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ON THE DIVISOR FUNCTION OVER PIATETSKI-SHAPIRO SEQUENCES

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Abstract. Let $[x]$ be an integer part of x and $d(n)$ be the number of positive divisor of n. Inspired by some results of M. Jutila (1987), we prove that for $1 < c < \frac{6}{5}$,

$$\sum_{n\leqslant x} d([n^c]) = cx\log x + (2\gamma - c)x + O\Big(\frac{x}{\log x}\Big),$$

where γ is the Euler constant and $[n^c]$ is the Piatetski-Shapiro sequence. This gives an improvement upon the classical result of this problem.

Keywords: divisor function; Piatetski-Shapiro sequence; exponential sum

MSC 2020: 11B83, 11L07, 11N25, 11N37

1. INTRODUCTION

The Piatetski-Shapiro sequences are sequences of the form

$$([n^c])_{n=1}^{\infty},$$

where $c > 1$ and $c \notin \mathbb{N}$. Let $[x]$ be the largest integer not exceeding x. Using the prime number theorem and some elementary calculation, we can easily prove that

$$\sum_{\substack{n\leqslant x\\ [n^c]\in\mathbb{P}}} 1 \sim \frac{x}{c\log x} \quad \text{as } x\to\infty \tag{1.1}$$

for $0 < c \leqslant 1$ and $\mathbb{P}$ is the set of primes.

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For $c > 1$, a classical result of finding primes in such sparse sequences is attributed to Piatetski-Shapiro, who proved that (1.1) holds if c is a fixed number lying in the range $1 < c < \frac{12}{11}$. Naturally, one would like the range of c to be as large as possible. When c is a positive integer larger than 2, $[n^c]$ is no longer to be a prime, so the left-hand side of equation (1.1) vanishes. In this direction, many experts have made significant contributions, see, e.g., [1], [4], [6], [7], [8], [10] and the references therein. At present, the best result is obtained by Rivat and Sargos (see [11]), who proved that (1.1) holds for $1 < c < \frac{2817}{2426}$. On the other hand, Rivat and Wu in [12] have proven that for $c \in (1, \frac{243}{205})$, there are infinitely many Piatetski-Shapiro primes.

In this paper, we are interested in the divisors of the sequence $[n^c]$. Let $d(n)$ be the number of positive integer solutions to equation $x_1 x_2 = n$. The estimation of the error term of the asymptotic formula of sum $\sum\limits_{n \leqslant x} d(n)$ is called the *Dirichlet divisor problem*, which is a famous problem in number theory. In 1999, Arkhipov, Soliba and Chubarikov in [2] proved that when $1 < c < \frac{8}{7}$,

$$\sum_{n \leqslant x} d([n^c]) = xQ(\log x) + O\Big(\frac{x}{\log x}\Big),$$

where $Q(x)$ is a polynomial of degree 1. Later, Lü and Zhai in [9] improved the range of c to $1 < c < \frac{495}{433}$ by involving the theory of exponent pairs. One may note that $\frac{495}{433} \approx 1.143187$ and $\frac{8}{7} \approx 1.142857$.

In this paper, we consider the asymptotic formula for $\sum\limits_{n \leqslant x} d([n^c])$, where $d(n)$ denotes the number of positive divisor of n. On this subject, we have the following result. We can give a further improvement upon the range of c.

Theorem 1.1. *Let $1 < c < \frac{6}{5}$. Then we have*

$$\sum_{n \leqslant x} d([n^c]) = cx \log x + (2\gamma - c)x + O\Big(\frac{x}{\log x}\Big), \tag{1.2}$$

where γ is the Euler constant.

Notations 1.1. Throughout the paper, $c > 1$ is a fixed number and we set $\beta = 1/c$. The symbols η and ε are small positive real numbers, where ε may not necessarily be the same at different occurrences. As usual, $e(z) = \exp(2\pi i z) = e^{2\pi i z}$. The symbol $k \sim K$ means $\frac{1}{2}K \leqslant k \leqslant 2K$. We write $f = O(g)$ or $f \ll g$ to mean $|f| \leqslant c_0 g$ for some unspecified positive constant c_0. We denote $f \asymp g$ to mean that $f \ll g$ and $g \ll f$.

2. Preliminaries

In this section, we quote the results needed later. Firstly, we need the following asymptotic formula for the divisor function $d(n)$.

Lemma 2.1. *Let* $x \geqslant 1$, *then*

$$\sum_{n \leqslant x} d(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}),$$

where γ *is the Euler constant.*

We shall use the following approximation of the saw-tooth function $\psi(x) = x - [x] - \frac{1}{2} \in [-\frac{1}{2}, \frac{1}{2})$.

