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REMARKS ON THE BALANCED METRIC ON HARTOGS TRIANGLES WITH INTEGRAL EXPONENT

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Abstract. In this paper we study the balanced metrics on some Hartogs triangles of exponent $\gamma \in \mathbb{Z}^+$, i.e.,

$$\Omega_n(\gamma) = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \colon |z_1|^{1/\gamma} < |z_2| < \ldots < |z_n| < 1\}$$

equipped with a natural Kähler form $\omega_{g(\mu)} := \frac{1}{2}(\mathrm{i}/\pi)\partial\overline{\partial}\Phi_n$ with

$$\Phi_n(z) = -\mu_1 \ln(|z_2|^{2\gamma} - |z_1|^2) - \sum_{i=2}^{n-1} \mu_i \ln(|z_{i+1}|^2 - |z_i|^2) - \mu_n \ln(1 - |z_n|^2),$$

where $\mu = (\mu_1, \ldots, \mu_n)$, $\mu_i > 0$, depending on n parameters. The purpose of this paper is threefold. First, we compute the explicit expression for the weighted Bergman kernel function for $(\Omega_n(\gamma), g(\mu))$ and we prove that $g(\mu)$ is balanced if and only if $\mu_1 > 1$ and $\gamma\mu_1$ is an integer, μ_i are integers such that $\mu_i \geqslant 2$ for all $i = 2, \ldots, n-1$, and $\mu_n > 1$. Second, we prove that $g(\mu)$ is Kähler-Einstein if and only if $\mu_1 = \mu_2 = \ldots = \mu_n = 2\lambda$, where λ is a nonzero constant. Finally, we show that if $g(\mu)$ is balanced then $(\Omega_n(\gamma), g(\mu))$ admits a Berezin-Engliš quantization.

Keywords: balanced metric; Kähler-Einstein metric; Berezin-Engliš quantization

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1. Introduction

Suppose that M is a domain in $\mathbb{C}^n$, let Φ be a strictly plurisubharmonic function on M. Therefore, we can endow M with a Kähler metric g, associated with a Kähler form ω expressed as

$$\omega = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\Phi.$$

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The function Φ is also called Kähler potential of g. For $\alpha > 0$ consider the complex Hilbert space $H_\alpha(M)$ defined as

$$H_\alpha(M) := \left\{ f \in \mathcal{O}(M) \colon \ \frac{1}{\pi^n} \int_M |f|^2 \exp\{-\alpha\Phi\} \frac{\omega^n}{n!} < \infty \right\},$$

where $\mathcal{O}(M)$ denotes the space of holomorphic functions on M. If $H_\alpha(M) \neq \{0\}$, let $K_\alpha(z, \overline{z})$ be the weighted reproducing kernel of $H_\alpha(M)$. Then we give the definition of balanced metric.

Definition 1.1. The metric αg on M is said to be a balanced metric if Rawnsley's ε-function defined by

$$\varepsilon_{(\alpha,g)}(z) := \exp\{-\alpha\Phi\} K_\alpha(z, \overline{z}) \quad (z \in M)$$

is constant on M.

One should notice that Rawnsley's ε-function depends only on the Kähler form ω, not on the choice of the Kähler potential Φ. Moreover, this terminology can be naturally extended to the more general setting of functions replaced by sections of line bundle (e.g., see [5]). In this setting, Rawnsley's ε-function is also called the weighted Bergman kernel.

As far as we know, the notation "balanced" may be firstly used by Donaldson [9] in the case of a compact polarized Kähler manifold, who also showed that there exist balanced metrics on any compact projective Kähler manifold with finite automorphism group. Then Arezzo-Loi [1] generalized the definition of balanced metric to the noncompact setting. Furthermore, there is also an asymptotic expansion for $\varepsilon_{(\alpha,g)}(z)$ in terms of the parameter α, namely

$$\varepsilon_{(\alpha,g)}(z) \sim \alpha^n + a_1(z)\alpha^{n-1} + a_2(z)\alpha^{n-2} + \dots$$

as $n \to \infty$. This was proved for compact manifolds by Catlin [6] and Zelditch [22], and for noncompact manifolds by Ma-Marinescu [17], [18]. In some particular case it was also proved by Engliš [10], [11]. Furthermore, the coefficients $a_j(z)$ depend on the curvature and its covariant derivatives of the metric g (e.g., [18]).

In the past decades, there were many deep studies of the existence and uniqueness of balanced metrics in the compact case. Unfortunately, much less seems to be known concerning existence and uniqueness of balanced metrics in noncompact manifolds, even on the domains in $\mathbb{C}^n$.

In 2012, Loi-Zedda [16] gave the necessary and sufficient conditions for the existence of balanced metrics on Cartan-Hartogs domains. Feng-Tu [13] firstly showed the existence of balanced metrics on nonhomogeneous domain. Afterward, Bi-Feng-Tu [2] proved the existence of balanced metrics for a class of unbounded nonhomogeneous domains. For more studies about balanced metrics, the reader is referred to [12], [15], and [20]. Recently, Bi-Su [4] showed that there exist balanced metrics on generalized Hartogs triangles. In their paper, they also posed a question whether one can find balanced metrics on generalized Hartogs triangles with exponent γ.

