$-coherent $R$-module is graded-$S$-coherent.

(2) Recall from [5] that an $R$-module $M$ is said to be graded-coherent if it is finitely generated and every finitely generated homogeneous submodule of $M$ is finitely presented. Clearly, any graded-coherent $R$-module is a graded-$S$-coherent $R$-module since every finitely presented module is $S$-finitely presented. Hence, all graded-Noetherian ring (see [9]), graded-valuation domain (see [1]), and graded Prüfer domain (see [2]) are graded-$S$-coherent rings.

(3) Obviously, every $S$-coherent graded $R$-module is a graded-$S$-coherent $R$-module and every $S$-coherent graded ring is a graded-$S$-coherent ring. But the converse is not true in general, as shown by the following example.

**Example 3.3.** If $A$ is a countable direct product of $\mathbb{Q}[[t,u]]$'s, consider the polynomial ring graded by $\mathbb{N}$ via $(A[X])_n = AX^n$ for every $n \in \mathbb{N}$, and let $S = \{1\}$. Then $A[X]$ is graded-$S$-coherent but not $S$-coherent.

*Proof.* By [5], Example 3.2, the polynomial ring $A[X]$ is graded-coherent and then it is graded-$S$-coherent, but it is not $S$-coherent. In fact, by [17], Proposition 18, $A[X]$ is not coherent since there exists an ideal $I$ which is not finitely generated and it is the intersection of two finitely generated ideals. Now, since $S = \{1\}$, $I$ is not even $S$-finite, and then $A[X]$ is not $S$-coherent by [7], Theorem 3.8 (3). $\square$

The following result studies the behavior of graded-$S$-coherence of graded modules in short exact sequences. It is the graded version of [7], Proposition 3.2.

**Proposition 3.4.** *Let* $0 \to P \xrightarrow{\alpha} N \xrightarrow{\beta} M \to 0$ *be an exact sequence of graded* $R$*-modules. Then the following statements hold:*

(1) *If* $P$ *is finitely generated,* $N$ *is graded-*$S$*-coherent and* $\beta$ *has a cancellable degree, then* $M$ *is graded-*$S$*-coherent.*

(2) *If* $M$ *is graded-coherent,* $P$ *is graded-*$S$*-coherent and* $\alpha$ *has a cancellable degree, then* $N$ *is graded-*$S$*-coherent.*

(3) *If* $N$ *is graded-*$S$*-coherent and* $P$ *is finitely generated, then* $P$ *is graded-*$S$*-coherent.*

Proof. (1) Since $N$ is finitely generated and $\beta\colon N \to M$ is surjective, we have that $M$ is finitely generated. Let $M_1$ be a finitely generated homogeneous submodule of $M$. Since $\beta$ has a cancellable degree, the submodule $\beta^{-1}(M_1)$ of $N$ is homogeneous, and we have the following exact sequence of graded $R$-modules:

$$0 \to P \to \beta^{-1}(M_1) \to M_1 \to 0.$$

Now, $\beta^{-1}(M_1)$ is finitely generated since $M_1$ and $P$ are so. Therefore, since $N$ is graded-$S$-coherent, $\beta^{-1}(M_1)$ is $S$-finitely presented and so using [7], Theorem 2.4 (4), $M_1$ is $S$-finitely presented, as desired.

(2) Since $M$ and $P$ are finitely generated modules, we have that $N$ is finitely generated. Now, let $N_1$ be a finitely generated homogeneous submodule of $N$. Since $\alpha$ has a cancellable degree, consider the exact sequence of graded $R$-modules $0 \to \operatorname{Ker}(\beta_{|N_1}) \xrightarrow{\alpha} N_1 \xrightarrow{\beta} \beta(N_1) \to 0$. Then, since $\beta(N_1)$ is a finitely generated homogeneous submodule of the graded-coherent module $M$, $\beta(N_1)$ is finitely presented. Hence, $\operatorname{Ker}(\beta_{|N_1})$ is finitely generated, and since $P$ is graded-$S$-coherent, $\operatorname{Ker}(\beta_{|N_1})$ is $S$-finitely presented since it is homogeneous. Therefore, by [7], Theorem 2.4 (2), $N_1$ is $S$-finitely presented.

