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EQUIVARIANT ONE-PARAMETER DEFORMATIONS OF ASSOCIATIVE ALGEBRA MORPHISMS

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Abstract. We introduce equivariant formal deformation theory of associative algebra morphisms. We also present an equivariant deformation cohomology of associative algebra morphisms and using this we study the equivariant formal deformation theory of associative algebra morphisms. We discuss some examples of equivariant deformations and use the Maurer-Cartan equation to characterize equivariant deformations.

Keywords: group action; Hochschild cohomology; equivariant formal deformation; equivariant cohomology

MSC 2020: 16E40, 16S80, 55N91

1. Introduction

Origin of the idea of deformation theory goes back to a paper of Riemann on abelian functions published in 1857. Kodaira and Spencer initiated deformation theory of complex analytic structures, see [9], [10], [11]. Gerstenhaber introduced algebraic deformation theory in a series of papers, see [2], [3], [4], [5], [6]. He studied the deformation theory of associative algebras. Deformation theory of associative algebra morphisms was introduced by Gerstenhaber and Schack, see [7], [8]. Nijenhuis and Richardson introduced deformation theory of Lie algebras, see [15], [16]. Deformation theory of dialgebras has been studied in [12]. Recently, the deformation theory of dialgebra morphisms and Leibniz algebra morphisms have been treated in [13], [17]. Equivariant deformation theory of associative algebras has been studied in [14].

In this paper, we introduce the equivariant deformation cohomology of associative algebra morphisms and equivariant formal deformation theory of associative algebra morphisms. We use the equivariant deformation cohomology of associative algebra morphisms to study the equivariant formal deformation theory. Organization of the

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paper is as follows. In Section 2, we recall some definitions and results. In Section 3, we introduce equivariant deformation complex and equivariant deformation cohomology of an associative algebra morphism. In Section 4, we present equivariant deformation of an associative algebra morphism. In this section we prove that obstructions to equivariant deformations are cocycles. We give an example of the equivariant deformation of associative algebra morphisms. In Section 5, we study the equivalence of two equivariant deformations and rigidity of an equivariant associative algebra morphism. In Section 6, we discuss the Maurer-Cartan equation and use it to characterize equivariant deformations. We give a geometric interpretation of equivariant deformations of associative algebra morphisms in this section. Some of the ideas in this section are motivated by the results in [1], [16].

2. Preliminaries

In this section, we recall definitions of associative algebra, associative algebra morphisms, Hochschild cohomology and equivariant deformation cohomology of an associative algebra. Also, we recall definitions of a module over an associative algebra and module over an associative algebra morphism. Throughout the paper we denote a fixed field by k and a finite group by G.

Definition 2.1. An associative algebra A is a k-module equipped with a k-bilinear map μ satisfying

$$\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$$

for all $a, b, c \in A$.

Let A be an associative k-algebra. A bimodule M over A is a k-module M with two actions (left and right) of A, $\mu\colon A \times M \to M$ and $\mu\colon M \times A \to M$ (for simplicity we denote both the actions by same symbol, one can distinguish both of them from the context) such that $\mu(x, \mu(y, z)) = \mu(\mu(x, y), z)$ whenever one of x, y, z is from M and others are from A.

Let A and B be associative k-algebras. An associative algebra morphism $\varphi\colon A \to B$ is a k-linear map satisfying

$$\varphi(\mu(a, b)) = \mu(\varphi a, \varphi b)$$

for all $a, b \in A$.

Example 2.1. Let $M_n(\mathbb{C})$ be the collection of all $n \times n$ matrices with entries in $\mathbb{C}$. Then $M_n(\mathbb{C})$ is an associative algebra over $\mathbb{C}$ with respect to addition and multiplication of matrices.

Example 2.2. Let X be a nonempty set and A be the set of all complex valued functions defined on X. Then A is an associative algebra over $\mathbb{C}$ with respect to $+$ and $\cdot$ defined by

$$(\alpha+\beta)(x)=\alpha(x)+\beta(x), \quad (\alpha\cdot\beta)(x)=\alpha(x)\beta(x)$$

for all $\alpha,\beta\in A$, $x\in\mathbb{C}$.