Following this line, we consider the so-called Hartogs triangles of exponent γ in $\mathbb{C}^n$, which are defined by

$$\Omega_n(\gamma) = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \colon\ |z_1|^{1/\gamma} < |z_2| < \dots < |z_n| < 1\}, \tag{1.1}$$

where γ is a positive constant. Throughout this paper, we assume that γ is a positive integer. Then we can define a strictly plurisubharmonic function $\Phi_n(z_1, \dots, z_n)$ on $\Omega_n(\gamma)$ which is expressed by

$$\Phi_n(z_1, \dots, z_n) = -\mu_1 \ln(|z_2|^{2\gamma} - |z_1|^2) - \sum_{i=2}^{n-1} \mu_i \ln(|z_{i+1}|^2 - |z_i|^2) - \mu_n \ln(1 - |z_n|^2), \tag{1.2}$$

where $\mu_i > 0$ $(1 \leqslant i \leqslant n)$. Therefore, the associated Kähler form $\omega_{g(\mu)}$ is

$$\omega_{g(\mu)} = \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial}\Phi_n.$$

Then we can give the sufficient and necessary conditions for the metric $g(\mu)$ to be balanced.

Theorem 1.2. *The Kähler metric $g(\mu)$ on Hartogs triangles $\Omega_n(\gamma)$ of exponent γ is balanced if and only if $\mu_1 > 1$ and $\gamma\mu_1$ is an integer, μ_i are integers such that $\mu_i \geqslant 2$ for all $i = 2, \dots, n-1$, and $\mu_n > 1$.*

The existence of Kähler-Einstein metric on various manifolds has been one of the central problems in complex geometry. When M is a homogeneous domain, it is well-known that the Bergman metric of M is a Kähler-Einstein metric. But for a nonhomogeneous domain, this is still an open problem. Therefore, from this point, it is natural to study the existence of Kähler-Einstein metric on Hartogs triangles of exponent γ.

Theorem 1.3. *The Hartogs triangles $(\Omega_n(\gamma), g(\mu))$ of exponent γ are Kähler-Einstein manifolds if and only if $\mu_1 = \mu_2 = \dots = \mu_n = 2\lambda$, where λ is a nonzero constant.*

By definition, when $\mu_j = 2$, $j = 1, \dots, n$, $g(\mu)$ reduces to the Bergman metric. Therefore by the above theorem, one can see that the Bergman metric on $(\Omega_n(\gamma), g(\mu))$ is also a Kähler-Einstein metric. Furthermore, we know that the existence of Kähler-Einstein metric is closely related to the existence of solution of the Monge-Ampère equation (see [7]). From the proof of Theorem 1.3, we can also construct an explicit solution of the Monge-Ampère equation on $\Omega_n(\gamma)$. More precisely, we prove the following result:

Theorem 1.4. *For $v > 0$, $\mu_i > 0$, $i = 1, \dots, n$, the function*

$$\Phi'(z) = \frac{1}{n+1} \ln v|z_2|^{2\gamma} \prod_{i=2}^{n-1} |z_{i+1}|^2 - \mu_1 \ln(|z_2|^{2\gamma} - |z_1|^2) \\ - \sum_{i=2}^{n-1} \mu_i \ln(|z_{i+1}|^2 - |z_i|^2) - \mu_n \ln(1 - |z_n|^2)$$

is the explicit solution of the Monge-Ampère equation

$$\det\Big(\frac{\partial^2 \Phi}{\partial z_i \overline{\partial z_j}}\Big) = \mathrm{e}^{(n+1)\Phi}$$

with the boundary condition $\Phi(z) = \infty$, $z \in \partial\Omega_n(\gamma)$, on Hartogs triangles $\Omega_n(\gamma)$ if and only if $v = \mu_1 \dots \mu_n$, $\mu_i = 2/(n+1)$.

As we get to this point, another interesting question arises: can one find a canonical metric on Kähler manifolds such that the metric is both balanced and Kähler-Einstein? One can check that the Bergman metric on the unit ball is balanced and Kähler-Einstein. Moreover, Bi-Hou [3] obtained similar results on generalized Hartogs triangles with exponent $\gamma = 1$. Therefore the following result can also be regarded as a natural generalization of Bi-Hou's result. In fact, by Theorem 1.2 and 1.3, we can easily get the following corollary.

Corollary 1.5. *The metrics $g(\mu)$ on $\Omega_n(\gamma)$ are both Kähler-Einstein and balanced if and only if $\mu_1 = \dots = \mu_n \geqslant 2$ are integers. In particular, the Bergman metric on Hartogs triangles $\Omega_n(\gamma)$ is both Kähler-Einstein and balanced.*

Moreover, the relations between the holomorphic isometric immersion and the balanced metric also imply the following result.

Corollary 1.6. *If:*

(i) $\alpha\mu_i$ *are integers such that* $\alpha\mu_i \geqslant 2$ *for all* $i = 1, \ldots, n-1$ *and* $\alpha\mu_n > 1$, $\alpha > 0$,

(ii) $\mu_1 = \mu_2 = \ldots = \mu_n = 2\lambda$, *where* λ *is a nonzero constant,*

then Hartogs triangles $(\Omega_n(\gamma), \alpha g(\mu))$ *of exponent* γ *are Kähler-Einstein submanifolds of the infinite dimensional complex projective space* $\mathbb{CP}^\infty$.

Furthermore, once we find the existence of balanced metric on $(\Omega_n(\gamma), g(\mu))$, we can also establish the Berezin-Engliš quantization on $(\Omega_n(\gamma), g(\mu))$ by using the ideas in Bi-Su [4]. Recently, Hou-Bi [14] showed that the generalized Hartogs triangle endowed with the Kähler metric admits a Berezin quantization by using Calabi's diastasis function and Rawnsley's ε-function.