(3) This is an immediate consequence of Remark 3.2 (1). □

**Remark 3.5.** Let $G$ be an abelian group, $R$ be a graded ring and $S$ a multiplicatively closed subset of $R_0$. Then, recall from [14] that $R$ is said to be graded $S$-Noetherian if every homogeneous ideal of $R$ is $S$-finite. Note that every graded $S$-Noetherian ring is graded-$S$-coherent. Indeed, this follows by applying Proposition 2.3 of [7] and from the fact that when $R$ is graded $S$-Noetherian, every finitely generated graded $R$-module is graded $S$-Noetherian.

We next give our definition which generalizes the definition in [14].

**Definition 3.6.** Let $R$ be a graded ring and $S$ a multiplicatively closed subset of homogeneous elements of $R$ (not necessarily of $R_0$). Then $R$ is said to be graded-$S$-Noetherian if every homogeneous ideal of $R$ is $S$-finite.

The next result presents the graded version of Proposition 2 (f) in [3]. For an ideal $I$ of $R$, $IR_S \cap R$ means the $S$-saturation of $I$.

**Proposition 3.7.** *Let $G$ be a group, $R$ be a graded ring and $S \subseteq R$ a multiplicatively closed subset of homogeneous elements of $R$. Then $R$ is graded-$S$-Noetherian if and only if $R_S$ is graded-Noetherian and for every finitely generated homogeneous ideal $I$ of $R$, $IR_S \cap R = (I : s)$ for some $s \in S$.*

Proof. Suppose that $R$ is graded-$S$-Noetherian and choose a homogeneous ideal $J$ of $R_S$. Then there exists an ideal $I$ of $R$ such that $I \cap S = \emptyset$ and $J = IR_S$. Then note that $IR_S \cap R$ is a homogeneous ideal of $R$ which is graded-$S$-Noetherian, then there exists $s \in S$ such that $s(IR_S \cap R) \subseteq K \subseteq IR_S \cap R$ for a finitely generated ideal $K$ of $R$. This implies that

$$[s(IR_S \cap R)]R_S = IR_S \subseteq KR_S \subseteq [IR_S \cap R]R_S = IR_S.$$

Thus, $J = KR_S$ is finitely generated, that is, $R_S$ is a graded-Noetherian ring. Now, suppose that $I$ is a finitely generated homogeneous ideal of $R$. Then it is clear that $(I : s') \subseteq IR_S \cap R$ for every $s' \in S$. Put $J = IR_S \cap R$. Then $J$ is $S$-finite, there exist $x_1, x_2, \ldots, x_n \in R$ and $t \in S$ such that $tJ \subseteq Rx_1 + \ldots + Rx_n \subseteq J$. Since $x_i \in J = IR_S \cap R$, we get $t_i x_i \in I$ for some $t_i \in S$, where $i = 1, 2, \ldots, n$. Fix $s = tt_1t_2 \ldots t_n$. Then note that for every $x \in J$ we have $sx \in I$, which implies that $J \subseteq (I : s)$, as needed.

For the reverse implication, assume that $R_S$ is graded-Noetherian and let $J$ be a homogeneous ideal of $R$. Then $JR_S \cap R = LR_S \cap R$ for a finitely generated homogeneous ideal $L \subseteq J$. If $s \in S$ satisfies $LR_S \cap R = (L : s)$, then $sJ \subseteq L$, which completes our proof. □

The following proposition investigates a change of rings result.