Example 2.3. Let V be a vector space over k. Define the tensor module by

$$T(V)=k\oplus V\oplus\ldots\oplus V^{\otimes n}\oplus\ldots$$

The tensor module $T(V)$ is an associative algebra with the concatenation product $T(V)\otimes T(V)\to T(V)$ given by

$$v_1\ldots v_p\otimes v_{p+1}\ldots v_{p+q}=v_1\ldots v_p v_{p+1}\ldots v_{p+q}.$$

Definition 2.2. Let A be an associative k-algebra and M be a bimodule over A. Let $C^n(A;M)=\hom_k(A^{\otimes n},M)$ for all integers $n\geqslant 0$. Also, define a k-linear map $\delta^n\colon C^n(A;M)\to C^{n+1}(A;M)$ given by

$$\begin{aligned}\delta^n f(x_1,\ldots,x_{n+1})=x_1f(x_2,\ldots,x_{n+1})+\sum_{i=1}^{n}(-1)^i f(x_1,\ldots,x_ix_{i+1},\ldots,x_{n+1})\\+(-1)^{n+1}f(x_1,\ldots,x_n)x_{n+1}.\end{aligned}$$

This gives a cochain complex $(C^*(A;M),\delta)$, cohomology of which is denoted by $H^*(A;M)$ and called as *Hochschild cohomology* of A with coefficients in M. Then A is a bimodule over itself in an obvious way. So we can consider the Hochschild cohomology $H^*(A;A)$.

Let A be an associative k-algebra with product $\mu(a,b)=ab$ and G be a finite group. The group G is said to act on A from the left if there exists a function

$$\varphi\colon G\times A\to A, \quad (g,a)\mapsto\varphi(g,a)=ga$$

satisfying the following conditions:

(1) $ex=x$ for all $x\in A$, where $e\in G$ is the group identity.
(2) $g_1(g_2x)=(g_1g_2)x$ for all $g_1,g_2\in G$ and $x\in A$.
(3) For every $g\in G$, the left translation $\varphi_g=\varphi(g,)\colon A\to A$, $a\to ga$ is a linear map.
(4) For all $g\in G$ and $a,b\in A$, $\mu(ga,gb)=g\mu(a,b)=g(ab)$, that is, μ is equivariant with respect to the diagonal action on $A\times A$.

The action above is denoted by (G, A), see [14]. Put

$$C^n_G(A; M) = \{c \in C^n(A; M)\colon\ c(gx_1, \dots, gx_n) = gc(x_1, \dots, x_n)\ \forall\, g \in G\}.$$

An element in $C^n_G(A; M)$ is called an *invariant n-cochain.* Clearly, $C^n_G(A; M)$ is a submodule of $C^n(A; M)$.

From [14] we have following lemma.

Lemma 2.1. *If an n-cochain c is invariant then $\delta^n(c)$ is also invariant. In other words,*

$$c \in C^n_G(A; M) \Rightarrow \delta^n(c) \in C^{n+1}_G(A; M).$$

From [14], we have $(C^*_G(A; M), \delta)$ is a cochain complex. Cohomology of this complex is called an equivariant deformation cohomology of A.

Definition 2.3. Let A and B be associative k-algebras, and $\varphi\colon\ A \to B$ be an associative algebra morphism. Let M and N be k-modules. A k-linear map $T\colon\ M \to N$ is said to be a left (or right) module over φ if following conditions are satisfied.

(1) M and N are left (or right) modules over A and B, respectively.

(2) T is a left (or right) A-module morphism when N is viewed as a left (or right) A-module by virtue of the morphism $\varphi\colon\ A \to B$.

A k-linear map $T\colon\ M \to N$ is said to be a bimodule over φ if T is a left as well as right module over φ.

From [14], we recall equivariant deformation of an associative algebra morphism.

Definition 2.4. Let A be an associative k-algebra with a (G, A) action. Put $A_t = \left\{\sum\limits_{i=0}^{\infty} a_i t^i\colon\ a_i \in A\right\}$. An equivariant formal one-parameter deformation of A is a k-bilinear multiplication $m_t\colon\ A_t \times A_t \to A_t$ satisfying the following properties:

(1) $m_t(a, b) = \sum\limits_{i=0}^{\infty} m_i(a, b)t^i$ for all $a, b \in A$, where $m_i\colon\ A \times A \to A$ are k-bilinear and $m_0(a, b) = ab$ is the original multiplication on A, and m_t is associative.

