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ON THE ZEROS OF A QUATERNIONIC POLYNOMIAL: AN EXTENSION OF THE ENESTRÖM-KAKEYA THEOREM

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Abstract. We present some results on the location of zeros of regular polynomials of a quaternionic variable. We derive new bounds of Eneström-Kakeya type for the zeros of these polynomials by virtue of a maximum modulus theorem and the structure of the zero sets of a regular product established in the newly developed theory of regular functions and polynomials of a quaternionic variable. Our results extend some classical results from complex to the quaternionic setting as well.

Keywords: quaternionic polynomial; Eneström-Kakeya theorem; zero-sets of a regular product

MSC 2020: 30E10, 30G35, 16K20

1. Introduction

The study of the distribution of zeros of a restricted coefficient polynomial in the plane has been intensively studied since the beginning of the twentieth century, and several breakthroughs have been made in the recent years. One of the most known results about the distribution of zeros of a complex polynomial with important applications in the geometric function theory is the following Eneström-Kakeya theorem, see [12].

Theorem A1. *If $T(z)=\sum\limits_{v=0}^{n} a_v z^v$ is a polynomial of degree n (where z is a complex variable) with real coefficients satisfying*

$$a_n \geqslant a_{n-1} \geqslant \ldots \geqslant a_1 \geqslant a_0 \geqslant 0,$$

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then all the zeros of $T(z)$ *lie in*

$$|z| \leqslant 1.$$

In the literature (for example, see [9], [10]) there exist several extensions of the Eneström-Kakeya theorem. An exhaustive survey on the Eneström-Kakeya theorem and its various generalizations is given in the comprehensive books of Marden (see [12]) and Milovanović et al., see [13]. In 1967, Joyal, Labelle and Rahman in [10] extended Theorem A1 to the polynomials whose coefficients are monotonic but not necessarily nonnegative in the form of the following result.

Theorem A2. *If* $T(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n *(where* z *is a complex variable) with real coefficients satisfying*

$$a_n \geqslant a_{n-1} \geqslant \ldots \geqslant a_1 \geqslant a_0,$$

then all the zeros of $T(z)$ *lie in*

$$|z| \leqslant \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

The extension of Theorem A1 to complex coefficients was established by Govil and Rahman (see [9]) in the form of the following result.

Theorem A3. *If* $T(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n *with complex coefficients, where* $\operatorname{Re}(a_v) = \alpha_v$, $\operatorname{Im}(a_v) = \beta_v$ *for* $0 \leqslant v \leqslant n$ *and satisfying*

$$\alpha_n \geqslant \alpha_{n-1} \geqslant \ldots \geqslant \alpha_1 \geqslant \alpha_0 \geqslant 0, \quad \alpha_n \neq 0,$$

then all the zeros of $T(z)$ *lie in*

$$|z| \leqslant 1 + \frac{2}{\alpha_n} \sum_{v=0}^{n} |\beta_v|.$$

The location of zeros of complex polynomials has been vastly studied with more focus on polynomials with restricted coefficients. The Eneström-Kakeya theorem and its various generalizations as mentioned above are the classic and significant examples of this kind. But we could not find much about the distribution of zeros of polynomials with quaternionic variable and quaternionic coefficients. The main purpose of this paper is to present extensions to the quaternionic setting of some classical results of Eneström-Kakeya type as discussed above.

2. Preliminary knowledge

In order to introduce the framework in which we will work, let us introduce some preliminaries on quaternions and regular functions of a quaternionic variable which will be useful in the sequel. The quaternions were formally introduced by W. R. Hamilton in 1843. This theory of quaternions is by now very well developed in many directions and some background about this type of hyper-complex numbers can be found, for example, in [17]. The noncommutative division ring $\mathbb{H}$ of quaternions consists of elements of the form $q = x_0 + x_1 i + x_2 j + x_3 k$; $x_0, x_1, x_2, x_3 \in \mathbb{R}$, where the imaginary units i, j, k satisfy $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. Every element $q = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$ is composed of the real part $\mathrm{Re}(q) = x_0$ and the imaginary part $\mathrm{Im}(q) = x_1 i + x_2 j + x_3 k$. The conjugate of q is denoted by $\overline{q}$ and is defined as $\overline{q} = x_0 - x_1 i - x_2 j - x_3 k$ and the norm of q is $|q| = \sqrt{q\overline{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$. The inverse of each nonzero element q of $\mathbb{H}$ is given by $q^{-1} = |q|^{-2}\overline{q}$.

