Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 73 (2023), No. 2, 581–602

Persistent URL: `http://dml.cz/dmlcz/151675`

Czechoslovak Mathematical Journal, 73 (148) (2023), 581–602

WEIGHTED w-CORE INVERSES IN RINGS

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Received April 1, 2022. Published online December 29, 2022.

Abstract. Let R be a unital $*$-ring. For any $a, s, t, v, w \in R$ we define the weighted w-core inverse and the weighted dual s-core inverse, extending the w-core inverse and the dual s-core inverse, respectively. An element $a \in R$ has a weighted w-core inverse with the weight v if there exists some $x \in R$ such that $awxvx = x$, $xvawa = a$ and $(awx)^* = awx$. Dually, an element $a \in R$ has a weighted dual s-core inverse with the weight t if there exists some $y \in R$ such that $ytysa = y$, $asaty = a$ and $(ysa)^* = ysa$. Several characterizations of weighted w-core invertible and weighted dual s-core invertible elements are given when weights v and t are invertible Hermitian elements. Also, the relations among the weighted w-core inverse, the weighted dual s-core inverse, the e-core inverse, the dual f-core inverse, the weighted Moore-Penrose inverse and the (v, w)-(b, c)-inverse are considered.

Keywords: inverse along an element; $\{e, 1, 3\}$-inverse; $\{f, 1, 4\}$-inverse; weighted Moore-Penrose inverse; (v, w)-(b, c)-inverse; w-core inverse; dual v-core inverse

MSC 2020: 15A09, 06A06, 16W10

1. Introduction

In 2010, Baksalary and Trenkler in [1] introduced the generalized inverse $A^-_{\varrho^*,x}$ of a complex matrix, which was initially investigated by Rao and Mitra in [24]. They called it the *core inverse* in [1]. Also, the dual core inverse (see [1]) was given. Then Rakić et al. in [23] generalized the core inverse and the dual core inverse of complex matrices to an element in a unital $*$-ring. Later, several types of extended core inverses, such as DMP inverses (see [14]), core-EP inverses (see [22]) (a.k.a. pseudo core inverses in rings, see [10]), e-core inverses (see [18]), pseudo e-core inverses (see [27]) and W-weighted core-EP inverses (see [9]) are introduced. Recently, the present authors in [28] introduced the w-core inverse in a $*$-semigroup. It should be noted

This research is supported by the National Natural Science Foundation of China (No. 11801124) and China Postdoctoral Science Foundation (No. 2020M671068).

DOI: 10.21136/CMJ.2022.0134-22

that one of the main properties of the w-core inverse is that it encompasses several well-known outer inverses such as the core inverse, the e-core inverse, the core-EP inverse and the Moore-Penrose inverse.

An involution $a \mapsto a^*$ in a ring R is an anti-isomorphism of degree 2, that is, $(a^*)^* = a$, $(a+b)^* = a^* + b^*$ and $(ab)^* = b^*a^*$ for all $a, b \in R$. Further, R is called a *unital $*$-ring* if R is a unital ring with involution. An element $a \in R$ is Hermitian if $a^* = a$.

The aim of this paper is to introduce a class of weighted edition of w-core inverses in a unital $*$-ring R, called the *weighted w-core inverse*. Also, the weighted dual s-core inverse is defined. Several properties of them and relations with other types of generalized inverses are derived. For instance, it is shown that a is weighted w-core invertible with weight e if and only if w is invertible along a and a is $\{e, 1, 3\}$-invertible for any $a, w, e \in R$ with e an invertible Hermitian element. Also, it is proved that a is weighted w-core invertible with weight e if and only if there exists an (unique) idempotent $p \in R$ such that $(ep)^* = ep$, $pa = 0$ and $u = p + aw \in R^{-1}$. Then, new characterizations for both weighted w-core invertibility and weighted dual v-core invertibility are given by units, and their expressions are shown. Finally, we present the relations between weighted w-core inverses with weight v and Drazin's recently introduced (v, w)-(b, c)-inverses in a ring R, see [8].

This paper is organized as follows. In Section 2, we give the definitions of weighted w-core inverses and weighted dual s-core inverses. Then, several properties of them are presented. In particular, it is proved in Theorem 2.8 that for any $a, v, w \in R$, a is weighted w-core invertible with weight v if and only if there exists a unique $x \in R$ such that $awxva = a$, $awxvx = x$ and $(awx)^* = awx$. In Section 3, the existence criteria for the weighted w-core inverse and the weighted dual s-core inverse are derived provided that weights v and t are invertible Hermitian elements in rings. In what follows, we assume that e and f are invertible Hermitian elements in R. Also, we characterize both the weighted w-core invertible and the weighted dual s-core invertible elements by units. In Section 4, the relations among the weighted w-core inverse, the weighted dual s-core inverse, the e-core inverse, the dual f-core inverse, the weighted Moore-Penrose inverse and the (v, w)-(b, c)-inverse are derived.

For the convenience of readers, some basic concepts of generalized inverses are presented below.

Let R be an associative ring with unity 1. An element $a \in R$ is called *(von Neumann) regular* if there exists $x \in R$ such that $axa = a$. Such an x is called an *inner inverse* of a, and is denoted by a^-. By $a\{1\}$ we denote the set of all inner inverses of a. The left annihilator and right annihilator of a are defined by ${}^0a = \{x \in R\colon\ xa = 0\}$ and $a^0 = \{x \in R\colon\ ax = 0\}$, respectively. It is known from [23] that $aR = bR$ implies ${}^0a = {}^0b$, and dually $Ra = Rb$ implies $a^0 = b^0$.

An element $a \in R$ is Drazin invertible (see [6]) if there exists $x \in R$ and a nonnegative integer k such that $ax = xa$, $xax = x$ and $a^k = a^{k+1}x$. Such x is called a *Drazin inverse* of a. It is unique if it exists. The smallest nonnegative integer k is called the *Drazin index* of a, and is denoted by $\operatorname{ind}(a)$. If $\operatorname{ind}(a) = 1$, the Drazin inverse of a is the group inverse of a, and it is denoted by $a^\#$. By R^D and $R^\#$ we denote the sets of all Drazin invertible and group invertible elements in R, respectively. It is well known (see [11]) that $a \in R^\#$ if and only if $a \in a^2R \cap Ra^2$. In particular, if $a = a^2x = ya^2$ for some $x, y \in R$, then $a^\# = yax = y^2a = ax^2$.

In 2011, Mary introduced the concept of the inverse along an element. Later, Drazin extended the inverse along an element to the inverse along two elements, i.e., the (b, c)-inverse, see [7]. Given any $a, b, c \in R$, an element $a \in R$ is called (b, c)-*invertible* if there exists some $y \in R$ satisfying $y \in (bRy) \cap (yRc)$, $yab = b$ and $cay = c$. The (b, c)-inverse of a is unique if it exists. It is known from [7] that a is (b, c)-invertible if and only if $yay = y$, $yR = bR$ and $Ry = Rc$ if and only if $b \in Rcab$ and $c \in cabR$. In particular, a is invertible along d if and only if a is (d, d)-invertible if and only if $d \in dadR \cap Rdad$. The inverse of a along d is unique if it exists, and is denoted by $a^{\| d}$. By $R^{\| d}$ we denote the set of all invertible elements along d. In Theorem 11 of [15], and Corollary 3.4 of [16], Mary showed that $a \in R^\#$ if and only if $a^{\| a}$ exists if and only if $1^{\| a}$ exists. In these cases, $a^{\| a} = a^\#$ and $1^{\| a} = aa^\#$. One also knows from Theorem 2.2 of [16] that if $d = dadx = ydad$ for some $x, y \in R$, then $a^{\| d} = dx = yd$. More results on the inverse along an element can be referred to [2], [3], [4].

Throughout this paper, R is a unital $*$-ring. Recall that an element $a \in R$ is called *weighted Moore-Penrose invertible* (see [21]) if there exists an element $x \in R$ such that

(1) $axa = a$,

(2) $xax = x$,

(3) $(eax)^* = eax$,

(4) $(fxa)^* = fxa$,

where e and f are two invertible Hermitian elements, and such an x is called a *weighted Moore-Penrose inverse* of a. It is unique if it exists, and is denoted by $a^\dagger_{e,f}$. Moreover, any $x \in R$ satisfying (1) $axa = a$ and (3) $(eax)^* = eax$ is called an $\{e, 1, 3\}$-*inverse* of a, and is denoted by $a_e^{(1,3)}$. Also, any $x \in R$ satisfying (1) $axa = a$ and (4) $(fxa)^* = fxa$ is called an $\{f, 1, 4\}$-*inverse* of a, and is denoted by $a_f^{(1,4)}$. The sets of all weighted Moore-Penrose invertible, $\{e, 1, 3\}$-invertible and $\{f, 1, 4\}$-invertible elements in R are denoted by $R^\dagger_{e,f}$, $R_e^{\{1,3\}}$ and $R_f^{\{1,4\}}$, respectively. In particular, $a \in R^\dagger_{e,f}$ if and only if $a \in R_e^{\{1,3\}} \cap R_f^{\{1,4\}}$. In this case, $a^\dagger_{e,f} = a_f^{(1,4)}aa_e^{(1,3)}$. If $e = f = 1$, then the weighted Moore-Penrose inverse is just the classical Moore-Penrose inverse (see [20]), the $\{e, 1, 3\}$-inverse is the $\{1,3\}$-inverse and the $\{f,1,4\}$-inverse is the $\{1,4\}$-inverse. The Moore-Penrose inverse, $\{1,3\}$-inverse and $\{1,4\}$-inverse of a are denoted by $a^\dagger$, $a^{(1,3)}$ and $a^{(1,4)}$, respectively. We denote by $R^\dagger$, $R^{\{1,3\}}$ and $R^{\{1,4\}}$ the sets of all Moore-Penrose invertible, $\{1,3\}$-invertible and $\{1,4\}$-invertible elements in R, respectively.