(2) For every $g \in G$, $m_i(ga, gb) = gm_i(a, b)$ for all $a, b \in A$, $i \geqslant 1$, that is $m_i \in \operatorname{Hom}^G_k(A \otimes A, A)$ for all $i \geqslant 1$.

3. Equivariant deformation complex of an associative algebra morphism

In this section, we introduce equivariant deformation complex of an associative algebra morphism. In the subsequent sections we show that the second and third cohomologies of this complex control deformation.

Definition 3.1. Let G be a finite group, A and B be associative algebras with actions (G, A) and (G, B), respectively. A G-equivariant associative algebra morphism $\varphi\colon A \to B$ is defined to be an associative algebra morphism such that $\varphi(ga) = g\varphi(a)$ for all $a \in A$, $g \in G$.

Next, we give some examples of G-equivariant associative algebra morphisms.

Example 3.1. Let X, Y be G-sets and $f\colon Y \to X$ be a G-equivariant map. Let $A = \{\alpha\colon X \to \mathbb{R}\}$, $B = \{\alpha\colon Y \to \mathbb{R}\}$ be the vector spaces of all real valued functions on X, Y, respectively. Observe that A, B are associative algebras with the product $\alpha\beta(x) = \alpha(x)\beta(x)$. As in [14], G acts on A and B by $(g, \alpha) \mapsto g\alpha$, where $(g\alpha)(x) = \alpha(gx)$. Define $\varphi\colon A \to B$ by $\varphi(\alpha)(y) = \alpha(fy)$ for all $y \in Y$, $\alpha \in A$. Clearly, φ is a G-equivariant associative algebra morphism.

Example 3.2. Consider the associative algebra $M_n(\mathbb{C})$ discussed in Example 2.1. Let U, V be unitary matrices in $M_n(\mathbb{C})$ such that U is self inverse and $UV = VU$. Consider a $\mathbb{Z}_2$ action on $M_n(\mathbb{C})$ given by $\bar{0}P = P$, $\bar{1}P = UPU^* = UPU$. Here Q^* denotes the conjugate transpose of any matrix Q in $M_n(\mathbb{C})$. Now define a map $\varphi\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by

$$\varphi(P) = VPV^*$$

for all $P \in M_n(\mathbb{C})$. Clearly, φ is a $\mathbb{Z}_2$-equivariant associative algebra morphism. In particular, we can choose $U = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and get a $\mathbb{Z}_2$-equivariant associative algebra morphism $\varphi\colon M_2(\mathbb{C}) \to M_2(\mathbb{C})$. Let ψ be the restriction of this φ on $M_2(\mathbb{R})$. Observe that for any $P \in M_2(\mathbb{R})$, $x \in \mathbb{R}^2$, $\psi P(x^t)$ is obtained by reflecting x through the line $u - v = 0$, then applying P, and again reflecting through the same line.

Definition 3.2. Let $\varphi\colon A \to B$ be a G-equivariant associative algebra morphism.

(1) An equivariant left (or right) module over φ is defined to be a G-equivariant k-linear map $T\colon M \to N$ which is a left (or right) module over φ.
(2) An equivariant bimodule over φ is defined to be a G-equivariant k-linear map $T\colon M \to N$ which is a bimodule over φ. In particular, φ is an equivariant bimodule over itself.

Definition 3.3. Let $\varphi\colon A \to B$ be a G-equivariant associative algebra morphism and $T\colon M \to N$ be an equivariant bimodule over φ. We put

$$C^n_G(\varphi; T) = C^n_G(A; M) \oplus C^n_G(B; N) \oplus C^{n-1}_G(A; N)$$

for all $n \in \mathbb{N}$ and $C^0_G(\varphi; T) = 0$. Also, we define $d^n\colon C^n_G(\varphi; T) \to C^{n+1}_G(\varphi; T)$ by

$$d^n(u, v, w) = (\delta^n u, \delta^n v, Tu - v\varphi - \delta^{n-1} w)$$

for all $(u,v,w) \in C_G^n(\varphi;T)$. Here δ^n's denote coboundaries of the cochain complexes $C_G^*(A;M)$, $C_G^*(B;N)$ and $C_G^*(A;N)$, Tu denotes the composition $T\circ u$ of T and u, and the map $v\varphi\colon A^{\otimes n} \to N$ is defined by $v\varphi(x_1,x_2,\dots,x_n) = v(\varphi(x_1), \varphi(x_2),\dots,\varphi(x_n))$. From Lemma 2.1, one can easily verify that d^n is well defined.