Corollary 1.7. *Let* $\Omega_n(\gamma)$ *be Hartogs triangles of exponent* γ *endowed with the Kähler metric* $g(\mu)$. *If* μ_i *are positive rational numbers for all* $i = 1, \ldots, n-1$ *and* $\mu_n > 0$, *then* $(\Omega_n(\gamma), g(\mu))$ *admits a Berezin-Engliš quantization.*

The paper is organized as follows. In Section 2, we present some results which is helpful for calculating the explicit forms of the weighted Bergman kernel. In Section 3, we give the proofs of Theorem 1.2, Theorem 1.3, and Theorem 1.4. In Section 4, we complete the proof of the corresponding applications, mainly the proof of Corollary 1.7.

2. Preliminaries

Firstly, we give several vital lemmas.

Lemma 2.1. *Let* $(\Omega_n(\gamma), g(\mu))$ *be Hartogs triangles of exponent* γ. *Then we have*

$$\det(g(\mu)) = \mu_1 \frac{|z_2|^{2\gamma}}{(|z_2|^{2\gamma} - |z_1|^2)^2} \prod_{i=2}^{n-1} \mu_i \frac{|z_{i+1}|^2}{(|z_{i+1}|^2 - |z_i|^2)^2} \mu_n \frac{1}{(1 - |z_n|^2)^2}. \tag{2.1}$$

Proof. By definition, we have

$$g_{i\bar{j}}(\mu) = \frac{\partial^2 \Phi_n}{\partial z_i \partial \overline{z_j}}$$

and

$$\Phi_n(z) = -\mu_1 \ln(|z_2|^{2\gamma} - |z_1|^2) - \sum_{i=2}^{n-1} \mu_i \ln(|z_{i+1}|^2 - |z_i|^2) - \mu_n \ln(1 - |z_n|^2).$$

From a direct calculation, we can see that

$$g(\mu) = (g_{i\bar{j}}(\mu))_{n\times n} = \begin{pmatrix} g_{1,\bar{1}}(\mu) & g_{1,\bar{2}}(\mu) & \dots & 0 \\ g_{2,\bar{1}}(\mu) & g_{2,\bar{2}}(\mu) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g_{n-1,\bar{n}}(\mu) \\ 0 & 0 & \dots & g_{n,\bar{n}}(\mu) \end{pmatrix}, \tag{2.2}$$

where

$$g_{1,\bar{1}}(\mu) = \mu_1 \frac{|z_2|^{2\gamma}}{(|z_2|^{2\gamma} - |z_1|^2)^2}, \quad g_{1,\bar{2}}(\mu) = -\gamma\mu_1 \frac{\overline{z_1}|z_2|^{2\gamma}}{\overline{z_2}(|z_2|^{2\gamma} - |z_1|^2)^2},$$
$$g_{2,\bar{1}}(\mu) = -\gamma\mu_1 \frac{z_1|z_2|^{2\gamma}}{z_2(|z_2|^{2\gamma} - |z_1|^2)^2},$$
$$g_{2,\bar{2}}(\mu) = \mu_1\gamma^2 \frac{|z_1|^2|z_2|^{2\gamma-2}}{(|z_2|^{2\gamma} - |z_1|^2)^2} + \mu_2 \frac{|z_3|^2}{(|z_3|^2 - |z_2|^2)^2},$$
$$g_{n-1,\bar{n}}(\mu) = -\mu_{n-1} \frac{\overline{z_{n-1}}z_n}{(|z_n|^2 - |z_{n-1}|^2)^2},$$
$$g_{n,\bar{n}}(\mu) = \mu_{n-1} \frac{|z_{n-1}|^2}{(|z_n|^2 - |z_{n-1}|^2)^2} + \mu_n \frac{1}{(1 - |z_n|^2)^2}.$$

By a straightforward computation, it is not difficult to obtain that

$$\det(g(\mu)) = \mu_1 \frac{|z_2|^{2\gamma}}{(|z_2|^{2\gamma} - |z_1|^2)^2} \prod_{i=2}^{n-1} \mu_i \frac{|z_{i+1}|^2}{(|z_{i+1}|^2 - |z_i|^2)^2} \mu_n \frac{1}{(1 - |z_n|^2)^2}.$$

□

Lemma 2.2 (see D'Angelo [8], Lemma 1)**.** *Suppose $\alpha \in (\mathbb{R}_+)^n$. Then we have*

$$\int_{B^n_+} \gamma^{2\alpha-1}\,\mathrm{d}V(\gamma) = \frac{\beta(\alpha)}{2^n|\alpha|}, \quad \int_{S^{n-1}_+} \omega^{2\alpha-1}\,\mathrm{d}\sigma(\omega) = \frac{\beta(\alpha)}{2^{n-1}}.$$

For $\alpha \in (\mathbb{R}_+)^n$, $\beta(\alpha)$ is defined by

$$\beta(\alpha) = \frac{\prod_{i=1}^n \Gamma(\alpha_i)}{\Gamma(|\alpha|)},$$

where Γ is the usual Euler gamma function. Here $\mathrm{d}V$ is the Euclidean n-dimensional volume form, $\mathrm{d}\sigma$ is the Euclidean $(n-1)$-dimensional volume form, and the subscript $+$ means that all the variables are positive.