Lemma 2.2. *For* $0 < |t| < 1$, *let*

$$W(t) = \pi t(1 - |t|) \cot \pi t + |t|.$$

Fix a positive integer J. *For* $x \in \mathbb{R}$ *define*

$$\psi^*(x) := - \sum_{1 \leqslant |j| \leqslant J} (2\pi \mathrm{i} j)^{-1} W\Big(\frac{j}{J+1}\Big) e(jx)$$

and

$$\delta(x) := \frac{1}{2(J+1)} \sum_{|j| \leqslant J} \Big(1 - \frac{|j|}{J+1}\Big) e(jx). \tag{2.1}$$

Then δ *is nonnegative, and we have*

$$|\psi^*(x) - \psi(x)| \leqslant \delta(x)$$

for all real numbers x.

Proof. See Vaaler [13], Theorem 18. □

We shall also use the following estimate for a sum involving function δ.

Lemma 2.3. *Fix* $0 < \beta < 1$. *Assume that* $1 \leqslant N < N_1 \leqslant 2N$. *Define the function* δ *as in* (2.1). *Then*

$$\sum_{N < n \leqslant N_1} \delta(-n^{\beta}) \ll J^{-1} N + J^{1/2} N^{\beta/2}.$$

Proof. See [3], Chapter 4, page 48. □

To estimate the exponential sums, we need the following lemma.

Lemma 2.4. *Let* $2 \leqslant M < M' \leqslant 2M$, *and let* f *be a holomorphic function in the domain*

$$D = \{z\colon\ |z-x| < cM \quad \text{for some } x \in [M, M']\},$$

where c *is a positive constant. Suppose that* $f(x)$ *is real for* $M \leqslant x \leqslant M'$, *and that either*

$$f(z) = Bz^{\alpha}(1 + O(F^{-1/3})) \quad \text{for } z \in D,$$

where $\alpha \neq 0, 1$ *is a fixed real number, and*

$$F = |B|M^{\alpha}, \quad \text{or} \quad f(z) = B\log z(1 + o(F^{-1/3})) \quad \text{for } z \in D,$$

where $F = |B|$.

Let $g \in C^1[M, M']$, *and suppose that* $M \leqslant x \leqslant M'$,

$$|g(x)| \ll G, \quad |g'(x)| \ll G'.$$

Suppose also that $M^{3/4} \ll F \ll M^{3/2}$, *then*

$$\Biggl|\sum_{M \leqslant m \leqslant M'} d(m)g(m)e(f(m))\Biggr| \ll (G + FG')M^{1/2}F^{1/3+\varepsilon}.$$

P r o o f. See Jutila [5], Lemma 4.6. □

3. Proof of Theorem 1.1

Throughout the proof, let $\beta = 1/c$. Then $[n^c] = m$ is equivalent to

$$-(m+1)^{\beta} < -n \leqslant -m^{\beta}.$$

Therefore, we have

$$\begin{aligned} S := \sum_{n \leqslant x} d([n^c]) &= \sum_{m \leqslant x^c} ([-m^{\beta}] - [-(m+1)^{\beta}])d(m) + O(x^{\varepsilon}) \\ &= S_1 + S_2 + O(x^{\varepsilon}), \end{aligned} \tag{3.1}$$

where

$$S_1 = \sum_{m \leqslant x^c} ((m+1)^{\beta} - m^{\beta})d(m) \quad \text{and} \quad S_2 = \sum_{m \leqslant x^c} (\psi(-(m+1)^{\beta}) - \psi(-m^{\beta}))d(m)$$

with $\psi(x)$ being the saw-tooth function in Lemma 2.2. Using partial summation, Lemma 2.1 and the Taylor expansion

$$(x+1)^{\beta} - x^{\beta} = \beta x^{\beta-1} + O(x^{\beta-2})$$

for $x \geqslant 1$, we deduce that

$$\begin{aligned} S_1 &= \sum_{m \leqslant x^c} ((m+1)^{\beta} - m^{\beta}) d(m) = \beta \sum_{m \leqslant x^c} d(m) m^{\beta-1} + O\biggl(\sum_{m \leqslant x^c} d(m) m^{\beta-2}\biggr) \\ &= cx \log x + (2\gamma - c)x + O_c(\sqrt{x}), \end{aligned}$$

where the O-term only depends on c. Replacing x^c by M and breaking into dyadic intervals, the remaining task is to prove that for small $\eta > 0$,

$$S_2 \ll (\log 2M) \sum_{m \sim M} (\psi(-(m+1)^{\beta}) - \psi(-m^{\beta})) d(m) \ll M^{\beta-\eta}.$$