Proposition 3.1. *$(C_G^*(\varphi;T),d)$ is a cochain complex.*

Proof. We have

$$\begin{aligned} d^{n+1}d^n(u,v,w) &= d^{n+1}(\delta^n u,\delta^n v, Tu - v\varphi - \delta^{n-1}w) \\ &= (\delta^{n+1}\delta^n u, \delta^{n+1}\delta^n v, T(\delta^n u) - (\delta^n v)\varphi - \delta^n(Tu - v\varphi - \delta^{n-1}w)). \end{aligned}$$

One can easily see that $\delta^n(Tu - v\varphi) = T(\delta^n u) - (\delta^n v)\varphi$. So, since $\delta^{n+1}\delta^n u = \delta^{n+1}\delta^n v = \delta^{n+1}\delta^n w = 0$, we have $d^{n+1}d^n = 0$. Hence, we obtain the result. □

We call the cochain complex $(C_G^*(\varphi,\varphi),d)$ as the equivariant deformation complex of φ and the corresponding cohomology as the equivariant deformation cohomology of φ. We denote the equivariant deformation cohomology by $H_G^n(\varphi,\varphi)$, that is $H_G^n(\varphi,\varphi) = H^n(C_G^*(\varphi,\varphi),d)$. The next proposition relates $H_G^*(\varphi,\varphi)$ to $H_G^*(A,A)$, $H_G^*(B,B)$ and $H_G^*(A,B)$.

Proposition 3.2. *If $H_G^n(A,A) = 0$, $H_G^n(B,B) = 0$ and $H_G^{n-1}(A,B) = 0$, then $H_G^n(\varphi,\varphi) = 0$.*

Proof. Let $(u,v,w) \in C_G^n(\varphi,\varphi)$ be a cocycle, that is $d^n(u,v,w) = (\delta^n u, \delta^n v, \varphi u - v\varphi - \delta^{n-1}w) = 0$. This implies that $\delta^n u = 0$, $\delta^n v = 0$, $\varphi u - v\varphi - \delta^{n-1}w = 0$. $H_G^n(A,A) = 0 \Rightarrow u = \delta^{n-1}u_1$, and $H_G^n(B,B) = 0 \Rightarrow \delta^{n-1}v_1 = v$ for some $u_1 \in C_G^{n-1}(\varphi,\varphi)$ and $v_1 \in C_G^{n-1}(\varphi,\varphi)$. So $0 = \varphi u - v\varphi - \delta^{n-1}w = \varphi(\delta^{n-1}u_1) - (\delta^{n-1}v_1)\varphi - \delta^{n-1}w = \delta^{n-1}(\varphi u_1) - \delta^{n-1}(v_1\varphi) - \delta^{n-1}w = \delta^{n-1}(\varphi u_1 - v_1\varphi - w)$. So $\varphi u_1 - v_1\varphi - w \in C_G^{n-1}(A,B)$ is a cocycle. Now, $H_G^{n-1}(A,B) = 0 \Rightarrow \varphi u_1 - v_1\varphi - w = \delta^{n-2}w_1 \Rightarrow \varphi u_1 - v_1\varphi - \delta^{n-2}w_1 = w$. Thus, $(u,v,w) = (\delta^{n-1}u_1, \delta^{n-1}v_1, \varphi u_1 - v_1\varphi - \delta^{n-2}w_1) = d^{n-1}(u_1,v_1,w_1)$ for some $(u_1,v_1,w_1) \in C_G^{n-1}(\varphi,\varphi)$. Thus, every cocycle in $C_G^n(\varphi,\varphi)$ is a coboundary. Hence, we conclude that $H_G^n(\varphi,\varphi) = 0$. □

4. Equivariant deformation of an associative algebra morphism

Definition 4.1. Let A and B be associative k-algebras with actions (G,A) and (G,B), respectively. An equivariant deformation of a G-equivariant associative algebra morphism $\varphi\colon A \to B$ is a triple (μ_t,ν_t,φ_t), in which:

(1) $\mu_t = \sum\limits_{i=0}^{\infty} \mu_i t^i$ is an equivariant formal one-parameter deformation for A.