Lemma 2.3 (see D'Angelo [8], Lemma 2)**.** *Let $x = (x_1, \ldots, x_n) \in (\mathbb{R})^n$, $\|x\|^2 < 1$ and $s \in \mathbb{R}$ with $s > 0$. Then we have*

$$\sum_{q\in\mathbb{N}^n} \frac{\Gamma(|q|+s)}{\Gamma(s)\prod_{i=1}^n \Gamma(q_i+1)} x^{2q} = \frac{1}{(1-\|x\|^2)^s}.$$

Lemma 2.4. *Let $z = (z_1, \ldots, z_n) \in \Omega_n(\gamma)$ and $p = (p_1, \ldots, p_n) \in \mathbb{Z}^n$. Then we have*

$$\|z^p\|^2_{L^2_{\Phi_n}} = \prod_{i=1}^n (\mu_i) B(p_1+1, \mu_1 - 1) \prod_{i=2}^n B\Big(\gamma(p_1+\mu_1) + \sum_{j=2}^i (p_j+\mu_j) - \mu_i + 1, \mu_i - 1\Big),$$

where $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x$ is the beta function.

P r o o f. By definition, we can see that

(2.3)
$$\begin{aligned}\|z^p\|^2_{L^2_{\Phi_n}} &= \frac{1}{\pi^n}\int_{\Omega_n(\gamma)} |z|^{2p} \exp\{-\Phi_n\} \frac{\omega^n}{n!} \\ &= \frac{1}{\pi^n}\int_{\Omega_n(\gamma)} |z|^{2p}(|z_2|^{2\gamma} - |z_1|^2)^{\mu_1} \prod_{i=2}^{n-1} (|z_{i+1}|^2 - |z_i|^2)^{\mu_i}(1-|z_n|^2)^{\mu_n} \\ &\quad \times \mu_1 \frac{|z_2|^{2\gamma}}{(|z_2|^{2\gamma} - |z_1|^2)^2} \prod_{i=2}^{n-1} \mu_i \frac{|z_{i+1}|^2}{(|z_{i+1}|^2 - |z_i|^2)^2} \mu_n \frac{1}{(1-|z_n|^2)^2}\,\mathrm{d}m(z),\end{aligned}$$

where $\mathrm{d}m(z)$ is the Euclidean measure. To calculate it, we first introduce polar coordinates and then the right-hand side of formula (2.3) becomes

$$\begin{aligned}\prod_{i=1}^n 2\mu_i \int_{0\leqslant t_1^{1/\gamma} < t_2 < \ldots < t_n < 1} & t_1^{2p_1+1}(t_2^{2\gamma} - t_1^2)^{\mu_1-2} t_2^{2p_2+2\gamma+1}(t_3^2 - t_2^2)^{\mu_2-2} \\ &\times \prod_{i=3}^{n-1} t_i^{2p_i+3}(t_{i+1}^2 - t_i^2)^{\mu_i-2} t_n^{2p_n+3}(1-t_n^2)^{\mu_n-2}\,\mathrm{d}t_1\ldots\mathrm{d}t_n.\end{aligned}$$

Next, set $s_j = t_j^2$ $(1 \leqslant j \leqslant n)$ and then we get

$$\begin{aligned}\prod_{i=1}^n \mu_i \int_{0\leqslant s_1^{1/\gamma} < s_2 < \ldots < s_n < 1} & s_1^{p_1}(s_2^{\gamma} - s_1)^{\mu_1-2} s_2^{p_2+\gamma}(s_3 - s_2)^{\mu_2-2} \\ &\times \prod_{i=3}^{n-1} s_i^{p_i+1}(s_{i+1} - s_i)^{\mu_i-2} s_n^{p_n+1}(1-s_n)^{\mu_n-2}\,\mathrm{d}s_1\ldots\mathrm{d}s_n.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\|z^p\|^2_{L^2_{\Phi_n}} &= \prod_{i=1}^{n}\mu_i \int_0^{s_2^\gamma} s_1^{p_1}(s_2^\gamma - s_1)^{\mu_1-2}\,\mathrm{d}s_1 \int_0^{s_3} s_2^{p_2+\gamma}(s_3-s_2)^{\mu_2-2}\,\mathrm{d}s_2 \\
&\quad\times \prod_{i=3}^{n-1}\int_0^{s_{i+1}} s_i^{p_i+1}(s_{i+1}-s_i)^{\mu_i-2}\,\mathrm{d}s_i \int_0^1 s_n^{p_n+1}(1-s_n)^{\mu_n-2}\,\mathrm{d}s_n \\
&= \prod_{i=1}^{n}\mu_i \int_0^1 \Big(\frac{s_1}{s_2^\gamma}\Big)^{p_1}\Big(1-\frac{s_1}{s_2^\gamma}\Big)^{\mu_1-2}\,\mathrm{d}\Big(\frac{s_1}{s_2^\gamma}\Big) \int_0^{s_3} s_2^{p_2+\gamma(p_1+\mu_1)}(s_3-s_2)^{\mu_2-2}\,\mathrm{d}s_2 \\
&\quad\times \prod_{i=3}^{n-1}\int_0^{s_{i+1}} s_i^{p_i+1}(s_{i+1}-s_i)^{\mu_i-2}\,\mathrm{d}s_i \int_0^1 s_n^{p_n+1}(1-s_n)^{\mu_n-2}\,\mathrm{d}s_n \\
&= \prod_{i=1}^{n}\mu_i B(p_1+1,\mu_1-1) \int_0^{s_3} s_2^{p_2+\gamma(p_1+\mu_1)}(s_3-s_2)^{\mu_2-2}\,\mathrm{d}s_2 \\
&\quad\times \prod_{i=3}^{n-1}\int_0^{s_{i+1}} s_i^{p_i+1}(s_{i+1}-s_i)^{\mu_i-2}\,\mathrm{d}s_i \int_0^1 s_n^{p_n+1}(1-s_n)^{\mu_n-2}\,\mathrm{d}s_n.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\|z^p\|^2_{L^2_{\Phi_n}} &= \prod_{i=1}^{n}\mu_i B(p_1+1,\mu_1-1) \int_0^1 \Big(\frac{s_2}{s_3}\Big)^{p_2+\gamma(p_1+\mu_1)}\Big(1-\frac{s_2}{s_3}\Big)^{\mu_2-2}\,\mathrm{d}\Big(\frac{s_2}{s_3}\Big) \\
&\quad\times \int_0^{s_4} s_3^{p_3+p_2+\mu_2+\gamma(p_1+\mu_1)}(s_4-s_3)^{\mu_3-2}\,\mathrm{d}s_3 \\
&\quad\times \prod_{i=4}^{n-1}\int_0^{s_{i+1}} s_i^{p_i+1}(s_{i+1}-s_i)^{\mu_i-2}\,\mathrm{d}s_i \int_0^1 s_n^{p_n+1}(1-s_n)^{\mu_n-2}\,\mathrm{d}s_n \\
&= \prod_{i=1}^{n}\mu_i B(p_1+1,\mu_1-1)B(p_2+\gamma(p_1+\mu_1)+1,\mu_2-1) \\
&\quad\times \int_0^{s_4} s_3^{p_3+p_2+\mu_2+\gamma(p_1+\mu_1)}(s_4-s_3)^{\mu_3-2}\,\mathrm{d}s_3 \\
&\quad\times \prod_{i=4}^{n-1}\int_0^{s_{i+1}} s_i^{p_i+1}(s_{i+1}-s_i)^{\mu_i-2}\,\mathrm{d}s_i \int_0^1 s_n^{p_n+1}(1-s_n)^{\mu_n-2}\,\mathrm{d}s_n.
\end{aligned}$$