For $r > 0$ we define the ball $B(0, r) = \{q \in \mathbb{H} \colon\ |q| < r\}$. By $\mathbb{B}$ we denote the open unit ball in $\mathbb{H}$ centered at the origin, i.e.,

$$\mathbb{B} = \{q = x_0 + x_1 i + x_2 j + x_3 k \colon\ x_0^2 + x_1^2 + x_2^2 + x_3^2 < 1\},$$

and by $\mathbb{S}$ the unit sphere of purely imaginary quaternions, i.e.,

$$\mathbb{S} = \{q = x_1 i + x_2 j + x_3 k \colon\ x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Notice that if $I \in \mathbb{S}$, then $I^2 = -1$. Thus, for any fixed $I \in \mathbb{S}$ we define

$$\mathbb{C}_I = \{x + Iy \colon\ x, y \in \mathbb{R}\},$$

which can be identified with a complex plane. The real axis belongs to $\mathbb{C}_I$ for every $I \in \mathbb{S}$ and so a real quaternion $q = x_0$ belongs to $\mathbb{C}_I$ for any $I \in \mathbb{S}$. For any nonreal quaternion $q \in \mathbb{H} \setminus \mathbb{R}$, there exist, and are unique, $x, y \in \mathbb{R}$ with $y > 0$ and $I \in \mathbb{S}$ such that $q = x + Iy$.

The functions we will consider in this paper are the so-called slice regular functions (as polynomials) of a quaternionic variable. These regular functions of a quaternionic variable have been introduced and intensively studied in the past decade, and they have proven to be a fertile topic in analysis, and their rapid development has been largely driven by the applications to operator theory. We refer the reader to [2], [4], [5], [6], [7], [8] and the reference therein, for a deeper treatment of these functions and their applications. In [6], Gentili and Struppa, inspired by a work of Cullen (see [3]) on analytic intrinsic functions of quaternions, introduced the following definition of regularity for functions of a quaternionic variable.

Definition 2.1. Let U be an open set in $\mathbb{H}$. A real differentiable function $f\colon U \to \mathbb{H}$ is said to be left slice regular, or simply slice regular, if for every $I \in \mathbb{S}$, its restriction f_I of f to the complex plane $\mathbb{C}_I$ satisfies

$$\overline{\partial}_I f(x+Iy) := \frac{1}{2}\Big(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\Big) f_I(x+Iy) = 0.$$

Since for all $n \geqslant 1$ and for all $I \in \mathbb{S}$ we have

$$\frac{1}{2}\Big(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\Big)(x+Iy)^n = 0,$$

it follows by definition that the monomial $P(q) = q^n$ is regular. Because addition and right multiplication by a constant preserves regularity, all polynomials of the form $T(q) = \sum\limits_{v=0}^{n} q^v a_v$, $a_v \in \mathbb{H}$ for $v = 0, 1, 2, \ldots, n$, (with coefficients on the right and indeterminate on the left) are regular. In the theory of polynomials of this kind over skew-fields, one defines a different product (we use the symbol $*$ to denote such a product) which guarantees that the product of regular functions is regular. For polynomials, for example, this product is defined as follows:

Two quaternionic polynomials of this kind can be multiplied according to the convolution product (Cauchy multiplication rule): given $T_1(q) = \sum\limits_{i=0}^{n} q^i a_i$, $T_2(q) = \sum\limits_{j=0}^{m} q^j b_j$, we define

$$(T_1 * T_2)(q) := \sum_{\substack{i=0,1,\ldots,n\\ j=0,1,\ldots,m}} q^{i+j} a_i b_j.$$

If T_1 has real coefficients, the so-called $*$ multiplication coincides with the usual pointwise multiplication. These slice regular functions are nowadays a widely studied topic, important especially in replicating many properties of holomorphic functions of a complex variable. One of the basic properties of holomorphic functions of a complex variable is the discreteness of their zero sets (except when the function vanishes identically). Given a regular function of a quaternionic variable, all its restrictions to complex lines are holomorphic and hence either have a discrete zero set or vanishes identically. In the preliminary steps, the structure of the zero sets of a quaternionic regular function and the factorization property of zeros was described. In this regard, Gentili and Stoppato (see [5] and also [7]) gave a necessary and sufficient condition for a quaternionic regular function to have a zero at a point in terms of the coefficients of the power series expansion of the function. This extends to quaternionic power series the theory presented in [11] for polynomials. The following result, which completely describes the zero sets of a regular product of two polynomials in terms of the zero sets of the two factors, is from [11], see also [5] and [7].