(2) $\nu_t = \sum\limits_{i=0}^{\infty} \nu_i t^i$ is an equivariant formal one-parameter deformations for B.

(3) $\varphi_t\colon A_t \to B_t$ is a G-equivariant associative algebra morphism of the form $\varphi_t = \sum\limits_{i=0}^{\infty} \varphi_i t^i$, where $\varphi_i\colon A \to B$ are G-equivariant associative algebra morphisms for all $i \geqslant 0$ and $\varphi_0 = \varphi$.

Therefore, a triple $(\mu_t, \nu_t, \varphi_t)$, as given above, is an equivariant deformation of φ provided the following properties are satisfied.

(i) $\mu_t(\mu_t(a,b),c) = \mu_t(a,\mu_t(b,c))$ for all $a,b,c \in A$;
(ii) $\mu_i(ga,gb) = g\mu_i(a,b)$ for all $a,b \in A$ and $g \in G$;
(iii) $\nu_t(\nu_t(a,b),c) = \nu_t(a,\nu_t(b,c))$ for all $a,b,c \in B$;
(iv) $\nu_i(ga,gb) = g\nu_i(a,b)$ for all $a,b \in B$ and $g \in G$;
(v) $\varphi_t(\mu_t(a,b)) = \nu_t(\varphi_t(a),\varphi_t(b))$ for all $a,b \in A$;
(vi) $\varphi_i(ga) = g\varphi_i(a)$ for all $a \in A$ and $g \in G$.

The conditions (i), (iii) and (v) are equivalent to the following conditions.

$$\sum_{i+j=r} \mu_i(\mu_j(a,b),c) = \sum_{i+j=r} \mu_i(a,\mu_j(b,c)) \quad \forall\, a,b,c \in A,\ r \geqslant 0. \tag{4.1}$$

$$\sum_{i+j=r} \nu_i(\nu_j(a,b),c) = \sum_{i+j=r} \nu_i(a,\nu_j(b,c)) \quad \forall\, a,b,c \in B,\ r \geqslant 0. \tag{4.2}$$

$$\sum_{i+j=r} \varphi_i(\mu_j(a,b)) = \sum_{i+j+k=r} \nu_i(\varphi_j(a),\varphi_k(b)) \quad \forall\, a,b \in A,\ r \geqslant 0. \tag{4.3}$$

Now we define equivariant deformations of finite order.

Definition 4.2. Let A and B be associative k-algebras with actions (G,A) and (G,B), respectively. An equivariant deformation of order n of a G-equivariant associative algebra morphism $\varphi\colon A \to B$ is a triple $(\mu_t, \nu_t, \varphi_t)$, in which:

(1) $\mu_t = \sum\limits_{i=0}^{n} \mu_i t^i$ is an equivariant formal one-parameter deformation of order n for A.

(2) $\nu_t = \sum\limits_{i=0}^{n} \nu_i t^i$ is an equivariant formal one-parameter deformation of order n for B.

(3) $\varphi_t\colon A_t \to B_t$ is a G-equivariant associative algebra morphism of the form $\varphi_t = \sum\limits_{i=0}^{n} \varphi_i t^i$, where $\varphi_i\colon A \to B$ is a G-equivariant associative algebra morphism for all $i \geqslant 0$ and $\varphi_0 = \varphi$.

Example 4.1. Let $\varphi\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be the G-equivariant associative algebra morphism as defined in Example 3.2. Define $\varphi_1\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by

$$\varphi_1(P) = \varphi(UP).$$

Also, define $\mu_1\colon M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by

$$\mu_1(P, Q) = U\mu(UP, UQ).$$

Put $\mu_t = \mu + \mu_1 t$, $\nu_t = \mu$ and $\varphi_t = \varphi + \varphi_1 t$, where μ is the matrix multiplication in $M_n(\mathbb{C})$. We have