Therefore, by induction, we conclude that

$$\begin{aligned}
\|z^p\|^2_{L^2_{\Phi_n}} &= \prod_{i=1}^{n}\mu_i B(p_1+1,\mu_1-1) \\
&\quad\times \prod_{i=2}^{n} B\Big(\gamma(p_1+\mu_1)+\sum_{j=2}^{i}(p_j+\mu_i)-\mu_i+1,\mu_i-1\Big).
\end{aligned}$$

□

Furthermore, by the proof of the above lemma, we can get the following result.

Remark 2.5. $H_{\Phi_n}(\Omega_n(\gamma)) \neq \{0\}$ if and only if $\mu_i > 1$ for all $i = 1, \ldots, n$.

3. Proofs of main theorems

Now we give the proof of the existence of balanced metric.

P r o o f of Theorem 1.2. According to the definition of balanced metric, if

$$\varepsilon_{(1,g)}(z) = \exp\{-\Phi_n\} K_1(z, \overline{z})$$

is a constant, then the metric $g(\mu)$ is balanced. So we need to calculate the Bergman kernel $K_1(z, z)$. It is easy to see that $\{z^p / \|z^p\|_{L^2_{\Phi_n}}\}$ forms a complete orthonormal basis of $H_1(\Omega_n(\gamma))$, where the multi-index $p = (p_1, \ldots, p_n)$ ranges over all integers that satisfy the following inequality for all $i = 1, \ldots, n$,

$$\gamma\kappa_1 + \sum_{j=2}^{i} \kappa_j - \mu_i \geqslant 0, \qquad \text{where } \kappa_1 = p_1 + \mu_1,\ \kappa_j = p_j + \mu_j.$$

Let N denote the set of all the multi-indices $p = (p_1, \ldots, p_n)$ satisfying such inequalities. Hence by Lemma 2.4, we have

$$K_1(z, \overline{z}) = \sum_{p \in N} \frac{|z^p|^2}{\|z^p\|^2_{L^2_{\Phi_n}}} = \frac{1}{\prod_{i=1}^n \mu_i} \sum_{p_1=0}^{\infty} \frac{|z_1|^{2p_1}}{B(p_1 + 1, \mu_1 - 1)} \times \ldots$$
$$\times \sum_{p_n = -\gamma\kappa_1 - \sum_{j=2}^{n-1} \kappa_j}^{\infty} \frac{|z_n|^{2p_n}}{B(\gamma\kappa_1 + \sum_{j=2}^{n} \kappa_j - \mu_n + 1, \mu_n - 1)}.$$