**Proposition 3.8.** *Let $\varphi\colon R \to L$ be a graded ring homomorphism, $M$ be a graded $L$-module and $S$ a multiplicatively closed subset of homogeneous elements of $R$ such that $0 \notin \varphi(S)$. Then if the graded $R$-module $L$ is finitely generated and $M$ is graded-$S$-coherent as an $R$-module, then $M$ is graded-$\varphi(S)$-coherent as an $L$-module.*

Proof. Let $N$ be a finitely generated homogeneous $L$-submodule of $M$. Then so is $N$ as an $R$-submodule of $M$. Hence, $N$ is $S$-finitely presented over $R$ since $M$ is graded-$S$-coherent over $R$, then, by [7], Proposition 2.6, $N$ is $\varphi(S)$-finitely presented over $L$ since the $R$-module $L$ is finitely generated. □

**Proposition 3.9.** *Let $R$ be a graded ring, $I$ be an $S$-finite homogeneous ideal of $R$, where $S$ is a multiplicatively closed subset of homogeneous elements of $R$ and $M$ be a graded $R/I$-module. Assume that $I\cap S=\emptyset$ so that $V:=\{s+I\in R/I; s\in S\}$ is a multiplicatively closed subset of homogeneous elements of $R/I$. Then the following statements hold:*

(1) *$M$ is graded-$V$-coherent as an $R/I$-module if and only if it is graded-$S$-coherent as an $R$-module.*
(2) *If $R$ is a graded-$S$-coherent ring, then $R/I$ is a graded-$V$-coherent ring. The converse holds if $R/I$ is a graded-$V$-coherent ring and $I$ is graded-$S$-coherent as an $R$-module.*

Proof. (1) ($\Rightarrow$) Let $M$ be a graded-$V$-coherent $R/I$-module and let $N$ be a finitely generated submodule of $M$. Since $M$ is graded-$V$-coherent as an $R/I$-module, $N$ is $V$-finitely presented $R/I$-module. Now, since $I$ is $S$-finite, by [7], Proposition 2.7, $N$ is $S$-finitely presented $R$-module and so $M$ is graded-$S$-coherent $R$-module.

($\Leftarrow$) Similar to the proof of the direct implication.

(2) The direct implication follows from (1). The reverse implication is a direct application of Proposition 3.4 (2). □

The following theorem clarifies the situation for the product of graded $S$-coherent modules.

**Theorem 3.10.** *Let $M_i$ be a graded $R_i$-module and $S_i$ a multiplicatively closed set of homogeneous elements of $R_i$ for $i=1,2,\dots,n$. Suppose that $R=R_1\times R_2\times\dots\times R_n$, $M=M_1\times M_2\times\dots\times M_n$ and $S=S_1\times S_2\times\dots\times S_n$. The following statements are equivalent.*

(1) *$M_i$ is a graded-$S_i$-coherent $R_i$-module for each $i=1,2,\dots,n$.*
(2) *$M$ is a graded-$S$-coherent $R$-module.*

Proof. By mathematical induction on $n$, it suffices to prove the case $n=2$.

(1) $\Rightarrow$ (2): Suppose that $N$ is a finitely generated homogeneous submodule of $M$. Then we can write $N=N_1\times N_2$ for a finitely generated homogeneous submodule $N_i$ of $M_i$ for each $i=1,2$. Since $M_i$ is a graded-$S_i$-coherent $R_i$-module for each $i=1,2$, there exists an exact sequence of $R_i$-modules $0\to K_i\xrightarrow{\alpha_i}F_i\xrightarrow{\beta_i}N_i\to 0$, where $K_i$ is $S_i$-finite and $F_i$ is finitely generated free $R_i$-module. Put $K=K_1\times K_2$, $F=F_1\times F_2$. Consider the $R$-homomorphisms $\alpha_1\times\alpha_2\colon K\to F$ defined by $\alpha(k_1,k_2)=(\alpha_1(k_1),\alpha_2(k_2))$ and also $\beta_1\times\beta_2\colon F\to N$ defined by $\beta(f_1,f_2)=(\beta_1(f_1),\beta_2(f_2))$. Then $0\to K\xrightarrow{\alpha_1\times\alpha_2}F\xrightarrow{\beta_1\times\beta_2}N\to 0$ is an exact sequence of $R$-modules, where $K$ is $S$-finite and $F$ is finitely generated free. Thus, $N$ is $S$-finitely presented, namely, $M$ is a graded-$S$-coherent $R$-module.