$$\mu_1(1P, 1Q) = UPU^*QU^*, \tag{4.4}$$

$$1\mu_1(P, Q) = UPU^*QU^*, \tag{4.5}$$

$$\begin{aligned}(\mu + \mu_1 t)((\mu + \mu_1 t)(P, Q), R)&\\ &= PQR + \{\mu_1(P, Q)R + \mu_1(PQ, R)\}t + \mu_1(\mu_1(P, Q), R)\\ &= PQR + \{PUQR + PQUR\}t + PUQURt^2,\end{aligned} \tag{4.6}$$

$$\begin{aligned}(\mu + \mu_1 t)(P, (\mu + \mu_1 t)(Q, R))&\\ &= PQR + \{\mu_1(P, QR) + P\mu_1(Q, R)\}t + \mu_1(P, \mu_1(Q, R))t^2\\ &= PQR + \{PQUR + PUQR\}t + PUQURt^2,\end{aligned} \tag{4.7}$$

$$\begin{aligned}(\varphi + \varphi_1 t)(\mu + \mu_1 t)(P, Q)&\\ &= VPQV^* + \{VUPQV^* + VPUQV^*\}t + VUPUQV^*t^2,\end{aligned} \tag{4.8}$$

$$\begin{aligned}(\varphi + \varphi_1 t)(P)(\varphi + \varphi_1 t)(Q)&\\ &= VPQV^* + \{VPUQV^* + VUPQV^*\}t + VUPUQV^*t^2,\end{aligned} \tag{4.9}$$

$$(\varphi + \varphi_1 t)(1P) = UVPV^* + UUVPV^*U^*t = 1(\varphi + \varphi_1 t)(P). \tag{4.10}$$

From the expressions (4.4)–(4.10), we conclude that $(\mu_t, \nu_t, \varphi_t)$ is a $\mathbb{Z}_2$-equivariant formal deformation of φ of order 1.

Remark 4.1.

- ▹ For $r = 0$, the conditions (4.1)–(4.3) are equivalent to the fact that A and B are associative algebras and φ is an associative algebra morphism, respectively.
- ▹ For $r = 1$, (4.1)–(4.3) are equivalent to $\delta^2\mu_1 = 0$, $\delta^2\nu_1 = 0$ and $\varphi\mu_1 - \nu_1\varphi - \delta^1\varphi_1 = 0$. Thus, for $r = 1$, (4.1)–(4.3) are equivalent to saying that $(\mu_1, \nu_1, \varphi_1) \in C^2_G(\varphi, \varphi)$ is a cocycle. In general, for $r \geqslant 2$, $(\mu_r, \nu_r, \varphi_r)$ is just a 2-cochain in $C^2_G(\varphi, \varphi)$.

Definition 4.3. The 2-cochain $(\mu_1, \nu_1, \varphi_1)$ in $C^2_G(\varphi, \varphi)$ is called the *infinitesimal* of equivariant deformation $(\mu_t, \nu_t, \varphi_t)$. In general, if $(\mu_i, \nu_i, \varphi_i) = 0$ for $1 \leqslant i \leqslant n-1$ and $(\mu_n, \nu_n, \varphi_n)$ is a nonzero cochain in $C^2_G(\varphi, \varphi)$, then $(\mu_n, \nu_n, \varphi_n)$ is called n-infinitesimal of deformation $(\mu_t, \nu_t, \varphi_t)$.

Proposition 4.1. *The infinitesimal $(\mu_1, \nu_1, \varphi_1)$ of the equivariant deformation $(\mu_t, \nu_t, \varphi_t)$ is a 2-cocycle in $C^2_G(\varphi, \varphi)$. In general, the n-infinitesimal $(\mu_n, \nu_n, \varphi_n)$ is a cocycle in $C^2_G(\varphi, \varphi)$.*

P r o o f. For $n=1$, the proof is obvious from Remark 4.1. For $n>1$, the proof is similar. $\square$