Theorem A4. *Let f and g be given quaternionic polynomials. Then $(f*g)(q_0)=0$ if and only if $f(q_0)=0$ or $f(q_0)\neq 0$ implies $g(f(q_0)^{-1}q_0 f(q_0))=0$.*

Gentili and Struppa in [6] introduced a maximum modulus theorem for regular functions, which includes convergent power series and polynomials in the form of the following result.

Theorem A5 (Maximum Modulus Theorem)**.** *Let $B=B(0,r)$ be a ball in $\mathbb{H}$ with centre 0 and radius $r>0$, and let $f\colon B\to\mathbb{H}$ be a regular function. If $|f|$ has a relative maximum at a point $a\in B$, then f is a constant on B.*

It is important to mention that an algebraic proof of the Fundamental Theorem of Algebra for regular polynomials with coefficients in $\mathbb{H}$ can be found, for instance, in [14], [15]. Its topological proof is given in [8]. This led to the complete identification of the zeros of polynomials in terms of their factorization, for reference see [16]. Thus, it became an interesting perspective to think about the regions containing all the zeros of a regular polynomial of quaternionic variable. Very recently, Carney et al. in [1] extended the Eneström-Kakeya theorem and its various generalizations from complex polynomials to quaternionic polynomials by making use of Theorems A4 and A5. Firstly, they established the following quaternionic analogue of Theorem A1.

Theorem A6. *If $T(q)=\sum\limits_{v=0}^{n} q^v a_v$ is a polynomial of degree n (where q is a quaternionic variable) with real coefficients satisfying*

$$a_n \geqslant a_{n-1} \geqslant \ldots \geqslant a_1 \geqslant a_0 \geqslant 0,$$

then all the zeros of $T(q)$ lie in

$$|q| \leqslant 1.$$

In the same paper, Carney et al. in [1] also established the following quaternionic analogue of Theorem A3.

Theorem A7. *If $T(q)=\sum\limits_{v=0}^{n} q^v a_v$ is a quaternionic polynomial of degree n, where $a_v=\alpha_v+\beta_v i+\gamma_v j+\delta_v k$ for $v=0,1,2,\ldots,n$, satisfying*

$$\alpha_n \geqslant \alpha_{n-1} \geqslant \ldots \geqslant \alpha_1 \geqslant \alpha_0 \geqslant 0, \quad \alpha_n \neq 0,$$

then all the zeros of $T(q)$ lie in

$$|q| \leqslant 1+\frac{2}{\alpha_n}\sum_{v=0}^{n}(|\beta_v|+|\gamma_v|+|\delta_v|).$$

The above Theorem A6 was further generalized recently by Tripathi, see [18]. Apart from the above results and a recent paper of Tripathi, we could not find much about the distribution of zeros of polynomials with quaternionic variable and quaternionic coefficients. The main purpose of this paper is to continue the topic of extending various results of Eneström-Kakeya type from complex to the quaternionic setting by making use of a recently established maximum modulus theorem (see Theorem A5) and the structure of the zero sets of a regular product of two polynomials (see Theorem A4) of a quaternionic variable. The obtained results also produce various generalizations of Theorems A6 and A7.

3. Main results

We begin by presenting the following Eneström-Kakeya type result for the distribution of zeros of a polynomial with quaternionic variable. The obtained result also provides generalizations of Theorem A6.