(2) ⇒ (1): Suppose that $M$ is a graded-$S$-coherent $R$-module. Consider the graded ring epimorphism $\pi_1\colon R \to R_1$. Note that $R_1$ is a finitely generated $R$-module and $0 \notin \pi_1(S) = S_1$. Take a finitely generated homogeneous $N_1$ of $M$. Then $N = N_1 \times (0)$ ($\cong N_1$) is a finitely generated homogeneous submodule of $M$. Since $M$ is a graded-$S$-coherent $R$-module, $N$ ($\cong N_1$) is an $S$-finitely presented $R$-module. Then by [7], Proposition 2.6, $N_1$ ($\cong N$) is an $S_1$-finitely presented $R_1$-module. Therefore, $M_1$ is a graded-$S_1$-coherent $R_1$-module. Likewise, $M_2$ is a graded-$S_2$-coherent $R_2$-module.

□

As a consequence of the previous theorem, we have the following result.

**Corollary 3.11.** *Let $R = \prod_{i=1}^{n} R_i$ be a direct product of graded rings $R_i$ ($n \in \mathbb{N}$) and $S = \prod_{i=1}^{n} S_i$ be a cartesian product of multiplicatively closed sets $S_i$ of homogeneous elements of $R_i$. Then $R$ is graded-$S$-coherent if and only if $R_i$ is graded-$S_i$-coherent for every $i \in \{1, \ldots, n\}$.*

Before giving the following two remarks, we first recall from [13] the idealization construction. For a ring $R$ and an $R$-module $E$ (both not necessarily non-trivially graded), the trivial ring extension of $A$ by $E$ is the ring $A := R \propto E$, whose underlying group is $A \oplus E$ with multiplication defined by $(r, e)(r', e') = (rr', re' + r'e)$.

**Remark 3.12.**

(1) As a particular case of Remark 3.2 (2), any graded-coherent ring is a graded-$S$-coherent ring, see [5]. The converse is not true in general. Consider the graded ring $A = \mathbb{Z} \propto (\mathbb{Z}_2)^{(\mathbb{N})}$ with its natural $\mathbb{Z}_2$-grading; $A_0 = \mathbb{Z} \propto 0$ and $A_1 = 0 \propto (\mathbb{Z}_2)^{(\mathbb{N})}$ and consider the multiplicative set of homogeneous elements $S = \{2^n\colon n \in \mathbb{N}\} \propto 0$. Since $(2, 0)$ is a homogeneous element and $(0 : (2, 0)) = 0 \propto (\mathbb{Z}_2)^{(\mathbb{N})}$ is not a finitely generated ideal, then, according to [5], Theorem 3.3, $A$ is not graded-coherent. Then, taking any homogeneous ideal $I$ of $A$, we have that $(2, 0)I$ is finitely generated. Hence, $A$ is graded-$S$-Noetherian and so graded-$S$-coherent by Remark 3.5.

(2) Recall from [7], Remark 3.4 (3) that if $M$ is an $S$-finitely presented $R$-module, then $M_S$ is a finitely presented $R_S$-module. Thus, if $R$ is a graded-$S$-coherent ring, $R_S$ is a graded-coherent ring.

Now, we give the graded version of the $S$-counterpart of the classical Chase's result [10], Theorem 2.2. For reference purposes, it will be helpful to recall the following elementary lemma.

**Lemma 3.13** ([12], Lemma 2.3.1). *Let $R$ be a ring, let $I = (u_1, u_2, \dots, u_r)$ be a finitely generated ideal of $R$ ($r \in \mathbb{N}$) and let $a \in R$. Set $J = I + Ra$. Let $F$ be a free module on generators $x_1, x_2, \dots, x_{r+1}$ and let $0 \to K \to F \xrightarrow{f} J \to 0$ be an exact sequence with $f(x_i) = u_i$ ($1 \leqslant i \leqslant r$) and $f(x_{r+1}) = a$. Then there exists an exact sequence $0 \to K \cap F' \to K \xrightarrow{g} (I : a) \to 0$, where $F' = \bigoplus_{i=1}^{r} Rx_i$.*

**Theorem 3.14.** *Let $R$ be a graded ring. The following assertions are equivalent:*

(1) *$R$ is graded-$S$-coherent.*

(2) *$(I : a)$ is an $S$-finite ideal of $R$ for every finitely generated homogeneous ideal $I$ of $R$ and for every homogeneous element $a \in R$.*