Following [23], an element $a \in R$ is core invertible if there exists some $x \in R$ such that $axa = a$, $xR = aR$ and $Rx = Ra^*$. Dually, if there exists an element $y \in R$ such that $aya = a$, $yR = a^*R$ and $Ry = Ra$, then y is called a *dual core inverse* of a. The core (or dual core) inverse of a is unique if it exists, and is denoted by $a^{\#}$ (or $a_{\#}$). By $R^{\#}$ and $R_{\#}$ we denote the sets of all core invertible and dual core invertible elements in R, respectively. Moreover, they proved in [23], Theorem 2.14 that the core inverse x of a can be characterized by the unique solution of the following five equations:

$$(1) \qquad axa = a,$$
$$(2) \qquad xax = x,$$
$$(3) \qquad ax^2 = x,$$
$$(4) \qquad xa^2 = a,$$
$$(5) \qquad (ax)^* = ax.$$

Dually, the dual core inverse y of a can be expressed by the unique solution of the following five equations:

$$(1') \qquad aya = a,$$
$$(2') \qquad yay = y,$$
$$(3') \qquad y^2a = y,$$
$$(4') \qquad a^2y = a,$$
$$(5') \qquad (ya)^* = ya.$$

Given any $a, e \in R$, a is called *e-core invertible* (see [18]) if there exists some $x \in R$ such that $axa = a$, $xR = aR$ and $Rx = Ra^*e$. The e-core inverse of a is unique if it exists, and is denoted by $a_e^{\#}$. Further, the writers in [18] characterized the existence of the e-core inverse of $a \in R$ by the unique element x satisfying $ax^2 = x$, $xa^2 = a$ and $(eax)^* = eax$. Dually, they showed that the dual f-core inverse of a is the unique element $a_{f,\#}$ satisfying $(a_{f,\#})^2a = a_{f,\#}$, $a^2a_{f,\#} = a$ and $(fa_{f,\#}a)^* = fa_{f,\#}a$. More results on e-core inverses and dual f-core inverses can be referred to [18] and [26].

Assume that S is a $*$-semigroup, that is a semigroup with an involution $*$ satisfying $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for every $x, y \in S$. The present authors in [28] recently introduced the w-core inverse by three equations in S, extending the classical core inverses. Given any $a, w \in S$, we say that a is w-core invertible (see [28]) if there exists some $x \in S$ such that $awx^2 = x$, $xawa = a$ and $(awx)^* = awx$. Such an x is called a *w-core inverse* of a. Moreover, the w-core inverse of a is unique if it exists, and is denoted by $a_w^{\oplus}$. Dually, the dual v-core inverse of a (see [28]), when exists, is denoted by the unique $a_{v,\oplus}$ such that $(a_{v,\oplus})^2 va = a_{v,\oplus}$, $avaa_{v,\oplus} = a$ and $(a_{v,\oplus}va)^* = a_{v,\oplus}va$. Therein, it is proved in Theorems 2.6 and 2.18 of [28] that a is w-core invertible if and only if w is invertible along a and a is $\{1,3\}$-invertible, and a is dual v-core invertible if and only if v is invertible along a and a is $\{1,4\}$-invertible. It is known that $a \in R^{\#}$ if and only if $a \in a^2R \cap Ra^2$ if and only if 1 is invertible along a. From the above, one knows that the core inverse is extended to the w-core inverse of quadratic level. More results on w-core inverses can be referred to [29].

As usual, the sets of all e-core invertible, dual f-core invertible, w-core invertible and dual v-core invertible elements in R are denoted by $R_e^{\oplus}$, $R_{f,\oplus}$, $R_w^{\oplus}$ and $R_{v,\oplus}$, respectively.

2. Weighted w-core inverses and weighted dual s-core inverses

We begin this section with the weighted w-core inverse and the weighted dual s-core inverse of an element in a unital $*$-ring.

Definition 2.1. Let $a, v, w \in R$. The element a has a weighted w-core inverse with weight v if there exists $x \in R$ such that
(1) $awxvx = x$,
(2) $xvawa = a$,
(3) $(awx)^* = awx$.
Such an x is called a *weighted w-core inverse* of a with weight v.

By Definition 2.1, one can observe that the weighted w-core inverse with weight 1 coincides with the w-core inverse and the weighted 1-core inverse with weight 1 coincides with the core inverse. The existence of weighted 1-core inverse with weight e coincides with the existence of e-core inverse, see Corollary 4.6 below.

As was stated in [28], all core invertible elements are w-core invertible. However, the converse statement may not be true. Herein, we claim that all w-core invertible elements are weighted w-core invertible. In general, the weighted w-core invertibility of an element does not imply its w-core invertibility as Example 2.2 below shows.

Example 2.2. Let $R = M_2(\mathbb{C})$ be the ring of all 2 by 2 complex matrices and let the involution $*$ be the transpose. Suppose $A = \begin{bmatrix} i & 0 \\ 1 & 0 \end{bmatrix}$, $V = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in R$. Then A is weighted W-core invertible with weight V and $\begin{bmatrix} (i+2)/5 & (-2i+1)/5 \\ (1-2i)/5 & (-2-i)/5 \end{bmatrix}$ is the weighted W-core inverse of A with weight V. However, $A \notin RA^*A$, i.e., $A \notin R^{\{1,3\}}$ (see Lemma 3.1 below), and hence $A \notin R_W^{\oplus}$ and $A \notin R^{\oplus}$.

In [28], the w-core inverse of $a \in R$ is characterized by five equations. It is natural to consider whether we can characterize the weighted w-core inverse by the solution of five equations. Indeed, Proposition 2.3 below illustrates this assumption.

Proposition 2.3. *Let $a, v, w, x \in R$. Then the following conditions are equivalent:*

(i) $awxvx = x$ *and* $xvawa = a$;

(ii) $awxva = a$, $xvawx = x$, $awxvx = x$ *and* $xvawa = a$.

P r o o f. (ii) $\Rightarrow$ (i) It is obvious.

(i) $\Rightarrow$ (ii) Assume that $awxvx = x$ and $xvawa = a$. Then we have $a = xvawa = (awxvx)vawa = awxv(xvawa) = awxva$ and $x = awxvx = (xvawa)wxvx = xvaw(awxvx) = xvawx$, as required. □

Proposition 2.3 states that the weighted w-core inverse x of a with the weight v can be characterized by the solution of the following five equations.

Proposition 2.4. *Let $a, v, w, x \in R$. Then the following conditions are equivalent:*

(i) *x is the weighted w-core inverse of a with weight v;*

(ii) $awxva = a$, $xvawx = x$, $awxvx = x$, $xvawa = a$ *and* $(awx)^* = awx$.

First and most fundamentally, the following result is presented.

Theorem 2.5. *Let $a, v, w \in R$. If a is weighted w-core invertible with weight v, then it has a unique weighted w-core inverse with weight v.*

P r o o f. Suppose that $x, y \in R$ are two weighted w-core inverses of a with weight v. By Proposition 2.4, we have $awxva = a = awyva$, $xvawx = x$, $yvawy = y$, $awxvx = x$, $awyvy = y$, $xvawa = a = yvawa$, $(awx)^* = awx$ and $(awy)^* = awy$. Then we get

$$\begin{aligned} x &= xvawx = xv(awyva)wx = xvaw(awyvy)vawx = (xvawa)wyvyvawx \\ &= (awyvy)vawx = yvawx = (yvawy)vawx = yv(awy)^* vawx \\ &= yvy^* w^* (awxva)^* vawx = yvy^* w^* a^* v^* (awxva)wx = yvy^* w^* a^* v^* awx \\ &= yv(awxvawy)^* = yv(awy)^* = yvawy = y. \end{aligned}$$

Hence, a has a unique weighted w-core inverse with weight v. □

As Theorem 2.5 above shows, the weighted w-core inverse of a with weight v is unique if it exists, and is denoted by $a^{\textcircled{\#}}_{v,w}$. We denote by $R^{\textcircled{\#}}_{v,w}$ the set of all weighted w-core invertible elements with weight v in R.

In what follows, we give the definition of the weighted dual s-core inverse with weight t in a ring R.

Definition 2.6. Let $a, s, t \in R$. The element a has a weighted dual s-core inverse with weight t if there exists $x \in R$ such that

(1′) $xtxsa = x$,

(2′) $asatx = a$,

(3′) $(xsa)^* = xsa$.

Such an x is called a weighted dual s-core inverse of a with weight t.

The weighted dual s-core inverse of a with weight t is unique if it exists, and is denoted by $a_{s,t,\textcircled{\#}}$. We denote by $R_{s,t,\textcircled{\#}}$ the set of all weighted dual s-core invertible elements with weight t in R.

The following result presents the relationship between the weighted w-core inverse and the weighted dual s-core inverse in R.

Proposition 2.7. *Let $a, v, w \in R$. Then a is weighted dual s-core invertible with weight t if and only if a^* is weighted s^*-core invertible with weight t^*. In this case, $(a_{s,t,\textcircled{\#}})^* = (a^*)^{\textcircled{\#}}_{t^*,s^*}$.*

Proof. Suppose $x \in R$ is the weighted dual s-core inverse of a with weight t. It follows from $xtxsa = x$, $asatx = a$ and $(xsa)^* = xsa$ that $a^*s^*x^*t^*x^* = x^*$, $x^*t^*a^*s^*a^* = a^*$ and $(a^*s^*x^*)^* = a^*s^*x^*$ by taking the involution. Therefore, x^* is the weighted s^*-core inverse of a^* with weight t^*. □

Let $\mathbb{M}_{n\times n}(\mathbb{C})$ be the ring of all $n \times n$ matrices over the complex field $\mathbb{C}$ with conjugate transpose as involution. Given $A \in \mathbb{M}_{n\times n}(\mathbb{C})$ with the core inverse existing, Wang and Liu in Theorem 2.1 of [25] proved the fact that $A^{\textcircled{\#}}$ is the unique solution of $AXA = A$, $AX^2 = X$ and $(AX)^* = AX$ in $\mathbb{M}_{n\times n}(\mathbb{C})$. In fact, one also can see from the proof of Theorem 2.1 of [25] that its converse statement also holds without the existence of $A^{\textcircled{\#}}$. We next generalize this result to the weighted w-core inverse in a unital $*$-ring.