Theorem 3.1. *If $T(q)=\sum\limits_{v=0}^{n} q^v a_v$ is a polynomial of degree n (where q is a quaternionic variable) with real coefficients satisfying*

$$la_n \geqslant a_{n-1} \geqslant \ldots \geqslant a_1 \geqslant \varrho a_0 > 0$$

for some $l \geqslant 1$ and $0 < \varrho \leqslant 1$, then all the zeros of $T(q)$ lie in

$$|q+l-1| \leqslant l + \frac{2a_0}{a_n}(1-\varrho).$$

For $\varrho = 1$, Theorem 3.1 gives the following generalization of Theorem A6.

Corollary 3.1. *If $T(q)=\sum\limits_{v=0}^{n} q^v a_v$ is a polynomial of degree n (where q is a quaternionic variable) with real coefficients satisfying*

$$la_n \geqslant a_{n-1} \geqslant \ldots \geqslant a_1 \geqslant a_0 > 0$$

for some $l \geqslant 1$, then all the zeros of $T(q)$ lie in

$$|q+l-1| \leqslant l.$$

Remark 3.1. Setting $l=1$ in Corollary 3.1, we get Theorem A6.

If we set $l = a_{n-1}/a_n \geqslant 1$ in Theorem 3.1, we immediately obtain the following result.

Corollary 3.2. *If* $T(q) = \sum\limits_{v=0}^{n} q^v a_v$ *is a polynomial of degree* n *(where* q *is a quaternionic variable) with real coefficients satisfying*

$$a_n \leqslant a_{n-1} \geqslant \ldots \geqslant a_1 \geqslant \varrho a_0 > 0$$

for some $0 < \varrho \leqslant 1$, *then all the zeros of* $T(q)$ *lie in*

$$\left|q + \frac{a_{n-1}}{a_n} - 1\right| \leqslant \frac{a_{n-1}}{a_n} + \frac{2a_0}{a_n}(1 - \varrho).$$

In the sequel, we prove a result for polynomials with quaternionic coefficients with monotone increasing real parts, giving generalizations of Theorem A7.

Theorem 3.2. *If* $T(q) = \sum\limits_{v=0}^{n} q^v a_v$ *is a quaternionic polynomial of degree* n *with quaternionic coefficients, where* $a_v = \alpha_v + \beta_v i + \gamma_v j + \delta_v k$ *for* $v = 0, 1, 2, \ldots, n$, *satisfying*

$$\varrho + \alpha_n \geqslant \alpha_{n-1} \geqslant \ldots \geqslant \alpha_1 \geqslant \alpha_0$$

for some $\varrho \geqslant 0$, *then all the zeros of* $T(q)$ *lie in*

$$\left|q + \frac{\varrho}{a_n}\right| \leqslant \frac{\varrho + \alpha_n + |\alpha_0| - \alpha_0 + 2\sum_{v=0}^{n}(|\beta_v| + |\gamma_v| + |\delta_v|)}{|a_n|}.$$

Setting $\varrho = (l-1)\alpha_n$, with $l \geqslant 1$ and assuming $\alpha_0 \geqslant 0$ in the above theorem, we get the following generalization of Theorem A7.

Corollary 3.3. *If* $T(q) = \sum\limits_{v=0}^{n} q^v a_v$ *is a quaternionic polynomial of degree* n *with quaternionic coefficients, where* $a_v = \alpha_v + \beta_v i + \gamma_v j + \delta_v k$ *for* $v = 0, 1, 2, \ldots, n$, *satisfying*

$$l\alpha_n \geqslant \alpha_{n-1} \geqslant \ldots \geqslant \alpha_1 \geqslant \alpha_0 \geqslant 0, \quad \alpha_n \neq 0$$

for some $l \geqslant 1$, *then all the zeros of* $T(q)$ *lie in*

$$\left|q + (l-1)\frac{\alpha_n}{a_n}\right| \leqslant \frac{l\alpha_n + 2\sum_{v=0}^{n}(|\beta_v| + |\gamma_v| + |\delta_v|)}{|a_n|} \leqslant l + \frac{2}{\alpha_n}\sum_{v=0}^{n}(|\beta_v| + |\gamma_v| + |\delta_v|).$$

Remark 3.2. If in Corollary 3.3 we set $l = 1$, we get Theorem A7.

Finally, we establish the following result for quaternionic polynomials with complex coefficients and with monotone increasing real and imaginary parts. The obtained result gives in particular a quaternionic analogue of Theorem A2.