(3) *$(0 : a)$ is an $S$-finite ideal of $R$ for every homogeneous element $a \in R$ and the intersection of two finitely generated homogeneous ideals of $R$ is an $S$-finite ideal of $R$.*

Proof. (1) $\Rightarrow$ (2): Let $I$ be a finitely generated homogeneous ideal of $R$ and let $a$ be a homogeneous element of $R$. Then $J = I + Ra$ is a finitely generated homogeneous ideal of the graded-$S$-coherent ring $R$, and so $J = I + Ra$ is $S$-finitely presented. Then there exist an exact sequence of $R$-modules

$$0 \to L \to K \to J \to 0,$$

where $L$ is $S$-finite. Thus, by Lemma 3.13, there exists a surjective homomorphism $L \to (I : a)$ which makes $(I : a)$ an $S$-finite ideal.

(2) $\Rightarrow$ (1): Let $I = \sum_{i=1}^{n} Rr_i$ be a finitely generated homogeneous ideal of $R$. We use induction on $n$ to prove that $I$ is $S$-finitely presented. For $n = 1$ we consider the exact sequence of $R$-modules: $0 \to (0 : r_1) \to R \to I \to 0$. By hypothesis (2), $(0 : r_1)$ is $S$-finite and so $I$ is $S$-finitely presented, as desired.

For $n > 1$, let $J = \sum_{i=1}^{n-1} Rr_i$. Consider the exact sequences of $R$-modules: $0 \to \ker f \hookrightarrow R^n \xrightarrow{f} I \to 0$ and $0 \to \ker g \hookrightarrow R^{n-1} \xrightarrow{g} J \to 0$ with $f(e_j) = g(e_j) = r_j$, $1 \leqslant j \leqslant n-1$ and $f(e_n) = r_n$, where $(e_j)_{j=1}^{n}$ is the canonical basis of $R^n$. Then, by Lemma 3.13, there exists an $R$-module homomorphism $\alpha\colon \ker f \to (J : r_n)$ such that the sequence of $R$-modules $0 \to \ker g \hookrightarrow \ker f \xrightarrow{\alpha} (J : r_n) \to 0$ is exact. By hypothesis (2), $(J : r_n)$ is $S$-finite, and by induction hypothesis, $J$ is $S$-finitely presented, so that $\ker g$ is $S$-finite. Therefore, $\ker f$ is $S$-finite, so that $I$ is $S$-finitely presented, as desired.

(1) $\Rightarrow$ (3): Consider the exact sequence of $R$-modules: $0 \to (0 : a) \to R \to Ra \to 0$, where $a$ is a homogeneous element of $R$. Since $Ra$ is a finitely generated homogeneous ideal of the graded-$S$-coherent ring $R$, $Ra$ is $S$-finitely presented. Therefore $(0 : a)$ is

$S$-finite, by [7], Proposition 2.3 as desired. Then, let $I$ and $J$ be two finitely generated homogeneous ideals of $R$. Then so is $I+J$ and hence $I+J$ is $S$-finitely presented since $R$ is graded-$S$-coherent. Therefore, $I \cap J$ is $S$-finite, by [7], Corollary 2.5.

(3) $\Rightarrow$ (1): Let $I = \sum_{i=1}^{n} Rr_i$ be a finitely generated homogeneous ideal of $R$. We use induction on $n$ to show that $I$ is $S$-finitely presented.

For $n = 1$, we consider the exact sequence of $R$-modules: $0 \to (0 : r_1) \to R \to I \to 0$. By hypothesis (3), $(0 : r_1)$ is $S$-finite and so $I$ is $S$-finitely presented, as desired. Let $n > 1$. By induction hypothesis, $\sum_{i=1}^{n-1} Rr_i$ and $Rr_n$ are $S$-finitely presented and, by hypothesis (3), the intersection $\left(\sum_{i=1}^{n-1} Rr_i\right) \cap Rr_n$ is $S$-finite. Therefore, $I = \sum_{i=1}^{n-1} Rr_i + Rr_n$ is $S$-finitely presented, by [7], Corollary 2.5, as desired. □

Recall from [15] that a homogeneous ideal $M$ of a graded ring $R$ is said to be a maximal homogeneous ideal if it is maximal among proper homogeneous ideals; equivalently, if every nonzero homogeneous element of $R/M$ is invertible and a graded ring is said to be graded-local if it has a unique maximal homogeneous ideal.