We can write the equations (4.1), (4.2) and (4.3) for $r=n+1$ using the definition of coboundary δ as

$$\delta^2\mu_{n+1}(a,b,c)=\sum_{\substack{i+j=n+1\\ i,j>0}}\mu_i(\mu_j(a,b),c)-\mu_i(a,\mu_j(b,c))\quad\forall\, a,b,c\in A, \tag{4.11}$$

$$\delta^2\nu_{n+1}(a,b,c)=\sum_{\substack{i+j=n+1\\ i,j>0}}\nu_i(\nu_j(a,b),c)-\nu_i(a,\nu_j(b,c))\quad\forall\, a,b,c\in B, \tag{4.12}$$

$$\begin{aligned}&\varphi(\mu_{n+1}(a,b))-\nu_{n+1}(\varphi(a),\varphi(b))-\delta^1\varphi_{n+1}(a,b)\\ &\quad=\sum{}'\,\nu_i(\varphi_j(a),\varphi_k(b))-\sum_{\substack{i+j=n+1\\ i,j>0}}\varphi_i(\mu_j(a,b))\end{aligned} \tag{4.13}$$

for all $a,b\in A$, where

$$\sum{}'=\sum_{\substack{i+j=n+1\\ i,j>0\\ k=0}}+\sum_{\substack{j+k=n+1\\ j,k>0\\ i=0}}+\sum_{\substack{k+i=n+1\\ k,i>0\\ j=0}}+\sum_{\substack{i+j+k=n+1\\ i,j,k>0}}. \tag{4.14}$$

By using the equations (4.11), (4.12) and (4.13) we have

(4.15)

$$\begin{aligned}&d^2(\mu_{n+1},\nu_{n+1},\varphi_{n+1})(a,b,c,x,y,z,p,q)\\ &\quad=\Bigg(\sum_{\substack{i+j=n+1\\ i,j>0}}\mu_i(\mu_j(a,b),c)-\mu_i(a,\mu_j(b,c)),\sum_{\substack{i+j=n+1\\ i,j>0}}\nu_i(\nu_j(x,y),z)-\nu_i(x,\nu_j(y,z)),\\ &\qquad\sum{}'\,\nu_i(\varphi_j(p),\varphi_k(q))-\sum_{\substack{i+j=n+1\\ i,j>0}}\varphi_i(\mu_j(p,q))\Bigg)\end{aligned}$$

for all $a,b,c,p,q\in A$ and $x,y,z\in B$.

Define a 3-cochain F_{n+1} by

(4.16)

$$\begin{aligned}&F_{n+1}(a,b,c,x,y,z,p,q)\\ &\quad=\Bigg(\sum_{\substack{i+j=n+1\\ i,j>0}}\mu_i(\mu_j(a,b),c)-\mu_i(a,\mu_j(b,c)),\sum_{\substack{i+j=n+1\\ i,j>0}}\nu_i(\nu_j(x,y),z)-\nu_i(x,\nu_j(y,z)),\\ &\qquad\sum{}'\,\nu_i(\varphi_j(p),\varphi_k(q))-\sum_{\substack{i+j=n+1\\ i,j>0}}\varphi_i(\mu_j(p,q))\Bigg).\end{aligned}$$

Lemma 4.1. *The 3-cochain F_{n+1} is invariant, that is $F_{n+1}\in C^3_G(\varphi,\varphi)$.*

P r o o f. To prove that F_{n+1} is invariant we show that

$$F_{n+1}(ga, gb, gc, gx, gy, gz, gp, gq) = gF_{n+1}(a, b, c, x, y, z, p, q)$$

for all $a, b, c, p, q \in A$ and $x, y, z \in B$. From Definition 4.1, we have

$$\mu_i(ga, gb) = g\mu_i(a, b), \quad \nu_i(gx, gy) = g\nu_i(x, y), \quad \varphi_i(ga) = g\varphi_i(a)$$

for all $a, b \in A$ and $x, y \in B$. So, we have, for all $a, b, c, p, q \in A$ and $x, y, z \in B$,