Theorem 3.3. *If* $T(q) = \sum\limits_{v=0}^{n} q^v a_v$ *is a polynomial of degree* n *with complex coefficients (where* q *is a quaternionic variable) such that* $\mathrm{Re}(a_v) = \alpha_v$, $\mathrm{Im}(a_v) = \beta_v$ *for* $v = 0, 1, 2, \ldots, n$, *and satisfying*

$$l\alpha_n \geqslant \alpha_{n-1} \geqslant \ldots \geqslant \alpha_1 \geqslant \alpha_0, \quad \beta_n \geqslant \beta_{n-1} \geqslant \ldots \geqslant \beta_1 \geqslant \beta_0 \geqslant 0$$

for some $l \geqslant 1$, *then all the zeros of* $T(q)$ *lie in*

$$\left|q + (l-1)\frac{\alpha_n}{a_n}\right| \leqslant \frac{l\alpha_n + \beta_n + |\alpha_0| - \alpha_0}{|a_n|}.$$

If we assume $\beta_v = 0$, $v = 0, 1, 2, \ldots, n$, in Theorem 3.3, we get the following result which as a special case yields quaternionic analogue of Theorem A2.

Corollary 3.4. *If* $T(q) = \sum\limits_{v=0}^{n} q^v a_v$ *is a polynomial of degree* n *(where* q *is a quaternionic variable) with real coefficients satisfying*

$$la_n \geqslant a_{n-1} \geqslant \ldots \geqslant a_1 \geqslant a_0$$

for some $l \geqslant 1$, *then all the zeros of* $T(q)$ *lie in*

$$|q + l - 1| \leqslant \frac{la_n + |a_0| - a_0}{|a_n|}.$$

Remark 3.3. Setting $l = 1$ in Corollary 3.4, we obtain quaternionic analogue of Theorem A2.

4. Proofs of the main results

Proof of Theorem 3.1. Consider the product

$$\begin{aligned}
T(q) * (1-q) &= a_0 + q(a_1 - a_0) + q^2(a_2 - a_1) + \ldots + q^n(a_n - a_{n-1}) - q^{n+1}a_n \\
&= a_0 + q(\varrho - 1)a_0 + q(a_1 - \varrho a_0) + \ldots + q^n(la_n - a_{n-1}) \\
&\quad - q^n la_n + q^n a_n - q^{n+1}a_n \\
&= \varphi(q) - q^n la_n + q^n a_n - q^{n+1}a_n,
\end{aligned}$$

where

$$\varphi(q) = a_0 + q(\varrho - 1)a_0 + q(a_1 - \varrho a_0) + \ldots + q^n(la_n - a_{n-1}).$$

By Theorem A4, $T(q) * (1-q) = 0$ if and only if either $T(q) = 0$ or $T(q) \neq 0$ implies $T(q)^{-1} q T(q) - 1 = 0$, that is, $T(q)^{-1} q T(q) = 1$. Thus, if $T(q) \neq 0$, this implies $q = 1$, so the only zeros of $T(q) * (1-q)$ are $q = 1$ and the zeros of $T(q)$.

For $|q| = 1$ we have

$$\begin{aligned}
|\varphi(q)| &= |a_0 + q(\varrho - 1)a_0 + q(a_1 - \varrho a_0) + \ldots + q^n(la_n - a_{n-1})| \\
&\leqslant |a_0| + |q(\varrho - 1)a_0| + |q(a_1 - \varrho a_0)| + \sum_{v=2}^{n-1} |q^v(a_v - a_{v-1})| + |q^n(la_n - a_{n-1})| \\
&= a_0 + (1-\varrho)a_0 + (a_1 - \varrho a_0) + \sum_{v=2}^{n-1} (a_v - a_{v-1}) + (la_n - a_{n-1}) \\
&= la_n + 2(1-\varrho)a_0.
\end{aligned}$$

Notice that we have

$$\max_{|q|=1} \left| q^n * \varphi\Big(\frac{1}{q}\Big) \right| = \max_{|q|=1} \left| q^n \varphi\Big(\frac{1}{q}\Big) \right| = \max_{|q|=1} \left| \varphi\Big(\frac{1}{q}\Big) \right| = \max_{|q|=1} |\varphi(q)|.$$