Theorem 2.8. *Let $a, w, v \in R$. Then $a \in R^{\textcircled{\#}}_{v,w}$ if and only if there exists a unique $x \in R$ such that $awxva = a$, $awxvx = x$ and $(awx)^* = awx$. In this case, $a^{\textcircled{\#}}_{v,w} = x$.*

Proof. We first assume $a \in R^{\textcircled{\#}}_{v,w}$. Then we have $awa^{\textcircled{\#}}_{v,w}va = a$, $awa^{\textcircled{\#}}_{v,w}va^{\textcircled{\#}}_{v,w} = a^{\textcircled{\#}}_{v,w}$ and $(awa^{\textcircled{\#}}_{v,w})^* = awa^{\textcircled{\#}}_{v,w}$. Hence, the existence is proved. We next prove the uniqueness. For any $x \in R$ satisfying $awxva = a$, $awxvx = x$ and $(awx)^* = awx$, we have

$$\begin{aligned} x &= awxvx = (a^{\textcircled{\#}}_{v,w}vawa)wxvx = a^{\textcircled{\#}}_{v,w}vaw(awxvx) = a^{\textcircled{\#}}_{v,w}vawx \\ &= a^{\textcircled{\#}}_{v,w}v(awx)^* = a^{\textcircled{\#}}_{v,w}v(awa^{\textcircled{\#}}_{v,w}vawx)^* = a^{\textcircled{\#}}_{v,w}vx^*w^*a^*v^*awa^{\textcircled{\#}}_{v,w} \\ &= a^{\textcircled{\#}}_{v,w}vx^*w^*a^*v^*aw(a^{\textcircled{\#}}_{v,w}vawa^{\textcircled{\#}}_{v,w}) = a^{\textcircled{\#}}_{v,w}v(awa^{\textcircled{\#}}_{v,w}vawx)^*vawa^{\textcircled{\#}}_{v,w} \\ &= a^{\textcircled{\#}}_{v,w}v(awx)^*vawa^{\textcircled{\#}}_{v,w} = a^{\textcircled{\#}}_{v,w}v(awxva)wa^{\textcircled{\#}}_{v,w} = a^{\textcircled{\#}}_{v,w}vawa^{\textcircled{\#}}_{v,w} = a^{\textcircled{\#}}_{v,w}. \end{aligned}$$

Conversely, assume that there exists a unique $x \in R$ such that $awxva = a$, $awxvx = x$ and $(awx)^* = awx$. Define $y = a - xvawa + x$, then $awyvy = aw(a - xvawa + x)v(a - xvawa + x) = a - xvawa + x$, $awy = aw(a - xvawa + x) = awx = (awx)^* = (awy)^*$ and $awyva = aw(a - xvawa + x)va = a$. As for the uniqueness of x, we obtain $x = y = a - xvawa + x$, i.e., $a = xvawa$, as required. □

3. Characterizations of weighted w-core inverses and weighted dual s-core inverses

In this section, we mainly investigate characterizations and representations of the weighted w-core inverse with weight e and the weighted dual s-core inverse with weight f in unital $*$-rings.

We begin with an auxiliary lemma.

Lemma 3.1 ([27], Propositions 2.1 and 2.2). *Let $a, e, f \in R$. We have the following results:*

(i) *a is $\{e, 1, 3\}$-invertible if and only if $a \in Ra^*ea$. Moreover, if $a = xa^*ea$ for some $x \in R$, then x^*e is an $\{e, 1, 3\}$-inverse of a;*

(ii) *a is $\{f, 1, 4\}$-invertible if and only if $a \in af^{-1}a^*R$. Moreover, if $a = af^{-1}a^*y$ for some $y \in R$, then $f^{-1}y^*$ is an $\{f, 1, 4\}$-inverse of a.*

Theorem 3.2. *Let $a, e, w \in R$. Then the following conditions are equivalent:*

(i) *a is weighted w-core invertible with weight e;*

(ii) *there exists $x \in R$ such that $awxex = x$, $xeawa = a$, $(awx)^* = awx$, $awxea = a$ and $xeawx = x$;*

(iii) *there exists $x \in R$ such that $awxea = a$, $xR = aR$ and $Rx = Ra^*$;*

(iv) *there exists $x \in R$ such that $awxea = a$, ${}^0x = {}^0a$, and $x^0 = (a^*)^0$;*

(v) *there exists $x \in R$ such that $awxea = a$, ${}^0x = {}^0a$, and $(a^*)^0 \subseteq x^0$;*

(vi) *$w \in R^{\|a}$ and $a \in R_e^{\{1,3\}}$.*

In this case, $a^{\oplus}_{e,w} = w^{\|a} a_e^{(1,3)} e^{-1}$.

Proof. (i) ⇒ (ii) by taking $v = e$ in Proposition 2.4.

(ii) ⇒ (iii) Given (ii), $x = awxex$ implies $x \in aR$, which together with $a = xeawa \in xR$ guarantees $xR = aR$. Note that $xeawx = x$ and $(awx)^* = awx$. Then $x = xeawx = xe(awx)^* = xex^*w^*a^* \in Ra^*$. Also, by $awxea = a$ and $(awx)^* = awx$ we have $a^* = (awxea)^* = a^*eawx \in Rx$. Therefore, $Rx = Ra^*$.

(iii) ⇒ (iv) and (iv) ⇒ (v) are clear.

(v) $\Rightarrow$ (vi) Given (v), from $awxea = a$ we have $(1 - awxe)a = 0$ and thus $1 - awxe \in {}^0a$. As ${}^0x = {}^0a$, $(1 - awxe)x = 0$, i.e., $x = awxex$. This means $a = awxea = aw(awxex)ea \in awaR$. Also, from $awxea = a$ we obtain $a^* = (awxea)^* = a^*e(awx)^*$ and $1 - e(awx)^* \in (a^*)^0$, which together with $(a^*)^0 \subseteq x^0$ ensures $x(1-e(awx)^*) = 0$, i.e., $x = xe(awx)^*$. So, $awx = (awx)e(awx)^*$, which implies $(awx)^* = awx$. Thus, $a^* = a^*e(awx)^* = a^*eawx$ and $a = awxea = (awx)^*ea = x^*w^*a^*ea \in Ra^*ea$. By Lemma 3.1 we have $a \in R_e^{\{1,3\}}$. Again, applying $(a^*)^0 \subseteq x^0$ and ${}^0x = {}^0a$, we have $a = xeawa \in Rawa$. Therefore, $w \in R^{\|a}$ and $a \in R_e^{\{1,3\}}$.

(vi) $\Rightarrow$ (i) Note that $w \in R^{\|a}$ and $a \in R_e^{\{1,3\}}$. To prove a is weighted w-core invertible with weight e, it is sufficient to prove that $x = w^{\|a}a_e^{(1,3)}e^{-1}$ is the weighted w-core inverse of a with weight e. Indeed, we have:

(1) Since $w \in R^{\|a}$ implies $w^{\|a} \in aR$, we have $w^{\|a} = as$ for some $s \in R$. Hence, $aa_e^{(1,3)}w^{\|a} = aa_e^{(1,3)}as = as = w^{\|a}$.

So, $awxex = aw(w^{\|a}a_e^{(1,3)}e^{-1})e(w^{\|a}a_e^{(1,3)}e^{-1}) = (aww^{\|a})a_e^{(1,3)}w^{\|a}a_e^{(1,3)}e^{-1} = aa_e^{(1,3)}w^{\|a}a_e^{(1,3)}e^{-1} = (aa_e^{(1,3)}w^{\|a})a_e^{(1,3)}e^{-1} = w^{\|a}a_e^{(1,3)}e^{-1} = x$.

(2) Note also that $w \in R^{\|a}$. Then $w^{\|a} \in Ra$ and $w^{\|a}a_e^{(1,3)}a = taa_e^{(1,3)}a = ta = w^{\|a}$ for some $t \in R$. So, $xeawa = w^{\|a}a_e^{(1,3)}e^{-1}eawa = (w^{\|a}a_e^{(1,3)}a)wa = w^{\|a}wa = a$.

(3) $(awx)^* = (aww^{\|a}a_e^{(1,3)}e^{-1})^* = (aa_e^{(1,3)}e^{-1})^* = (e^{-1}(eaa_e^{(1,3)})e^{-1})^* = e^{-1} \times (eaa_e^{(1,3)})^*e^{-1} = e^{-1}eaa_e^{(1,3)}e^{-1} = awx$.

The proof is completed. $\square$

We can also prove an analogous result relating to the weighted dual s-core inverse with weight f in a ring R.

Theorem 3.3. *Let $a, f, s \in R$. Then the following conditions are equivalent:*

(i) *a is weighted dual s-core invertible with weight f;*

(ii) *there exists $x \in R$ such that $xfxsa = x$, $asafx = a$, $(xsa)^* = xsa$, $afxsa = a$ and $xsafx = x$;*

(iii) *there exists $x \in R$ such that $afxsa = a$, $xR = a^*R$ and $Rx = Ra$;*

(iv) *there exists $x \in R$ such that $afxsa = a$, ${}^0x = {}^0(a^*)$ and $x^0 = a^0$;*

(v) *there exists $x \in R$ such that $afxsa = a$, ${}^0(a^*) \subseteq {}^0x$ and $x^0 = a^0$;*

(vi) *$s \in R^{\|a}$ and $a \in R_{f^{-1}}^{\{1,4\}}$.*

In this case, $a_{s,f,\oplus} = f^{-1}a_{f^{-1}}^{(1,4)}s^{\|a}$.

Remark 3.4. Replacing f by f^{-1} in Theorem 3.3 above, we obtain that $a \in R_{s,f^{-1},\oplus}$ if and only if $s \in R^{\|a}$ and $a \in R_f^{\{1,4\}}$. In this case, $a_{s,f^{-1},\oplus} = fa_f^{(1,4)}s^{\|a}$. This result will be used frequently in the sequel.