Notice that

$$\sum_{p_n = -\gamma\kappa_1 - \sum_{j=2}^{n-1} \kappa_j}^{\infty} \frac{|z_n|^{2p_n}}{B(\gamma\kappa_1 + \sum_{j=2}^{n} \kappa_j - \mu_n + 1, \mu_n - 1)}$$
$$= |z_n|^{-2(\gamma\kappa_1 + \sum_{j=2}^{n-1} \kappa_j)} \sum_{m=0}^{\infty} \frac{|z_n|^{2m}}{B(m+1, \mu_n - 1)}$$
$$= |z_n|^{-2(\gamma\kappa_1 + \sum_{j=2}^{n-1} \kappa_j)} \frac{\boldsymbol{\Gamma}(\mu_n)}{\boldsymbol{\Gamma}(\mu_n - 1)} \sum_{m=0}^{\infty} \frac{\boldsymbol{\Gamma}(m + \mu_n)}{\boldsymbol{\Gamma}(m+1)\boldsymbol{\Gamma}(\mu_n)} |z_n|^{2m}$$
$$= |z_n|^{-2(\gamma\kappa_1 + \sum_{j=2}^{n-1} \kappa_j)} (\mu_n - 1) \frac{1}{(1 - |z_n|^2)^{\mu_n}}.$$

Thus, we obtain

$$K_1(z,\overline{z}) = \frac{1}{\prod_{i=1}^n \mu_i} \frac{\mu_n - 1}{(1-|z_n|^2)^{\mu_n}} \sum_{p_1=0}^{\infty} \frac{|z_1|^{2p_1}}{B(p_1+1,\mu_1-1)} \times \dots$$
$$\times \sum_{p_{n-1}=-\gamma\kappa_1-\sum_{j=2}^{n-2}\kappa_j}^{\infty} \frac{|z_{n-1}|^{2p_{n-1}}|z_n|^{-2(\gamma\kappa_1+\sum_{j=2}^{n-1}\kappa_j)}}{B(\gamma\kappa_1+\sum_{j=2}^{n-1}\kappa_j-\mu_{n-1}+1,\mu_{n-1}-1)}.$$

Similarly, we can see that

$$\sum_{p_{n-1}=-\gamma\kappa_1-\sum_{j=2}^{n-2}\kappa_j}^{\infty} \frac{|z_{n-1}|^{2p_{n-1}}|z_n|^{-2(\gamma\kappa_1+\sum_{j=2}^{n-1}\kappa_j)}}{B(\gamma\kappa_1+\sum_{j=2}^{n-1}\kappa_j-\mu_{n-1}+1,\mu_{n-1}-1)}$$
$$= |z_{n-1}|^{-2(\gamma\kappa_1+\sum_{j=2}^{n-2}\kappa_j)}|z_n|^{-2\mu_{n-1}} \sum_{m=0}^{\infty} \frac{|z_{n-1}/z_n|^{2m}}{B(m+1,\mu_{n-1}-1)}$$
$$= |z_{n-1}|^{-2(\gamma\kappa_1+\sum_{j=2}^{n-2}\kappa_j)}|z_n|^{-2\mu_{n-1}}(\mu_{n-1}-1)\frac{1}{(1-|z_{n-1}/z_n|^2)^{\mu_{n-1}}}.$$

Then we get

$$K_1(z,\overline{z}) = \frac{1}{\prod_{i=1}^n \mu_i} \frac{(\mu_n-1)(\mu_{n-1}-1)}{(1-|z_n|^2)^{\mu_n}(1-|z_{n-1}/z_n|^2)^{\mu_{n-1}}} \frac{1}{|z_n|^{2\mu_{n-1}}}$$
$$\times \sum_{p_1=0}^{\infty} \frac{|z_1|^{2p_1}}{B(p_1+1,\mu_1-1)} \times \dots$$
$$\times \sum_{p_{n-2}=-\gamma\kappa_1-\sum_{j=2}^{n-3}\kappa_j}^{\infty} \frac{|z_{n-2}|^{2p_{n-2}}|z_{n-1}|^{-2(\gamma\kappa_1+\sum_{j=2}^{n-2}\kappa_j)}}{B(\gamma\kappa_1+\sum_{j=2}^{n-2}\kappa_j-\mu_{n-2}+1,\mu_{n-2}-1)}.$$

Therefore, by induction, we conclude that

$$K_1(z,\overline{z}) = \frac{\prod_{i=2}^n(\mu_i-1)}{\prod_{i=1}^n \mu_i} \prod_{i=2}^{n-1} \frac{1}{(1-|z_i/z_{i+1}|^2)^{\mu_i}} \frac{1}{(1-|z_n|^2)^{\mu_n}}$$
$$\times \prod_{i=2}^{n-1} \frac{1}{|z_{i+1}|^{2\mu_i}} \sum_{p_1=0}^{\infty} \frac{|z_1|^{2p_1}|z_2|^{-2\gamma\kappa_1}}{B(p_1+1,\mu_1-1)}$$
$$= \frac{\prod_{i=1}^n(\mu_i-1)}{\prod_{i=1}^n \mu_i} \prod_{i=2}^{n-1} \frac{1}{(1-|z_i/z_{i+1}|^2)^{\mu_i}} \frac{1}{(1-|z_n|^2)^{\mu_n}}$$
$$\times \prod_{i=2}^{n-1} \frac{1}{|z_{i+1}|^{2\mu_i}} \frac{1}{|z_2|^{2\gamma\mu_1}} \frac{1}{(1-|z_1|^2/|z_2|^{2\gamma})^{\mu_1}}$$
$$= \frac{\prod_{i=1}^n(\mu_i-1)}{\prod_{i=1}^n \mu_i} \frac{1}{(|z_2|^{2\gamma}-|z_1|^2)^{\mu_1}} \prod_{i=2}^{n-1} \frac{1}{(|z_{i+1}|^2-|z_i|^2)^{\mu_i}} \frac{1}{(1-|z_n|^2)^{\mu_n}}.$$

Now, we are able to show that

$$\varepsilon_{(1,g)}(z) = \exp\{-\Phi_n\}K_1(z,\overline{z}) = \frac{\prod_{i=1}^n(\mu_i - 1)}{\prod_{i=1}^n \mu_i}$$

is a positive constant, which means that the metric $g(\mu)$ is balanced.