It is clear that $q^n * \varphi(1/q)$ has the same bound on $|q| = 1$ as φ, that is,

$$\left| q^n * \varphi\Big(\frac{1}{q}\Big) \right| \leqslant la_n + 2(1-\varrho)a_0 \quad \text{for } |q| = 1.$$

Since $q^n * \varphi(1/q)$ is a polynomial and so is regular in $|q| \leqslant 1$, it follows by the Maximum Modulus Theorem (Theorem A5) that

$$\left| q^n * \varphi\Big(\frac{1}{q}\Big) \right| = \left| q^n \varphi\Big(\frac{1}{q}\Big) \right| \leqslant la_n + 2(1-\varrho)a_0 \quad \text{for } |q| \leqslant 1.$$

Hence,

$$\left| \varphi\Big(\frac{1}{q}\Big) \right| \leqslant \frac{la_n + 2(1-\varrho)a_0}{|q|^n} \quad \text{for } |q| \leqslant 1.$$

Replacing q by $1/q$, we see that

$$|\varphi(q)| \leqslant [la_n + 2(1-\varrho)a_0]|q|^n \quad \text{for } |q| \geqslant 1. \tag{4.1}$$

For $|q| \geqslant 1$ we have

$$\begin{aligned}
|T(q) * (1-q)| &= |\varphi(q) - q^n la_n + q^n a_n - q^{n+1} a_n| \geqslant |q|^n |a_n| |q + l - 1| - |\varphi(q)| \\
&\geqslant |q|^n [|a_n| |q + l - 1| - (la_n + 2(1-\varrho)a_0)] \quad \text{by (4.1)}.
\end{aligned}$$

Hence, if

$$|q + l - 1| > \frac{la_n + 2(1-\varrho)a_0}{a_n},$$

then $|T(q) * (1-q)| > 0$, that is, $T(q) * (1-q) \neq 0$.

Since the only zeros of $T(q) * (1-q)$ are $q=1$ and the zeros of $T(q)$, therefore $T(q) \neq 0$ for

$$|q+l-1| > \frac{la_n + 2(1-\varrho)a_0}{a_n}.$$

In other words, all the zeros of $T(q)$ lie in

$$|q+l-1| \leqslant \frac{la_n + 2(1-\varrho)a_0}{a_n},$$

and this completes the proof of Theorem 3.1. □

P r o o f of Theorem 3.2. Again consider

$$\begin{aligned} T(q) * (1-q) &= a_0 + q(a_1 - a_0) + \ldots + q^n(a_n - a_{n-1}) - q^{n+1}a_n \\ &= a_0 + q(a_1 - a_0) + \ldots + q^n(\varrho + a_n - a_{n-1}) - q^n\varrho - q^{n+1}a_n \\ &= \psi(q) - q^n\varrho - q^{n+1}a_n, \end{aligned}$$

where

$$\psi(q) = a_0 + q(a_1 - a_0) + \ldots + q^n(\varrho + a_n - a_{n-1}).$$

For $|q| = 1$ we have

$$\begin{aligned} |\psi(q)| &= \Big| a_0 + \sum_{v=1}^{n-1} q^v(a_v - a_{v-1}) + q^n(\varrho + a_n - a_{n-1}) \Big| \\ &= \Big| a_0 + \sum_{v=1}^{n-1} q^v\{(\alpha_v - \alpha_{v-1}) + (\beta_v - \beta_{v-1})i + (\gamma_v - \gamma_{v-1})j + (\delta_v - \delta_{v-1})k\} \\ &\quad + q^n\{(\varrho + \alpha_n - \alpha_{n-1}) + (\beta_n - \beta_{n-1})i + (\gamma_n - \gamma_{n-1})j + (\delta_n - \delta_{n-1})k\} \Big| \\ &\leqslant |a_0| + \sum_{v=1}^{n-1}\{(\alpha_v - \alpha_{v-1}) + |\beta_v| + |\beta_{v-1}| + |\gamma_v| + |\gamma_{v-1}| + |\delta_v| + |\delta_{v-1}|\} \\ &\quad + (\varrho + \alpha_n - \alpha_{n-1}) + |\beta_n| + |\beta_{n-1}| + |\gamma_n| + |\gamma_{n-1}| + |\delta_n| + |\delta_{n-1}| \\ &\leqslant |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + |\beta_n| + |\beta_{n-1}| + |\gamma_n| + |\gamma_{n-1}| + |\delta_n| + |\delta_{n-1}| \\ &\quad + \varrho + \alpha_n - \alpha_0 + \sum_{v=1}^{n}(|\beta_v| + |\beta_{v-1}| + |\gamma_v| + |\gamma_{v-1}| + |\delta_v| + |\delta_{v-1}|) \\ &\leqslant \varrho + \alpha_n + |\alpha_0| - \alpha_0 + 2\sum_{v=0}^{n}(|\beta_v| + |\gamma_v| + |\delta_v|). \end{aligned}$$