Lemma 3.5 ([16], Theorem 2.1). *Let $a, d \in R$. Then the following conditions are equivalent:*
(i) $a \in R^{\|a}$;
(ii) $a \in daR$ and $da \in R^{\#}$;
(iii) $a \in Rad$ and $ad \in R^{\#}$.
In this case, $a^{\|d} = d(ad)^{\#} = (da)^{\#}d$.

Applying Lemma 3.5, we obtain the following new representations of the weighted w-core inverse and the weighted dual s-core inverse in R.

Proposition 3.6. *Let $a, e, f, w, s \in R$. We have the following results:*
(i) $a \in R^{\oplus}_{e,w}$ *if and only if* $w \in R^{\|a}$ *and* $a \in R^{\{1,3\}}_e$. *In this case,* $a^{\oplus}_{e,w} = a(wa)^{\#}a^{(1,3)}_e e^{-1} = (aw)^{\#}aa^{(1,3)}_e e^{-1}$.
(ii) $a \in R_{s,f^{-1},\oplus}$ *if and only if* $s \in R^{\|a}$ *and* $a \in R^{\{1,4\}}_f$.
In this case, $a_{s,f^{-1},\oplus} = fa^{(1,4)}_f a(sa)^{\#} = fa^{(1,4)}_f (as)^{\#}a$.

In 2018, Li and Chen in Theorem 2.10 of [13] proved that $a \in R^{\oplus}$ if and only if $a \in R(a^*)^n a \cap Ra^n$. The present authors in Theorem 2.10 of [28] illustrated that $a \in R^{\oplus}_w$ if and only if $a \in R[(aw)^*]^n a \cap R(aw)^{n-1}a$. Inspired by these, we aim to characterize the weighted w-core invertibility by ideals, and to give its corresponding expressions.

Theorem 3.7. *Let $a, e, w \in R$ and $n \geqslant 2$ be a positive integer. Then the following conditions are equivalent:*
(i) $a \in R^{\oplus}_{e,w}$;
(ii) $awR = aR$ and $aw \in R^{\oplus}_e$;
(iii) $a \in R[(aw)^*]^n ea \cap R(aw)^{n-1}a$.
In this case, $a^{\oplus}_{e,w} = (aw)^{\oplus}_e e^{-1}$.

Proof. (i) $\Rightarrow$ (ii) Assume that $a \in R^{\oplus}_{e,w}$ and $x = a^{\oplus}_{e,w}$. Then, by Theorem 3.2, we have $xeawa = a$, $awxex = x$ and $(awx)^* = awx$, which imply $xeawaw = aw$, $awxexe = xe$ and $(eawxe)^* = eawxe$, and hence $aw \in R^{\oplus}_e$ and xe is the e-core inverse of aw, i.e., $xe = (aw)^{\oplus}_e$ and $x = (aw)^{\oplus}_e e^{-1}$. From Theorem 3.2, it is known that $a \in R^{\oplus}_{e,w}$ if and only if $w \in R^{\|a}$ and $a \in R^{\{1,3\}}_e$. Since $w \in R^{\|a}$, it follows that $aR = awaR \subseteq awR$.

(ii) $\Rightarrow$ (iii) As $aw \in R^{\oplus}_e$, there exists some $y \in R$ such that $awy^2 = y$, $y(aw)^2 = aw$, $(eawy)^* = eawy$, $awyaw = aw$ and $yawy = y$. Note that $aR = awR$, then $a = awt$ for some $t \in R$ and thus $a = awt = y(aw)^2t = yawa = y(y(aw)^2)a = y^2(aw)^2a = y^2(y(aw)^2)awa = y^3(aw)^3a = \ldots = y^{n-1}(aw)^{n-1}a \in R(aw)^{n-1}a$. Also,

$a = awt = awyawt = awya = e^{-1}(eawy)a = e^{-1}(eawy)^*a = e^{-1}y^*(aw)^*ea = e^{-1}(awy^2)^*(aw)^*ea = e^{-1}(y^2)^*((aw)^*)^2ea = e^{-1}(y)^*(awy^2)^*((aw)^*)^2ea = e^{-1} \times (y^3)^*((aw)^*)^3ea = \ldots = e^{-1}(y^n)^*((aw)^*)^n ea \in R((aw)^*)^n ea$, as required.

(iii) $\Rightarrow$ (i) Note that $a \in R[(aw)^*]^n ea$, then there exists $r \in R$ such that

$$a = r[(aw)^*]^n ea = r[(aw)^*]^{n-1} w^* a^* ea.$$

So, $a \in R_e^{\{1,3\}}$ and $(r[(aw)^*]^{n-1}w^*)^*e = w(aw)^{n-1}r^*e \in a\{e, 1, 3\}$ by Lemma 3.1. Moreover, $a = aa_e^{(1,3)}a = a(w(aw)^{n-1}r^*e)a \in awaR$.

Note also that $a \in R[(aw)^*]^n ea \cap R(aw)^{n-1}a$. Then $a \in Ra^*ea \cap Rawa$, which combines with $a \in awaR$ to ensure that a is weighted w-core invertible with weight e by Lemma 3.1 and Theorem 3.2. □

We next investigate the existence of the weighted w-core inverse of an element by idempotents and units. First, we give the following lemma.

Lemma 3.8 ([5]). *Let $a, b \in R$. Then $ab \in R^D$ if and only if $ba \in R^D$. In this case, $(ba)^D = b((ab)^D)^2 a$.*

Theorem 3.9. *Let $a, e, w \in R$. The following conditions are equivalent:*

(i) $a \in R^{\oplus}_{e,w}$;

(ii) *there exists a unique idempotent $p \in R$ such that $(ep)^* = ep$, $pa = 0$ and $u = p + aw \in R^{-1}$;*

(iii) *there exists an idempotent $p \in R$ such that $(ep)^* = ep$, $pa = 0$ and $u = p + aw \in R^{-1}$.*

In this case, $a^{\oplus}_{e,w} = u^{-1}(1-p)e^{-1}$.

P r o o f. (i) $\Rightarrow$ (ii) Suppose that $a \in R^{\oplus}_{e,w}$. Then by Theorem 3.7 we have $awR = aR$ and $aw \in R_e^{\oplus}$. Let $x = (aw)_e^{\oplus}$ and $p = 1 - awx$. Then, $p^2 = p$ and $(ep)^* = ep$. From $awR = aR$ we have $pa = pawt = (1 - awx)awt = 0$ for some $t \in R$. Define $v = x + 1 - xaw$. By a direct check, we have $vu = (x + 1 - xaw)(p + aw) = (x + 1 - xaw)(1 - awx + aw) = 1 = (1 - awx + aw)(x + 1 - xaw) = uv$. Therefore, $u \in R^{-1}$ and v is the inverse of u.

Next, we prove the uniqueness of p. Let $q \in R$ be the idempotent satisfying $(eq)^* = eq$, $qa = 0$ and $u = q + aw \in R^{-1}$, we have $(1 - q)(q + aw) = aw$ and hence $1 - q = aw(q + aw)^{-1}$. Then $p(1 - q) = paw(q + aw)^{-1} = 0$, i.e., $p = pq$.

Similarly, we have $(1 - p)(p + aw) = aw$ and $q = qp$. So, $ep = epq = (epq)^* = ((ep)e^{-1}(eq))^* = eqe^{-1}ep = eqp = eq$, i.e., $p = q$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) Given (iii), we have $paw = (pa)w = 0$ and $aw \in R_e^{\oplus}$. Indeed, suppose $x = (p+aw)^{-1}(1-p)$, we have $1-p = aw(p+aw)^{-1}$ and $p = p(p+aw)^{-1}$. Then we can easily check that $x(aw)^2 = aw$, $awx^2 = x$ and $(eawx)^* = eawx$. Clearly, x is the e-core inverse of aw. Hence, to prove $a \in R_{e,w}^{\oplus}$, it suffices to prove that $awR = aR$ by Theorem 3.7.

Since $p + aw \in R^{-1}$ and $pa = 0$, we have $R(p + aw)a = (R(p + aw))a = Ra$ and $R(p + aw)a = R(pa + awa) = Rawa$. Then $Ra = Rawa \subseteq Rwa \subseteq Ra$ and hence $Ra = Rwa = Rawa = (Rwa)wa = R(wa)^2$ and $wa = t(wa)^2$ for some $t \in R$. From $aw \in R_e^{\oplus}$ we have $aw \in R^{\#} \subseteq R^D$. Then we have $wa \in R^D$ by Lemma 3.8. Let ind(wa)=k, we have $(wa)^{k+1}R = (wa)^kR$. Pre-multiplying the equation $(wa)^{k+1}R = (wa)^kR$ by t^{k-1} gives $t^{k-1}(wa)^{k+1}R = t^{k-1}(wa)^kR$. Hence, we have $(wa)^2R = waR$ since $wa = t(wa)^2$.

We conclude that $wa \in R^{\#}$. By Lemma 3.5, combining with $wa \in R^{\#}$ and $Rwa = Ra$, we get $w \in R^{\|a}$ and so $aR = awaR = awR$. □

Applying Theorems 3.2 and 3.3, we can get the following result.

Proposition 3.10. *Let $a, w, s \in R$. Then $a \in R_{e,w}^{\oplus} \cap R_{s,f^{-1},\oplus}$ if and only if $w, s \in R^{\|a}$ and $a \in R_{e,f}^{\dagger}$.*

We next characterize the existence criteria of both the weighted w-core invertible and the weighted dual s-core invertible elements by units. We first present some known results.

Lemma 3.11 ([12]). *Given $a, b \in R$, $1 + ab$ is invertible if and only if $1 + ba$ is invertible. Moreover, $(1 + ba)^{-1} = 1 - b(1 + ab)^{-1}a$.*

Lemma 3.12 ([16], Theorem 3.2 and [17], Theorem 1.3). *Let $a \in R$ and $d \in R$ be regular with $d^- \in d\{1\}$. The following conditions are equivalent:*

(i) $a \in R^{\|d}$;

(ii) $u = da + 1 - dd^- \in R^{-1}$;

(iii) $v = ad + 1 - d^-d \in R^{-1}$.

In this case, $a^{\|d} = u^{-1}d = dv^{-1}$.