**Proposition 3.15.** *Assume that the grading monoid $G$ is cancellative and let $(R, M)$ be a graded-local ring such that $M^2 = 0$ and $S$ be a multiplicatively closed set of homogeneous elements of $R$. The following statements are equivalent.*

(1) *$R$ is a graded-$S$-coherent ring.*

(2) *$(0 : x)$ is an $S$-finite ideal for every homogeneous element $x \in R$.*

(3) *$M$ is $S$-finite.*

Proof. (1) $\Rightarrow$ (2): Follows from Theorem 3.14.

(2) $\Rightarrow$ (3): Assume that $M$ is not zero. Choose $0 \neq x \in M \cap h(R)$. Since $xM = 0$, we have $M \subseteq (0 : x) \subseteq M$, that is, $M = (0 : x)$. Then by assumption, $M$ is an $S$-finite ideal.

(3) $\Rightarrow$ (1): Suppose that $M$ is $S$-finite and $I$ is a finitely generated homogeneous ideal of $R$. Assume that $\{a_1, a_2, \ldots, a_n\}$ is a minimal generator set of homogeneous elements of $I$. Consider the exact sequence $0 \to \operatorname{Ker}\beta \hookrightarrow R^n \xrightarrow{\beta} I \to 0$, where $\beta\colon R^n \to I$ is defined by $\beta(x_1, x_2, \ldots, x_n) = x_1a_1 + x_2a_2 + \ldots + x_na_n$. Since $M^2 = 0$, we can easily get $\prod_{i=1}^{n} M \subseteq \operatorname{Ker}\beta$. Let $(x_1, x_2, \ldots, x_n) \in \operatorname{Ker}\beta$ be a homogeneous element. Then $x_1a_1 + x_2a_2 + \ldots + x_na_n = 0$. Assume that for some $i$, $x_i \notin M$. Then we have that $x_i$ is a unit, so we have $a_i = -x_i^{-1}(x_1a_1 + x_2a_2 + \ldots + x_{i-1}a_{i-1} + x_{i+1}a_{i+1} + \ldots + x_na_n)$. This gives $I = (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$, which

is a contradiction. So we have $\operatorname{Ker}\beta \subseteq \prod_{i=1}^{n} M$, that is, $\prod_{i=1}^{n} M = \operatorname{Ker}\beta$ (since $G$ is a cancellative monoid and so $\beta$ has a cancellable degree, thus $\operatorname{Ker}\beta$ is a homogeneous submodule and so every element of $\operatorname{Ker}\beta$ is a sum of its homogeneous elements). Since $M$ is $S$-finite, so is $\prod_{i=1}^{n} M = \operatorname{Ker}\beta$. Thus, $I$ is $S$-finitely presented, that is, $R$ is a graded-$S$-coherent ring. □

We end this section with the following result, which studies the transfer of graded-$S$-coherence under localizations.

**Proposition 3.16.** *Assume that the grading monoid $G$ is a group and let $R$ be a graded ring. If $R$ is a graded-$S$-coherent ring, then $R_V$ is a graded-$S_V$-coherent ring for every multiplicative set $V$ of homogeneous elements of $R$.*

Proof. $R_V$ is a graded ring since $G$ is a group and $V$ is a multiplicative set of homogeneous elements of $R$. Now, let $J$ be a finitely generated homogeneous ideal of $R_V$. Then there is a finitely generated homogeneous ideal $I$ of $R$ such that $J = I_V$. Since $R$ is graded-$S$-coherent, $I$ is $S$-finitely presented. Then, by [7], Lemma 3.10, the homogeneous ideal $J = I \bigotimes_R R_V$ of $R_V$ is $S_V$-finitely presented, as desired. □


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