Notice that

$$\max_{|q|=1}\Big|q^n * \psi\Big(\frac{1}{q}\Big)\Big| = \max_{|q|=1}\Big|q^n\psi\Big(\frac{1}{q}\Big)\Big| = \max_{|q|=1}\Big|\psi\Big(\frac{1}{q}\Big)\Big| = \max_{|q|=1}|\psi(q)|,$$

it follows that $q^n * \psi(1/q)$ has the same bound on $|q| = 1$ as ψ, that is,

$$\left|q^n * \psi\Big(\frac{1}{q}\Big)\right| = \left|q^n \psi\Big(\frac{1}{q}\Big)\right| \leqslant \varrho + \alpha_n - \alpha_0 + |\alpha_0| + 2\sum_{v=0}^{n}(|\beta_v| + |\gamma_v| + |\delta_v|) \quad \text{for } |q| = 1.$$

Then by the Maximum Modulus Theorem (Theorem A5) we have

$$\left|q^n \psi\Big(\frac{1}{q}\Big)\right| \leqslant \varrho + \alpha_n - \alpha_0 + |\alpha_0| + 2\sum_{v=0}^{n}(|\beta_v| + |\gamma_v| + |\delta_v|) \quad \text{for } |q| \leqslant 1.$$

Equivalently,

$$|\psi(q)| \leqslant \left[\varrho + \alpha_n - \alpha_0 + |\alpha_0| + 2\sum_{v=0}^{n}(|\beta_v| + |\gamma_v| + |\delta_v|)\right]|q|^n \quad \text{for } |q| \geqslant 1. \tag{4.2}$$

For $|q| \geqslant 1$ we have

$$\begin{aligned}
|T(q) * (1-q)| &= |\psi(q) - q^n\varrho - q^{n+1}a_n| \\
&\geqslant |q|^n|qa_n + \varrho| - |\psi(q)| \\
&\geqslant |q|^n\left[|qa_n + \varrho| - \left\{\varrho + \alpha_n - \alpha_0 + |\alpha_0| + 2\sum_{v=0}^{n}(|\beta_v| + |\gamma_v| + |\delta_v|)\right\}\right] \quad \text{by (4.2)}.
\end{aligned}$$

Hence, if

$$\left|q + \frac{\varrho}{a_n}\right| > \frac{\varrho + \alpha_n - \alpha_0 + |\alpha_0| + 2\sum_{v=0}^{n}(|\beta_v| + |\gamma_v| + |\delta_v|)}{|a_n|},$$

then $|T(q)*(1-q)| > 0$, that is, $T(q)*(1-q) \neq 0$. Since the only zeros of $T(q)*(1-q)$ are $q = 1$ and the zeros of $T(q)$, therefore $T(q) \neq 0$ for

$$\left|q + \frac{\varrho}{a_n}\right| > \frac{\varrho + \alpha_n - \alpha_0 + |\alpha_0| + 2\sum_{v=0}^{n}(|\beta_v| + |\gamma_v| + |\delta_v|)}{|a_n|}.$$

In other words, all the zeros of $T(q)$ lie in

$$\left|q + \frac{\varrho}{a_n}\right| \leqslant \frac{\varrho + \alpha_n - \alpha_0 + |\alpha_0| + 2\sum_{v=0}^{n}(|\beta_v| + |\gamma_v| + |\delta_v|)}{|a_n|}.$$