Lemma 3.13 ([27], Theorem 2.5 and Corollary 2.9). *Let $e, f \in R$ and $a \in R$ be regular with $a^- \in a\{1\}$. The following conditions are equivalent:*

(i) $a \in R_{e,f}^{\dagger}$;

(ii) $a \in af^{-1}a^*eaR$;

(iii) $a \in Raf^{-1}a^*ea$;

(iv) $u = af^{-1}a^*e + 1 - aa^- \in R^{-1}$;

(v) $v = f^{-1}a^*ea + 1 - a^-a \in R^{-1}$.

In this case, $a^{\dagger}_{e,f} = f^{-1}(eax)^* = (yaf^{-1})^*e = (u^{-1}af^{-1})^*e = f^{-1}(eav^{-1})^*$, *where* $x, y \in R$ *satisfy* $a = af^{-1}a^*eax = yaf^{-1}a^*ea$.

Theorem 3.14. *Let* $a, e, f, s, w \in R$ *with* $a^- \in a\{1\}$. *If* $s \in R^{\|a}$, *then the following conditions are equivalent:*

(i) $a \in R^{\oplus}_{e,w} \cap R_{s,f^{-1},\oplus}$;

(ii) $w \in R^{\|a}$ *and* $a \in R^{\dagger}_{e,f}$;

(iii) $u = awasaf^{-1}a^*e + 1 - aa^- \in R^{-1}$;

(iv) $v = asawaf^{-1}a^*ea + 1 - aa^- \in R^{-1}$;

(v) $u' = af^{-1}a^*eawas + 1 - aa^- \in R^{-1}$;

(vi) $v' = af^{-1}a^*easaw + 1 - aa^- \in R^{-1}$.

In this case, $a^{\oplus}_{e,w} = (v')^{-1}af^{-1}a^*easa(u^{-1}awasaf^{-1})^*$ *and* $a_{s,f^{-1},\oplus} = f(u^{-1} \times awasaf^{-1})^*e(u')^{-1}af^{-1}a^*eawa$.

Proof. (i) $\Leftrightarrow$ (ii) It follows from Theorems 3.2 and 3.3.

(ii) $\Rightarrow$ (iii) Note that $s, w \in R^{\|a}$. Then by Lemmas 3.11 and 3.12 we have $awaa^- + 1 - aa^- \in R^{-1}$ and $asaa^- + 1 - aa^- \in R^{-1}$. Lemma 3.13 shows that $af^{-1}a^*e + 1 - aa^- \in R^{-1}$ provided that $a \in R^{\dagger}_{e,f}$. Hence, $(awaa^- + 1 - aa^-)(asaa^- + 1 - aa^-)(af^{-1}a^*e + 1 - aa^-) = awasaf^{-1}a^*e + 1 - aa^- = u \in R^{-1}$.

(iii) $\Rightarrow$ (ii) Since $u = awasaf^{-1}a^*e + 1 - aa^- \in R^{-1}$, it follows that $ua = awasaf^{-1}a^*ea$ and $a = u^{-1}awasaf^{-1}a^*ea \in Raf^{-1}a^*ea$. Then by Lemma 3.13 we have $a \in R^{\dagger}_{e,f}$, $a^{\dagger}_{e,f} = (u^{-1}awasaf^{-1})^*e$ and $af^{-1}a^*e + 1 - aa^- \in R^{-1}$. The assumption $s \in R^{\|a}$ implies $asaa^- + 1 - aa^- \in R^{-1}$ by Lemmas 3.11 and 3.12. Therefore, by $u = awasaf^{-1}a^*e + 1 - aa^- = (awaa^- + 1 - aa^-)(asaa^- + 1 - aa^-)(af^{-1}a^*e + 1 - aa^-)$, we can get $awaa^- + 1 - aa^- = u(af^{-1}a^*e + 1 - aa^-)^{-1}(asaa^- + 1 - aa^-)^{-1} \in R^{-1}$, and consequently, $aw + 1 - aa^- \in R^{-1}$, which guarantees that $w \in R^{\|a}$ by Lemmas 3.11 and 3.12.

(ii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) can be proved by a similar way of (ii) $\Leftrightarrow$ (iii).

We next give the representations of $a^{\oplus}_{e,w}$ and $a_{s,f^{-1},\oplus}$. Since $v' = af^{-1}a^*easaw + 1 - aa^- \in R^{-1}$, we get $v'a = af^{-1}a^*easawa$ and $a = (v')^{-1}af^{-1}a^*easawa$. Similarly, by $u' = af^{-1}a^*eawas + 1 - aa^- \in R^{-1}$, we have $u'a = af^{-1}a^*eawasa$ and $a = (u')^{-1}af^{-1}a^*eawasa$. As $w^{\|a}$ and $s^{\|a}$ exist, then $w^{\|a} = (v')^{-1}af^{-1}a^*easa$ and $s^{\|a} = (u')^{-1}af^{-1}a^*eawa$. So, $a^{\oplus}_{e,w} = w^{\|a}a^{(1,3)}_e e^{-1} = w^{\|a}a^{\dagger}_{e,f}e^{-1} = (v')^{-1}af^{-1}a^*easa \times (u^{-1}awasaf^{-1})^*$ and $a_{s,f^{-1},\oplus} = fa^{(1,4)}_f s^{\|a} = fa^{\dagger}_{e,f}s^{\|a} = f(u^{-1}awasaf^{-1})^*e \times (u')^{-1}af^{-1}a^*eawa$. $\square$

According to Lemma 3.12 and Theorem 3.14, we easily obtain characterizations for both weighted w-core invertible elements with weight e and weighted dual s-core invertible elements with weight f^{-1} by means of the inverse along an element in R.

Corollary 3.15. *Let $a, e, f, s, w \in R$ with $a^- \in a\{1\}$. If $s \in R^{\|a}$, then the following conditions are equivalent:*

(i) $a \in R^{\oplus}_{e,w} \cap R_{s,f^{-1},\oplus}$;

(ii) $w \in R^{\|a}$ and $a \in R^{\dagger}_{e,f}$;

(iii) $wasaf^{-1}a^*e \in R^{\|a}$;

(iv) $sawaf^{-1}a^*ea \in R^{\|a}$;

(v) $f^{-1}a^*eawas \in R^{\|a}$;

(vi) $f^{-1}a^*easaw \in R^{\|a}$.

In this case, $a^{\oplus}_{e,w} = (wasaf^{-1}a^*e)^{\|a} a_e^{(1,3)} e^{-1}$ *and* $a_{s,f^{-1},\oplus} = fa_f^{(1,4)}(f^{-1}a^*eawas)^{\|a}$.

Setting $s = w$ in Theorem 3.14, we get the following characterization of both the weighted w-core invertible elements with weight e and weighted dual w-core invertible elements with weight f^{-1} by units.

Theorem 3.16. *Let $e, f, w \in R$ and $a \in R$ be regular with $a^- \in a\{1\}$. Then the following conditions are equivalent:*

(i) $w \in R^{\|a}$ and $a \in R^{\dagger}_{e,f}$;

(ii) $a \in R^{\oplus}_{e,w} \cap R_{w,f^{-1},\oplus}$;

(iii) $u = awaf^{-1}a^*e + 1 - aa^- \in R^{-1}$;

(iv) $v = f^{-1}a^*eawa + 1 - a^-a \in R^{-1}$;

(v) $s = waf^{-1}a^*ea + 1 - a^-a \in R^{-1}$;

(vi) $t = af^{-1}a^*eaw + 1 - aa^- \in R^{-1}$.

In this case, $a^{\oplus}_{e,w} = t^{-1}af^{-1}a^*$ *and* $a_{f^{-1},w,\oplus} = a^*eas^{-1}$.

Proof. (i) $\Leftrightarrow$ (ii) It follows from Theorems 3.2 and 3.3.

(i) $\Rightarrow$ (iii) Since $a \in R^{\dagger}_{e,f}$, one can get that $af^{-1}a^*e + 1 - aa^- \in R^{-1}$ by Lemma 3.13. Also, by Lemmas 3.11 and 3.12, $w \in R^{\|a}$ implies $aw + 1 - aa^- \in R^{-1}$ and hence $awaa^- + 1 - aa^- \in R^{-1}$. Therefore, $(awaa^- + 1 - aa^-)(af^{-1}a^*e + 1 - aa^-) = awaf^{-1}a^*e + 1 - aa^- = u \in R^{-1}$.

(iii) $\Rightarrow$ (i) As $u = awaf^{-1}a^*e + 1 - aa^- \in R^{-1}$, we have $ua = awaf^{-1}a^*ea$ and hence $a = u^{-1}awaf^{-1}a^*ea \in Raf^{-1}a^*ea$. Then by Lemma 3.13 we have $a \in R^{\dagger}_{e,f}$, $a^{\dagger}_{e,f} = (u^{-1}awaf^{-1})^*e$ and $af^{-1}a^*e + 1 - aa^- \in R^{-1}$. Therefore, $awaa^- + 1 - aa^- = u(af^{-1}a^*e + 1 - aa^-)^{-1} \in R^{-1}$, which implies $aw + 1 - aa^- \in R^{-1}$, i.e., $w \in R^{\|a}$ by Lemma 3.12.

(iii) $\Leftrightarrow$ (v) It is obvious by Lemma 3.11.