On the other hand, assume that $g(\mu)$ is balanced. Then there exists a constant $C\,(>0)$ satisfying

$$K_1(z,\overline{z}) = C\exp\{\Phi_n\} = C(|z_2|^{2\gamma} - |z_1|^2)^{-\mu_1}\prod_{i=2}^{n-1}(|z_{i+1}|^2 - |z_i|^2)^{-\mu_i}(1-|z_n|^2)^{-\mu_n}.$$

By Lemma 2.3, we obtain

$$(|z_{i+1}|^2 - |z_i|^2)^{-\mu_i} = \sum_{p_i=0}^{\infty}\frac{\Gamma(p_i+\mu_i)}{\Gamma(\mu_i)\Gamma(p_i+1)}|z_i|^{2p_i}|z_{i+1}|^{-2(p_i+\mu_i)}.$$

Thus for any $p_1 \in \mathbb{N}$, considering the coefficient of $|z_1|^{2p_1}$ in the expansion of $K_1(z,\overline{z})$, we can see that there exists a constant $\widetilde{C}$ such that the corresponding coefficient equals

$$\widetilde{C}\sum_{p_1=0}^{\infty}\frac{\Gamma(\kappa_1)}{\Gamma(\mu_1)\Gamma(p_1+1)}|z_1|^{2p_1}|z_2|^{-2\gamma\kappa_1}\prod_{i=2}^{n-1}\sum_{p_i=0}^{\infty}\frac{\Gamma(p_i+\mu_i)}{\Gamma(\mu_i)\Gamma(p_i+1)}|z_i|^{2p_i}|z_{i+1}|^{-2(p_i+\mu_i)}$$
$$\times\sum_{p_n=0}^{\infty}\frac{\Gamma(p_n+\mu_n)}{\Gamma(\mu_n)\Gamma(p_n+1)}|z_n|^{2p_n}.$$

Notice that $z_1^{p_1}$ belongs to the basis of $H_1(\Omega_n(\gamma))$ and p_i $(1 \leqslant i \leqslant n)$ are integers, and then we get that

$$\gamma\mu_1 = p_2 - \gamma p_1.$$

Similarly we also can get that for all $i = 2,\dots,n-1$,

$$\mu_i = p_{i+1} - p_i.$$

Then we conclude that $\gamma\mu_1, \mu_2, \dots, \mu_{n-1}$ are forced to be integers. The proof is complete. □

P r o o f of Theorem 1.3. Recall that a Kähler metric g is said to be a Kähler-Einstein metric if $\mathrm{Ric}_g = \lambda\omega_g$ for some nonzero constant λ. Furthermore, we know that the Ricci curvature Ric_g is given in local coordinates by

(3.1) $$\mathrm{Ric}_{i\bar{j}} = -\frac{\partial^2\ln(\det(g(\mu)))}{\partial z_i\partial\overline{z_j}} \quad (i,j = 1,\dots,n).$$

Then by a direct calculation, we get

$$\text{(3.2)} \qquad \mathrm{Ric}_g = (\mathrm{Ric}_{i\bar{j}})_{n\times n} = \begin{pmatrix} \mathrm{Ric}_{1,\bar{1}}(\mu) & \mathrm{Ric}_{1,\bar{2}}(\mu) & \dots & 0 \\ \mathrm{Ric}_{2,\bar{1}}(\mu) & \mathrm{Ric}_{2,\bar{2}}(\mu) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathrm{Ric}_{n-1,\bar{n}}(\mu) \\ 0 & 0 & \dots & \mathrm{Ric}_{n,\bar{n}}(\mu) \end{pmatrix},$$

where

$$\mathrm{Ric}_{1,\bar{1}}(\mu) = 2\frac{|z_2|^{2\gamma}}{(|z_2|^{2\gamma}-|z_1|^2)^2}, \quad \mathrm{Ric}_{1,\bar{2}}(\mu) = -2\gamma\frac{\overline{z_1}|z_2|^{2\gamma}}{\overline{z_2}(|z_2|^{2\gamma}-|z_1|^2)^2},$$

$$\mathrm{Ric}_{2,\bar{1}}(\mu) = -2\gamma\frac{z_1|z_2|^{2\gamma}}{z_2(|z_2|^{2\gamma}-|z_1|^2)^2},$$

$$\mathrm{Ric}_{2,\bar{2}}(\mu) = 2\gamma^2\frac{|z_1|^2|z_2|^{2\gamma-2}}{(|z_2|^{2\gamma}-|z_1|^2)^2} + 2\frac{|z_3|^2}{(|z_3|^2-|z_2|^2)^2},$$

$$\mathrm{Ric}_{n-1,\bar{n}}(\mu) = -2\frac{\overline{z_{n-1}}z_n}{(|z_n|^2-|z_{n-1}|^2)^2}, \quad \mathrm{Ric}_{n,\bar{n}}(\mu) = 2\frac{|z_{n-1}|^2}{(|z_n|^-|z_{n-1}|^2)^2} + 2\frac{1}{(1-|z_n|^2)^2}.$$