This completes the proof of Theorem 3.2. □

P r o o f of Theorem 3.3. We have

$$\begin{aligned}
T(q)*(1-q) &= a_0 + q(a_1-a_0)+\ldots+q^n(a_n-a_{n-1})-q^{n+1}a_n\\
&= a_0+\sum_{v=1}^{n-1} q^v(a_v-a_{v-1})+q^n\{(l\alpha_n-\alpha_{n-1})+(\beta_n-\beta_{n-1})i\}\\
&\quad - q^n l\alpha_n + q^n\alpha_n - q^{n+1}a_n\\
&= f(q)-q^n l\alpha_n+q^n\alpha_n-q^{n+1}a_n,
\end{aligned}$$

where

$$f(q)=a_0+\sum_{v=1}^{n-1} q^v(a_v-a_{v-1})+q^n\{(l\alpha_n-\alpha_{n-1})+(\beta_n-\beta_{n-1})i\}.$$

For $|q|=1$ we have

$$\begin{aligned}
|f(q)| &= \Bigl|a_0+\sum_{v=1}^{n-1} q^v(a_v-a_{v-1})+q^n\{(l\alpha_n-\alpha_{n-1})+(\beta_n-\beta_{n-1})i\}\Bigr|\\
&\leqslant |a_0|+\sum_{v=1}^{n-1}|q^v(a_v-a_{v-1})|+|q^n\{(l\alpha_n-\alpha_{n-1})+(\beta_n-\beta_{n-1})j\}|\\
&\leqslant |\alpha_0|+|\beta_0|+\sum_{v=1}^{n-1}|\alpha_v-\alpha_{v-1}|+\sum_{v=1}^{n-1}|\beta_v-\beta_{v-1}|+|l\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|\\
&= l\alpha_n+|\alpha_0|-\alpha_0+\beta_n.
\end{aligned}$$

Proceeding as in the proof of Theorem 3.1, for $|q|\geqslant 1$ we have

$$|f(q)|\leqslant (l\alpha_n+|\alpha_0|-\alpha_0+\beta_n)|q|^n. \tag{4.3}$$

For $|q|\geqslant 1$ we have

$$\begin{aligned}
|T(q)*(1-q)| &= |f(q)-q^n l\alpha_n+q^n\alpha_n-q^{n+1}a_n| \geqslant |q|^n|qa_n+(l-1)\alpha_n|-|f(q)|\\
&\geqslant |q|^n[|qa_n+(l-1)\alpha_n|-(l\alpha_n+|\alpha_0|-\alpha_0+\beta_n)] \quad \text{by (4.3)}.
\end{aligned}$$

Hence, if

$$\Bigl|q+(l-1)\frac{\alpha_n}{a_n}\Bigr| > \frac{l\alpha_n+|\alpha_0|-\alpha_0+\beta_n}{|a_n|},$$

then $|T(q)*(1-q)|>0$, and therefore, $T(q)*(1-q)\neq 0$. Since the only zeros of $T(q)*(1-q)$ are $q=1$ and the zeros of $T(q)$, hence, $T(q)\neq 0$ for

$$\Bigl|q+(l-1)\frac{\alpha_n}{a_n}\Bigr| > \frac{l\alpha_n+|\alpha_0|-\alpha_0+\beta_n}{|a_n|}.$$

In other words, all the zeros of $T(q)$ lie in

$$\left|q+(l-1)\frac{\alpha_n}{a_n}\right| \leqslant \frac{l\alpha_n+|\alpha_0|-\alpha_0+\beta_n}{|a_n|}.$$

This completes the proof of Theorem 3.3. □

5. Conclusion

The regular functions of a quaternionic variable have been introduced and intensively studied in the past decade. They have proven to be a fertile topic in analysis, and their rapid development has been largely driven by its numerous applications in many areas of scientific disciplines. We point out that after the study of the structure of zero sets and the Fundamental Theorem of Algebra for regular polynomials it became interesting to establish the regions containing all the zeros of a regular polynomial of quaternionic variable. In the literature, we could not find much about the distribution of zeros of polynomials with quaternionic variable and quaternionic coefficients. Here, we obtain regions containing all the zeros of a regular polynomial of quaternionic variable when the real and imaginary parts of its coefficients are restricted by virtue of the extended maximum modulus theorem and the structure of the zero sets of a regular product established in the newly developed theory of regular functions and polynomials of a quaternionic variable.

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