Analogously, we can prove (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (vi). Note that $w^{\|a} = (aw)^{\#}a$ and $t = af^{-1}a^*eaw + 1 - aa^-$, we have $aw = t^{-1}af^{-1}a^*e(aw)^2$ and hence $(aw)^{\#} =$

$(t^{-1}af^{-1}a^*e)^2aw$. It follows from Theorems 3.2 and 3.3 that

$$\begin{aligned}a^{\oplus}_{e,w} &= w^{\|a}a_e^{(1,3)}e^{-1} = (t^{-1}af^{-1}a^*e)^2awaa_e^{(1,3)}e^{-1}\\ &= (t^{-1}af^{-1}a^*e)^2aw(aww^{\|a})a_e^{(1,3)}e^{-1}\\ &= (t^{-1}af^{-1}a^*e)(t^{-1}af^{-1}a^*e(aw)^2)w^{\|a}a_e^{(1,3)}e^{-1}\\ &= t^{-1}af^{-1}a^*e(aww^{\|a})a_e^{(1,3)}e^{-1} = t^{-1}af^{-1}a^*eaa_e^{(1,3)}e^{-1}\\ &= t^{-1}a(e^{-1}eaa_e^{(1,3)}af^{-1})^* = t^{-1}a(af^{-1})^* = t^{-1}af^{-1}a^*.\end{aligned}$$

Note also that $s = waf^{-1}a^*ea + 1 - a^-a$ and $w^{\|a} = a(wa)^{\#}$, then we have $wa = (wa)^2f^{-1}a^*eas^{-1}$ and $(wa)^{\#} = wa(f^{-1}a^*eas^{-1})^2$. As a consequence,

$$\begin{aligned}a_{w,f^{-1},\oplus} &= fa_f^{(1,4)}w^{\|a} = fa_f^{(1,4)}awa(f^{-1}a^*eas^{-1})^2\\ &= fa_f^{(1,4)}(w^{\|a}wa)wa(f^{-1}a^*eas^{-1})^2\\ &= fa_f^{(1,4)}w^{\|a}((wa)^2f^{-1}a^*eas^{-1})f^{-1}a^*eas^{-1}\\ &= fa_f^{(1,4)}(w^{\|a}wa)f^{-1}a^*eas^{-1} = fa_f^{(1,4)}af^{-1}a^*eas^{-1}\\ &= (eaf^{-1}fa_f^{(1,4)}a)^*as^{-1} = (ea)^*as^{-1} = a^*eas^{-1}.\end{aligned}$$

The proof is completed. □

Corollary 3.17. *Let $a, e, f, w \in R$ with $a^- \in a\{1\}$. Then the following conditions are equivalent:*

(i) $w \in R^{\|a}$ *and* $a \in R^{\dagger}_{e,f}$;
(ii) $a \in R^{\oplus}_{e,w} \cap R_{w,f^{-1},\oplus}$;
(iii) $waf^{-1}a^*e \in R^{\|a}$;
(iv) $f^{-1}a^*eaw \in R^{\|a}$;
(v) $a \in awaf^{-1}a^*eaR \cap Rawaf^{-1}a^*ea$;
(vi) $a \in af^{-1}a^*eawaR \cap Raf^{-1}a^*eawa$. *In this case,* $a^{\oplus}_{e,w} = af^{-1}a^*eax(yawaf^{-1})^*$, $a_{w,f^{-1},\oplus} = a^*eax$, $w^{\|a} = awa(f^{-1}a^*eax)^2$, $a^{\dagger}_{e,f} = f^{-1}(yawa)^*e$, *where* $x, y \in R$ *satisfy* $a = awaf^{-1}a^*eax = yawaf^{-1}a^*ea$.

P r o o f. The results above can be easily obtained by Lemma 3.12 and Theorem 3.16. The representations for the $a^{\oplus}_{e,w}$, $a_{w,f^{-1},\oplus}$, $w^{\|a}$ and $a^{\dagger}_{e,f}$ are given below.

As $waf^{-1}a^*e \in R^{\|a}$, we have $a = awaf^{-1}a^*eax$ and $a = yawaf^{-1}a^*ea$ for some $x, y \in R$. From $a = awaf^{-1}a^*eax$ we obtain $wa = (wa)^2f^{-1}a^*eax$ and hence $(wa)^{\#} = wa(f^{-1}a^*eax)^2$. Then $w^{\|a} = a(wa)^{\#} = awa(f^{-1}a^*eax)^2$. Since $a = yawaf^{-1}a^*ea \in Ra^*ea$, we have $(yawaf^{-1})^*e \in a\{e, 1, 3\}$ by Lemma 3.1. Therefore,

$$\begin{aligned}a^{\dagger}_{e,f} &= a_f^{(1,4)}aa_e^{(1,3)} = f^{-1}fa_f^{(1,4)}a(yawaf^{-1})^*e\\ &= f^{-1}(yawaf^{-1}fa_f^{(1,4)}a)^*e = f^{-1}(yawa)^*e.\end{aligned}$$

Applying Theorems 3.2 and 3.3, we get

$$\begin{aligned} a^{\oplus}_{e,w} &= w^{\|a} a_e^{(1,3)} e^{-1} = awa(f^{-1}a^*eax)^2 a_e^{(1,3)} e^{-1} \\ &= (awaf^{-1}a^*eax)f^{-1}a^*eax(yawaf^{-1})^* ee^{-1} \\ &= af^{-1}a^*eax(yawaf^{-1})^*, \\ a_{w,f^{-1},\oplus} &= fa_f^{(1,4)} w^{\|a} = fa_f^{(1,4)} awa(f^{-1}a^*eax)^2 \\ &= fa_f^{(1,4)}(awaf^{-1}a^*eax)f^{-1}a^*eax \\ &= fa_f^{(1,4)} af^{-1}a^*eax = (af^{-1}fa_f^{(1,4)}a)^*eax = a^*eax. \end{aligned}$$

□

In 2011, Mary in [15] told us that $a^{\|a^*} = a^\dagger$, and it can be easily seen that $a \in R^\dagger$ if and only if $(a^*)^{\|a}$ exists. Setting $w = a^*$ in Theorem 3.16, we can easily obtain the following result, whose proof is left to the reader.

Corollary 3.18. *Let $a \in R$. Then the following conditions are equivalent:*

(i) $a \in R^{\oplus}_{e,a^*} \cap R_{a^*,f^{-1},\oplus}$;

(ii) $a \in R^\dagger_{e,f} \cap R^\dagger$.

In this case, $a^{\oplus}_{e,a^*} = (a^\dagger)^* a^\dagger e^{-1}$, $a_{a^*,f^{-1},\oplus} = fa^\dagger(a^\dagger)^*$, $a^\dagger = (a^{\oplus}_{e,a^*}ea)^* = (afa_{a^*,f^{-1},\oplus})^*$ *and* $a^\dagger_{e,f} = f^{-1}a_{a^*,f^{-1},\oplus}a^*aa^*a^{\oplus}_{e,a^*}e$.

4. Relations with other generalized inverses

In this section, the relations among the weighted w-core inverse, the weighted dual s-core inverse, the (v,w)-(b,c)-inverse, the e-core inverse, the dual f-core inverse and the weighted Moore-Penrose inverse are investigated. For given complex tensors $\mathcal{M}$, $\mathcal{N}$, $\mathcal{B}$, $\mathcal{C}$, the $(\mathcal{M},\mathcal{N})$-weighted $(\mathcal{B},\mathcal{C})$-inverse of a tensor was firstly defined by Mosić et al. in [19], extending the notation of $(\mathcal{B},\mathcal{C})$-inverse of a complex tensor. In [8], Drazin introduced the (v,w)-(b,c)-inverse in a semigroup. Given any semigroup S and any $a,b,c,v,w,y \in S$, an element a is the (v,w)-(b,c)-invertible (see [8]) if there exists $y \in R$ such that $y \in bSwy \cap yvSc$, $yvawb = b$ and $cvawy = c$ (and two other cases of mutual equivalence are also introduced). The (v,w)-(b,c)-inverse of a is unique if it exists, and is denoted by $a^{(b,c)}_{v,w}$. By $R^{(b,c)}_{v,w}$ we denote the set of all (v,w)-(b,c)-invertible elements in R.

More results on (v,w)-(b,c)-inverses can be referred to [8], [19].

Lemma 4.1 ([8], Proposition 2.3)**.** *Let $a,b,c,v,w,y \in R$. Then the following statements are equivalent:*

(i) *y is the (v,w)-(b,c)-inverse of a;*
(ii) *y is the (b,c)-inverse of vaw.*

One can apply Lemma 4.1 to transform study about (v,w)-(b,c)-inverses into the corresponding study about (b,c)-inverses.

Next, we present the relationship between the (v,w)-(b,c)-inverse and the weighted w-core inverse of a with weight v in R.

Theorem 4.2. *Let $a,v,w,s,t\in R$. Then we have:*
(i) *If $a\in R^{\oplus}_{v,w}$, then $a\in R^{(a,a^*)}_{(v,w)}$ and $a^{(a,a^*)}_{(v,w)}=a^{\oplus}_{v,w}$.*
(ii) *If $a\in R_{s,t,\oplus}$, then $a\in R^{(a^*,a)}_{(s,t)}$ and $a^{(a^*,a)}_{(s,t)}=a_{s,t,\oplus}$.*

Proof. (i) Suppose that $a\in R^{\oplus}_{v,w}$ and $x=a^{\oplus}_{v,w}$. It follows from Proposition 2.4 and Theorem 3.2 that $xvawx=x$, $xR=aR$ and $Rx=Ra^*$. Therefore, x is (v,w)-(a,a^*)-invertible of a by Lemma 4.1.

(ii) can be proved similarly. □

In view of Theorem 4.2, we naturally want to know whether a is weighted w-core invertible with weight v when it is (v,w)-(a,a^*)-invertible. If not, under what conditions can it be established. A counterexample and a characterization are given below.

Example 4.3. Let R be the ring of all 2×2 complex matrices with transpose as the involution $*$. Suppose $v=\begin{bmatrix}1&1\\0&1\end{bmatrix}$, $w=\begin{bmatrix}1&-1\\0&1\end{bmatrix}$ and $a=\begin{bmatrix}1&0\\0&1\end{bmatrix}\in R$. Then a is (v,w)-(a,a^*)-invertible and $a^{(a,a^*)}_{(v,w)}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$. If a is weighted w-core invertible with weight v and $x=a^{\oplus}_{v,w}$, from the equation $xvawa=a$ by Definition 2.1, we can get $x=a^{\oplus}_{v,w}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, but $awx=\begin{bmatrix}1&-1\\0&1\end{bmatrix}\neq(awx)^*$, which is a contradiction. Thus, $a\notin R^{\oplus}_{v,w}$.