For convenience of calculation, we write

$$S_2^* := \sum_{m \sim M} (\psi(-(m+1)^{\beta}) - \psi(-m^{\beta})) d(m).$$

By Lemma 2.2, for any $J > 0$ there exist functions ψ^* and δ ($\geqslant 0$) such that

$$\psi(x) = \psi^*(x) + O(\delta(x)),$$

where

$$\psi^*(x) = \sum_{1 \leqslant |j| \leqslant J} a(j) e(jx), \quad \delta(x) = \sum_{|j| \leqslant J} b(j) e(jx) \tag{3.2}$$

with

$$a(j) \ll j^{-1}, \quad b(j) \ll J^{-1}.$$

Hence,

$$\begin{aligned} S_2^* &= \sum_{m \sim M} d(m) (\psi^*(-(m+1)^{\beta}) - \psi^*(-m^{\beta})) \\ &\quad + O\biggl(M^{\varepsilon} \sum_{m \sim M} (\delta(-(m+1)^{\beta}) + \delta(-m^{\beta}))\biggr) \\ &= S_3 + O(M^{\varepsilon} S_4), \end{aligned}$$

say. By Lemma 2.3, we have

$$S_4 \ll J^{-1} M + J^{1/2} M^{\beta/2}.$$

We fix a small $\eta > 0$ and set

$$J := M^{1-\beta+\eta}, \tag{3.3}$$

then we obtain

$$S_4 \ll M^{\beta-\eta/2}$$

if $\frac{1}{2} < \beta < 1$.

Finally, we need to prove that

$$S_3 = \sum_{m\sim M} d(m)(\psi^*(-(m+1)^\beta) - \psi^*(-m^\beta)) \ll M^{\beta-\eta/2}, \tag{3.4}$$

provided that η is sufficiently small. By (3.2), we write

$$\begin{aligned} S_3 &= \sum_{m\sim M} d(m)(\psi^*(-(m+1)^\beta) - \psi^*(-m^\beta)) \\ &= \sum_{m\sim M} d(m)\Bigg(\sum_{1\leqslant|j|\leqslant J} a(j)e(-j(m+1)^\beta) - \sum_{1\leqslant|j|\leqslant J} a(j)e(-jm^\beta)\Bigg) \\ &= \sum_{m\sim M} d(m) \sum_{1\leqslant|j|\leqslant J} a(j)\varphi_j(m)e(-jm^\beta), \end{aligned}$$

where $\varphi_j(x) = e(j(x^\beta - (x+1)^\beta)) - 1$ and

$$\varphi_j(x) \ll jM^{\beta-1}, \quad \frac{\mathrm{d}\varphi_j(x)}{\mathrm{d}x} \ll jM^{\beta-2} \tag{3.5}$$

for $x \in (M, 2M]$. Using partial summation and (3.5), we have

$$\begin{aligned} S_3 &\ll \sum_{1\leqslant|j|\leqslant J} \frac{1}{j}\Bigg|\sum_{m\sim M} d(m)\varphi_j(m)e(-jm^\beta)\Bigg| \\ &\ll \sum_{1\leqslant|j|\leqslant J} \frac{1}{j} \max_{M<x\leqslant 2M} |\varphi_j(x)|\Bigg|\sum_{M<m\leqslant x} d(m)e(-jm^\beta)\Bigg| \\ &\quad + \int_M^{2M} \sum_{1\leqslant|j|\leqslant J} \frac{1}{j}\left|\frac{\mathrm{d}\varphi_j(x)}{\mathrm{d}x}\right|\Bigg|\sum_{M<m\leqslant x} d(m)e(-jm^\beta)\Bigg|\,\mathrm{d}x \\ &\ll M^{\beta-1}\max_{M_1} \sum_{1\leqslant|j|\leqslant J}\Bigg|\sum_{M<m\leqslant M_1} d(m)e(-jm^\beta)\Bigg| \end{aligned}$$

with $M < M_1 \leqslant 2M$.

We can infer that in order to get (3.4), it suffices to prove that

$$\sum_{1\leqslant|j|\leqslant J}\Bigg|\sum_{m\sim M} d(m)e(-jm^\beta)\Bigg| \ll M^{1-\eta/2}.$$

Taking the definition of J in (3.3) into account and dividing the summation interval $1 \leqslant |j| \leqslant J$ into $O(\log 2J)$ dyadic intervals, we see that the above bound holds if

$$K = \sum_{h\sim H}\Big|\sum_{m\sim M} d(m)e(-hm^{\beta})\Big| \ll M^{1-\eta} \tag{3.6}$$

for any $M \geqslant 1$ and $1 \leqslant H \leqslant M^{1-\beta+\eta}$.

By Lemma 2.4 with $f(z) = -hz^{\beta}$, we have

$$K \ll M^{1/2+\beta/3}H^{4/3},$$

where $M^{3/4-\beta} \ll H \ll M^{3/2-\beta}$. As $1 \leqslant H \leqslant M^{1-\beta+\eta}$, we have

$$K \ll M^{1/2+\beta/3}H^{4/3}$$

for $c < \frac{4}{3}$. Then we can conclude that

$$K \ll M^{1-\eta}$$

for $1 < c < \frac{6}{5}$. This completes the proof of Theorem 1.1. □

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