Comparing (2.2) and (3.2), we can easily find that the metric $g(\mu)$ is Kähler-Einstein if and only if

$$\mu_1 = \ldots = \mu_n = 2\lambda,$$

where λ is a nonzero constant. The proof is finished. □

P r o o f of Theorem 1.4. According to the definition and by a straightforward calculation, we get

$$\mathrm{e}^{(n+1)\Phi'} = \frac{v|z_2|^{2\gamma}\prod_{i=2}^{n-1}|z_{i+1}|^2}{(|z_2|^{2\gamma}-|z_1|^2)^{(n+1)\mu_1}\prod_{i=2}^{n-1}(|z_{i+1}|^2-|z_i|^2)^{(n+1)\mu_i}(1-|z_n|^2)^{(n+1)\mu_n}}.$$

By (1.2), we have

$$\Phi'(z) - \Phi_n(z) = \frac{1}{n+1}\ln v|z_2|^{2\gamma}\prod_{i=2}^{n-1}|z_{i+1}|^2$$

since $\ln v|z_2|^{2\gamma}\prod\limits_{i=2}^{n-1}|z_{i+1}|^2$ are pluriharmonic terms. Hence, we have

$$\partial\overline{\partial}\Phi' = \partial\overline{\partial}\Phi_n.$$

It follows that

$$\det\Big(\frac{\partial^2\Phi'}{\partial z_i\partial\overline{z_j}}\Big) = \det(g(\mu)).$$

Then by Lemma 2.1, we can see that $\Phi'(z)$ is the explicit solution of the Monge-Ampère equation on Hartogs triangles $\Omega_n(\gamma)$ if and only if $v = \mu_1\ldots\mu_n$, $\mu_i = 2/(n+1)$. The proof is finished. □

4. Proofs of corollaries

In order to prove Corollary 1.7 we briefly recall some results on the Berezin-Engliš quantization. Suppose that (M, g) is a Kähler manifold with a global Kähler potential, Engliš [10] gave a sufficient condition for (M, g) admitting the Berezin-Engliš quantization.

Theorem 4.1 (see [10]). *Let M be a Kähler manifold with a global defined Kähler potential. If:*

(I) *The function $\exp\{-D_g(z, w)\}$ is globally defined on $M \times M$,*

$$\exp\{-D_g(z, w)\} \leqslant 1,\ \exp\{-D_g(z, w)\} = 1$$

if and only if $z = w$, where $D_g(z, w)$ denotes Calabi's diastasis function.

(II) *There exists a subset $E \subset \mathbb{R}^+$ which has ∞ in its closure such that Rawnsley's ε-function $\varepsilon_{(\alpha,g)}(z)$ is a positive constant for $\alpha \in E$.*

Then (M, g) admits a Berezin-Engliš quantization.

Lemma 4.2 (Lemma 3 in [21]). *Let (M, g) be a noncompact Kähler manifold. Suppose that (M, g) admits a holomorphic isometric immersion into the infinite dimensional complex space $l^2(\mathbb{C})$ through an injective mapping f. Then (M, g) satisfies the condition* (I) *of Theorem* 4.1.

P r o o f of Corollary 1.7. By the proof of Corollary 1.6, we know that $(\Omega_n(\gamma), g(\mu))$ can be a Kähler (not necessarily Einstein) submanifold of $\mathbb{CP}^\infty$. Furthermore, the immersion $f\colon \Omega_n(\gamma) \to \mathbb{CP}^\infty$ can be written as

$$f = [\ldots, z^p, \ldots],$$

where the multi-index $p = (p_1, \ldots, p_n)$ ranges all integers such that

$$\gamma\kappa_1 + \sum_{j=2}^{i} \kappa_j - \mu_i \geqslant 0, \quad i = 1, \ldots, n.$$

Therefore, we have $f(\Omega_n(\gamma)) \subset l^2(\mathbb{C})$ and f is an injective mapping. In fact, one only needs to notice that the immersion f is constructed by the orthonormal basis of some Hilbert space. Then Lemma 4.2 yields that $(\Omega_n(\gamma), g(\mu))$ satisfies the condition (I) of Theorem 4.1.

In the following, we prove that $(\Omega_n(\gamma), g(\mu))$ also satisfies the condition (II). In fact, the methods came from [4]. For completeness, we write it down.

Let $E \subset \mathbb{R}^+$ be a set defined by

$$E = \{\alpha \in \mathbb{N}^+\colon\ \alpha\mu_k \geqslant 2 \text{ are integers for } 1 \leqslant k \leqslant n-1 \text{ and } \alpha\mu_n > 1\}.$$

Since μ_k for all $1 \leqslant k \leqslant n-1$ are positive rational numbers and $\mu_n > 0$, we obtain that ∞ is in the closure of the subset E. If $\alpha \in E$, we can get that $\alpha\mu_k$ for all $1 \leqslant k \leqslant n-1$ are integers and $\alpha\mu_n > 1$. Thus by Theorem 1.2, we conclude that $\alpha g(\mu)$ is a balanced metric on $\Omega_n(\gamma)$, which means that $\varepsilon_{(1,\alpha g(\mu))}(z)$ is a constant for any $\alpha \in E$. Then by Lemma 3.2 in Yang-Bi [19], we have that $\varepsilon_{(\alpha,g(\mu))}(z)$ is a constant for any $\alpha \in E$. It follows that E satisfies the condition (II) of Theorem 4.1. Therefore, we conclude that $(\Omega_n(\gamma), g(\mu))$ admits the Berezin-Engliš quantization by Theorem 4.1. The proof is complete. □

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