Theorem 4.4. *Let $a,v,w,s,t\in R$. Then:*
(i) *If v is Hermitian, then a is weighted w-core invertible with weight v if and only if a is (v,w)-(a,a^*)-invertible. In this case, the (v,w)-(a,a^*)-inverse of a coincides with the weighted w-core inverse of a with weight v.*
(ii) *If t is Hermitian, then a is weighted dual s-core invertible with weight t if and only if a is (s,t)-(a^*,a)-invertible. In this case, the (s,t)-(a^*,a)-inverse of a coincides with the weighted dual s-core inverse of a with weight t.*

Proof. (i) The "only if" part follows from Theorem 4.2. Thus, it suffices to prove the "if" part. Suppose a is (v,w)-(a,a^*)-invertible and $x\in R$ is the (v,w)-(a,a^*)-inverse of a. Then by Lemma 4.1, x is the (a,a^*)-inverse of vaw. Hence, we have $xvawa=a$, $a^*vawx=a^*$ and $x\in aR$.

By $a^* = a^*vawx$ and $v^* = v$, we get $a = (awx)^*va$, and thus $awx = (awx)^*vawx$, i.e., $(awx)^* = awx$. Therefore $a = (awx)^*va = awxva$. Since $x \in aR$, we get $x = at = awxvat = awxvx$ for some $t \in R$. Thus, x is the weighted w-core inverse of a with weight v.

(ii) The proof is similar to the proof of (i). □

We next give the connection between the e-core inverse and the weighted w-core inverse with weight e. Before this, the following lemma is presented.

Lemma 4.5 ([18], Theorems 2.1 and 2.2)**.** *Let $a, e, f \in R$. Then we have the following results:*

(i) *$a \in R_e^{\oplus}$ if and only if $a \in R^{\#} \cap R_e^{\{1,3\}}$. In this case, $a_e^{\oplus} = a^{\#}aa_e^{(1,3)}$.*

(ii) *$a \in R_{f,\oplus}$ if and only if $a \in R^{\#} \cap R_f^{\{1,4\}}$. In this case, $a_{f,\oplus} = a_f^{(1,4)}aa^{\#}$.*

Proposition 4.6. *Let $a, e \in R$. Then the following conditions are equivalent:*

(i) *$a \in R_e^{\oplus}$;*

(ii) *$a \in R^{\#} \cap R_e^{\{1,3\}}$;*

(iii) *$a \in R_{e,1}^{\oplus}$;*

(iv) *$a \in R_{e,a}^{\oplus}$;*

(v) *there exists some $x \in R$ such that $axea = a$, $xR = aR$ and $Rx = Ra^*$;*

(vi) *there exists some $y \in R$ such that $a^2yea = a$, $yR = aR$ and $Ry = Ra^*$.*

In this case, $a_e^{\oplus} = aa_{e,a}^{\oplus}e = a_{e,1}^{\oplus}e$, $a_{e,1}^{\oplus} = a_e^{\oplus}e^{-1}$ and $a_{e,a}^{\oplus} = a^{\#}a_e^{\oplus}e^{-1}$.

Proof. By Definition 2.1, Theorem 3.2 and Lemma 4.5, we can easily get the equivalences of (i) to (vi). We next give the representations of $a_e^{\oplus}$, $a_{e,1}^{\oplus}$ and $a_{e,a}^{\oplus}$.

One observes that $x, y \in R$ satisfying conditions (v) and (vi) are the weighted 1-core inverse of a with weight e and the weighted a-core inverse of a with weight e, respectively. Then $x = 1^{\|a}a_e^{(1,3)}e^{-1} = a^{\#}aa_e^{(1,3)}e^{-1} = a_e^{\oplus}e^{-1}$ and $y = a^{\|a}a_e^{(1,3)}e^{-1} = a^{\#}a_e^{(1,3)}e^{-1} = a^{\#}a^{\#}aa_e^{(1,3)}e^{-1} = a^{\#}a_e^{\oplus}e^{-1}$. So, $a_e^{\oplus} = a_{e,1}^{\oplus}e$ and $a_e^{\oplus} = a^{\#}aa_e^{(1,3)} = aa^{\#}a_e^{\oplus} = aa_{e,a}^{\oplus}e$. □

Proposition 4.7. *Let $a, f \in R$. Then the following conditions are equivalent:*

(i) *$a \in R_{f,\oplus}$;*

(ii) *$a \in R^{\#} \cap R_f^{\{1,4\}}$;*

(iii) *$a \in R_{1,f^{-1},\oplus}$;*

(iv) *$a \in R_{a,f^{-1},\oplus}$;*

(v) *there exists some $x \in R$ such that $afxa = a$, $xR = a^*R$ and $Rx = Ra$;*

(vi) *there exists some $y \in R$ such that $afya^2 = a$, $yR = a^*R$ and $Ry = Ra$.*

In this case, $a_{f,\oplus} = f^{-1}a_{1,f^{-1},\oplus} = f^{-1}a_{a,f^{-1},\oplus}a$, $a_{1,f^{-1},\oplus} = fa_{f,\oplus}$ and $a_{a,f^{-1},\oplus} = fa_{f,\oplus}a^{\#}$.

The following result proves that the weighted $f^{-1}a^*e$-core invertibility with weight e and the weighted dual $f^{-1}a^*e$-core invertibility with weight f^{-1} of a are consistent with the weighted Moore-Penrose invertibility.

Proposition 4.8. *Let $a, e, f \in R$. Then the following conditions are equivalent:*
(i) $a \in R^{\dagger}_{e,f}$;
(ii) $a \in R^{\oplus}_{e,f^{-1}a^*e}$;
(iii) $a \in R_{f^{-1}a^*e,f^{-1},\oplus}$.
In this case, $a^{\dagger}_{e,f} = (a^{\oplus}_{e,f^{-1}a^*e}eaf^{-1})^*e = f^{-1}(eaf^{-1}a_{f^{-1}a^*e,f^{-1},\oplus})^*$, $a^{\oplus}_{e,f^{-1}a^*e} = e^{-1}(a^{\dagger}_{e,f})^*fa^{\dagger}_{e,f}e^{-1}$ *and* $a_{f^{-1}a^*e,f^{-1},\oplus} = fa^{\dagger}_{e,f}e^{-1}(a^{\dagger}_{e,f})^*f$.

Proof. (i) $\Rightarrow$ (ii) Given $a \in R^{\dagger}_{e,f}$, by Lemma 3.13, we have $a \in af^{-1}a^*eaR$ and $a \in Raf^{-1}a^*ea$, and hence $(f^{-1}a^*e) \in R^{\|a}$. It is clear that $a \in R^{\dagger}_{e,f}$ gives $a \in R^{\{1,3\}}_e$. Therefore $a \in R^{\oplus}_{e,f^{-1}a^*e}$.

(ii) $\Rightarrow$ (iii) By Theorem 3.2, we know that $a \in R^{\oplus}_{e,f^{-1}a^*e}$ implies $(f^{-1}a^*e) \in R^{\|a}$ and hence $a \in af^{-1}a^*eaR \cap Raf^{-1}a^*ea$ and $a \in R^{\dagger}_{e,f}$ by Lemma 3.13. From Theorems 3.2 and 3.3, it is known that $a \in R^{\oplus}_{e,f^{-1}a^*e}$ if and only if both $(f^{-1}a^*e)^{\|a}$ and $a^{(1,3)}_e$ exist if and only if $(f^{-1}a^*e)^{\|a}$ exists if and only if both $(f^{-1}a^*e)^{\|a}$ and $a^{(1,4)}_f$ exist if and only if $a \in R_{f^{-1}a^*e,f^{-1},\oplus}$.

(iii) $\Rightarrow$ (i) Assume that $y \in R$ is the weighted dual $f^{-1}a^*e$-core inverse of a with weight f^{-1}. Then we have $a = a(f^{-1}a^*e)af^{-1}y \in af^{-1}a^*eaR$ by Definition 2.6. Therefore, $a \in R^{\dagger}_{e,f}$ and $f^{-1}(eaf^{-1}y)^*$ is the weighted Moore-Penrose inverse of a by Lemma 3.13.

Herein, the representations for the $a^{\dagger}_{e,f}$, $a^{\oplus}_{e,f^{-1}a^*e}$ and $a_{a,f^{-1},\oplus}$ can be easily calculated by Theorems 3.2 and 3.3 and Lemma 3.13.

If $a \in R^{\oplus}_{e,f^{-1}a^*e}$, then $a = a^{\oplus}_{e,f^{-1}a^*e}eaf^{-1}a^*ea \in Raf^{-1}a^*ea$, and hence $a \in R^{\dagger}_{e,f}$ and $a^{\dagger}_{e,f} = (a^{\oplus}_{e,f^{-1}a^*e}eaf^{-1})^*e$ by Lemma 3.13. If $a \in R^{\dagger}_{e,f}$, then $a^{\dagger}_{e,f} = f^{-1}(eax)^*$, where $x \in R$ satisfies $a = af^{-1}a^*eax \in af^{-1}a^*eaR$. By Theorems 3.2 and 3.3, we obtain $a^{\oplus}_{e,f^{-1}a^*e} = (f^{-1}a^*e)^{\|a}a^{(1,3)}_e e^{-1} = axa^{(1,3)}_e e^{-1} = ((ax)^*)^*a^{(1,3)}_e e^{-1} = (fa^{\dagger}_{e,f}e^{-1})^*a^{\dagger}_{e,f}e^{-1} = e^{-1}(a^{\dagger}_{e,f})^*fa^{\dagger}_{e,f}e^{-1}$.

Similarly, $a_{f^{-1}a^*e,f^{-1},\oplus} = fa^{\dagger}_{e,f}e^{-1}(a^{\dagger}_{e,f})^*f$. □

Lemma 4.1 above tells us the relationship between (v,w)-(b,c)-inverses and (b,c)-inverses. It is well known that the (b,c)-inverse encompasses the inverse along an element. Then we obtain that a is the (v,w)-(d,d)-invertible if and only if vaw is (d,d)-invertible if and only if vaw is invertible along d.

The following theorem illustrates that the equivalence relations among the weighted w-core inverse, the weighted dual s-core inverse and the weighted inverse along an element.

Theorem 4.9. *Let $a, w, s, e, f \in R$ and let $a \in R^{\dagger}_{e,f}$. Then:*

(i) *a is weighted w-core invertible with weight e if and only if eaw is invertible along $af^{-1}a^*$. In this case, the weighted w-core inverse of a with weight e coincides with the inverse of eaw along $af^{-1}a^*$.*

(ii) *a is weighted dual s-core invertible with weight f if and only if saf is invertible along a^*ea. In this case, the weighted dual s-core inverse of a with weight f coincides with the inverse of saf along a^*ea.*

Proof. (i) Suppose that a is weighted w-core invertible with weight e and $x \in R$ is the weighted w-core inverse of a with weight e. By Theorem 3.2, we have $awxea = a$, $xeawx = x$, $xeawa = a$, $awxex = x$ and $(awx)^* = awx$.

We next prove that x is the inverse of eaw along $d = af^{-1}a^*$. Assuming $x = a^{\oplus}_{e,w}$, we have

(1) $xeawd = xeawaf^{-1}a^* = (xeawa)f^{-1}a^* = af^{-1}a^* = d$ and $deawx = af^{-1}a^*eawx = af^{-1}a^*e(awx)^* = af^{-1}(awxea)^* = af^{-1}a^* = d$,

(2) $x = awxex = (af^{-1}fa^{\dagger}_{e,f}a)wxex = af^{-1}(fa^{\dagger}_{e,f}a)^*wxex = af^{-1}a^*(a^{\dagger}_{e,f})^* f^*wxex = d(a^{\dagger}_{e,f})^*f^*wxex \in dR$,

(3) $x = xeawx = xe(awx)^* = xe(wx)^*a^* = xe(wx)^*(af^{-1}fa^{\dagger}_{e,f}a)^* = xe(wx)^* \times (fa^{\dagger}_{e,f}a)^*f^{-1}a^* = xe(wx)^*fa^{\dagger}_{e,f}af^{-1}a^* = xe(wx)^*fa^{\dagger}_{e,f}d \in Rd$. Therefore x is the inverse of eaw along $af^{-1}a^*$.

For the converse, to illustrate that a is weighted w-core invertible with weight e, it suffices to find an element $y \in R$ (indeed, $y = (eaw)^{\parallel af^{-1}a^*}$) satisfying $awy = (awy)^*$, $yeawa = a$ and $awyey = y$. Note that $yeawaf^{-1}a^* = af^{-1}a^* = af^{-1}a^*eawy$ and $y \in af^{-1}a^*R$, consequently $y = af^{-1}a^*t$ for some $t \in R$. Then we have

(1) $awy = (e^{-1}eaa^{\dagger}_{e,f}a)wy = e^{-1}(eaa^{\dagger}_{e,f})^*awy = e^{-1}(a^{\dagger}_{e,f})^*a^*eawy = e^{-1} \times (f^{-1}fa^{\dagger}_{e,f}aa^{\dagger}_{e,f})^*a^*eawy = e^{-1}(a^{\dagger}_{e,f})^*fa^{\dagger}_{e,f}(af^{-1}a^*eawy) = e^{-1}(a^{\dagger}_{e,f})^*fa^{\dagger}_{e,f}a \times f^{-1}a^* = e^{-1}(af^{-1}fa^{\dagger}_{e,f}aa^{\dagger}_{e,f})^* = e^{-1}(aa^{\dagger}_{e,f})^* = e^{-1}(eaa^{\dagger}_{e,f})^*e^{-1}$, i.e., $(awy)^* = awy$,

(2) $yeawa = yeaw(aa^{\dagger}_{e,f}a) = yeawa(f^{-1}fa^{\dagger}_{e,f}a) = yeawaf^{-1}(fa^{\dagger}_{e,f}a)^* = (yeawaf^{-1}a^*)(a^{\dagger}_{e,f})^*f = af^{-1}a^*(a^{\dagger}_{e,f})^*f = af^{-1}(fa^{\dagger}_{e,f}a)^* = aa^{\dagger}_{e,f}a = a$,

(3) $awyey = awye(af^{-1}a^*t) = (e^{-1}(aa^{\dagger}_{e,f})^*)eaf^{-1}a^*t = e^{-1}(eaa^{\dagger}_{e,f})^*af^{-1}a^*t = e^{-1}eaa^{\dagger}_{e,f}af^{-1}a^*t = af^{-1}a^*t = y$. So, y is the weighted w-core inverse of a with weight e.

(ii) can be proved similarly. □

As a consequence of Theorem 4.9, we have the following corollary.

Corollary 4.10 ([28], Theorem 2.25)**.** *Let $a, w, v \in R$ and $a \in R^{\dagger}$. Then:*

(i) *$a \in R_w^{\oplus}$ if and only if aw is invertible along aa^*. In this case, the w-core inverse of a coincides with the inverse of aw along aa^*.*

(ii) *$a \in R_{v,\oplus}$ if and only if va is invertible along a^*a. In this case, the dual v-core inverse of a coincides with the inverse of va along a^*a.*

Acknowledgments. The authors are highly grateful to Professor Matej Brešar and the referee for his/her valuable comments and suggestions which greatly improved this paper.

References

[1] *O. M. Baksalary, G. Trenkler*: Core inverse of matrices. Linear Multilinear Algebra *58* (2010), 681–697. zbl MR doi
[2] *J. Benítez, E. Boasso*: The inverse along an element in rings. Electron. J. Linear Algebra *31* (2016), 572–592. zbl MR doi
[3] *J. Benítez, E. Boasso*: The inverse along an element in rings with an involution, Banach algebras and C^*-algebras. Linear Multilinear Algebra *65* (2017), 284–299. zbl MR doi
[4] *J. Benítez, E. Boasso, H. Jin*: On one-sided (b, c)-inverses of arbitrary matrices. Electron. J. Linear Algebra *32* (2017), 391–422. zbl MR doi
[5] *R. E. Cline*: An Application of Representation for the Generalized Inverse of a Matrix. MRC Technical Report 592. University of Wisconsin, Madison, 1965.
[6] *M. P. Drazin*: Pseudo-inverses in associative rings and semigroups. Am. Math. Mon. *65* (1958), 506–514. zbl MR doi
[7] *M. P. Drazin*: A class of outer generalized inverses. Linear Algebra Appl. *436* (2012), 1909–1923. zbl MR doi
[8] *M. P. Drazin*: Weighted (b, c)-inverses in categories and semigroups. Commun. Algebra *48* (2020), 1423–1438. zbl MR doi
[9] *D. E. Ferreyra, F. E. Levis, N. Thome*: Revisiting the core EP inverse and its extension to rectangular matrices. Quaest. Math. *41* (2018), 265–281. zbl MR doi
[10] *Y. Gao, J. Chen*: Pseudo core inverses in rings with involution. Commun. Algebra *46* (2018), 38–50. zbl MR doi
[11] *R. E. Hartwig, J. Luh*: A note on the group structure of unit regular ring elements. Pac. J. Math. *71* (1977), 449–461. zbl MR doi
[12] *N. Jacobson*: The radical and semi-simplicity for arbitrary rings. Am. J. Math. *67* (1945), 300–320. zbl MR doi
[13] *T. Li, J. Chen*: Characterizations of core and dual core inverses in rings with involution. Linear Multilinear Algebra *66* (2018), 717–730. zbl MR doi
[14] *S. B. Malika, N. Thome*: On a new generalized inverse for matrices of an arbitrary index. Appl. Math. Comput. *226* (2014), 575–580. zbl MR doi
[15] *X. Mary*: On generalized inverses and Green's relations. Linear Algebra Appl. *434* (2011), 1836–1844. zbl MR doi
[16] *X. Mary, P. Patrício*: The inverse along a lower triangular matrix. Appl. Math. Comput. *219* (2012), 886–891. zbl MR doi
[17] *X. Mary, P. Patrício*: Generalized inverses modulo $\mathcal{H}$ in semigroups and rings. Linear Multilinear Algebra *61* (2013), 1130–1135. zbl MR doi
[18] *D. Mosić, C. Deng, H. Ma*: On a weighted core inverse in a ring with involution. Commun. Algebra *46* (2018), 2332–2345. zbl MR doi

[19] *D. Mosić, P. S. Stanimirović, J. K. Sahoo, R. Behera, V. N. Katsikis*: One-sided weighted outer inverses of tensors. J. Comput. Appl. Math. *388* (2021), Article ID 113293, 23 pages. zbl MR doi
[20] *R. Penrose*: A generalized inverse for matrices. Proc. Camb. Philos. Soc. *51* (1955), 406–413. zbl MR doi
[21] *K. M. Prasad, R. B. Bapat*: The generalized Moore-Penrose inverse. Linear Algebra Appl. *165* (1992), 59–69. zbl MR doi
[22] *K. M. Prasad, K. S. Mohana*: Core-EP inverse. Linear Multilinear Algebra *62* (2014), 792–802. zbl MR doi
[23] *D. S. Rakić, N. Č. Dinčić, D. S. Djordjević*: Group, Moore-Penrose, core and dual core inverse in rings with involution. Linear Algebra Appl. *463* (2014), 115–133. zbl MR doi
[24] *C. R. Rao, S. K. Mitra*: Generalized Inverse of a Matrices and Its Application. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, New York, 1971. zbl MR
[25] *H. Wang, X. Liu*: Characterizations of the core inverse and the core partial ordering. Linear Multilinear Algebra *63* (2015), 1829–1836. zbl MR doi
[26] *H. Zhu, Q.-W. Wang*: Weighted pseudo core inverses in rings. Linear Multilinear Algebra *68* (2020), 2434–2447. zbl MR doi
[27] *H. Zhu, Q.-W. Wang*: Weighted Moore-Penrose inverses and weighted core inverses in rings with involution. Chin. Ann. Math., Ser. B *42* (2021), 613–624. zbl MR doi
[28] *H. Zhu, L. Wu, J. Chen*: A new class of generalized inverses in semigroups and rings with involution. To appear in Comm. Algebra. doi
[29] *H. Zhu, L. Wu, D. Mosić*: One-sided w-core inverses in rings with an involution. To appear in Linear Multilinear Algebra. doi

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