$$\begin{aligned}
&F_{n+1}(ga, gb, gc, gx, gy, gz, gp, gq)\\
&= \Big(\sum_{\substack{i+j=n+1\\ i,j>0}} \mu_i(\mu_j(ga, gb), gc) - \mu_i(ga, \mu_j(gb, gc)),\\
&\qquad \sum_{\substack{i+j=n+1\\ i,j>0}} \nu_i(\nu_j(gx, gy), gz) - \nu_i(gx, \nu_j(gy, gz)),\\
&\qquad {\sum}' \nu_i(\varphi_j(gp), \varphi_k(gq)) - \sum_{\substack{i+j=n+1\\ i,j>0}} \varphi_i(\mu_j(gp, gq)) \Big)\\
&= \Big(\sum_{\substack{i+j=n+1\\ i,j>0}} \mu_i(g\mu_j(a, b), gc) - \mu_i(ga, g\mu_j(b, c)),\\
&\qquad \sum_{\substack{i+j=n+1\\ i,j>0}} \nu_i(g\nu_j(x, y), gz) - \nu_i(gx, g\nu_j(y, z)),\\
&\qquad {\sum}' \nu_i(g\varphi_j(p), g\varphi_k(q)) - \sum_{\substack{i+j=n+1\\ i,j>0}} \varphi_i(g\mu_j(p, q)) \Big)\\
&= \Big(\sum_{\substack{i+j=n+1\\ i,j>0}} g\mu_i(\mu_j(a, b), c) - g\mu_i(a, \mu_j(b, c)),\\
&\qquad \sum_{\substack{i+j=n+1\\ i,j>0}} g\nu_i(\nu_j(x, y), z) - g\nu_i(x, \nu_j(y, z)),\\
&\qquad {\sum}' \nu_i(\varphi_j(p), \varphi_k(q)) - \sum_{\substack{i+j=n+1\\ i,j>0}} g\varphi_i(\mu_j(p, q)) \Big)\\
&= gF_{n+1}(a, b, c, x, y, z, p, q).
\end{aligned}$$

So we conclude that $F_{n+1} \in C^n_G(\varphi, \varphi)$. □

Definition 4.4. The 3-cochain $F_{n+1} \in C^n_G(\varphi, \varphi)$ is called the $(n+1)$st obstruction cochain for extending the given equivariant deformation of order n to an equivariant deformation of φ of order $(n+1)$. From now on, we denote F_{n+1} by $Ob_{n+1}(\varphi_t)$.

We have the following result.

Theorem 4.1. *The $(n+1)$st obstruction cochain $Ob_{n+1}(\varphi_t)$ is a 3-cocycle.*

Proof. We have

$$d^3 Ob_{n+1} = (\delta^3(O_1), \delta^3(O_2), \varphi O_1 - O_2\varphi - \delta^2(O_3)),$$

where O_1, O_2 and O_3 are given by

$$O_1(a,b,c) = \sum_{\substack{i+j=n+1\\ i,j>0}} \{\mu_i(\mu_j(a,b),c) - \mu_i(a,\mu_j(b,c))\},$$
$$O_2(x,y,z) = \sum_{\substack{i+j=n+1\\ i,j>0}} \{\nu_i(\nu_j(x,y),z) - \nu_i(x,\nu_j(y,z))\},$$
$$O_3(p,q) = \sum{}' \nu_i(\varphi_j(p),\varphi_k(q)) - \sum_{\substack{i+j=n+1\\ i,j>0}} \varphi_i(\mu_j(p,q)).$$

From [14], we have $\delta^3(O_1) = 0$, $\delta^3(O_2) = 0$. So, to prove that $d^3 Ob_{n+1} = 0$, it remains to show that $\varphi O_1 - O_2\varphi - \delta^2(O_3) = 0$. To prove that $\varphi O_1 - O_2\varphi - \delta^2(O_3) = 0$ we use similar ideas as have been used in [13] and [17]. We have

(4.17)
$$\begin{aligned}(\varphi O_1 - O_2\varphi)(x,y,z) = &\sum_{\substack{i+j=n+1\\ i,j>0}} \varphi\mu_i(\mu_j(x,y),z) - \sum_{\substack{i+j=n+1\\ i,j>0}} \varphi\mu_i(x,\mu_j(y,z))\\ &- \sum_{\substack{i+j=n+1\\ i,j>0}} \nu_i(\nu_j(\varphi x,\varphi y),\varphi z) + \sum_{\substack{i+j=n+1\\ i,j>0}} \nu_i(\varphi x,\nu_j(\varphi y,\varphi z))\end{aligned}$$

and

(4.18)
$$\begin{aligned}\delta^2(O_3)(x,y,z) = &\sum{}' \nu_0(\varphi(x),\nu_i(\varphi_j(y),\varphi_k(z))) - \sum_{\substack{i+j=n+1\\ i,j>0}} \nu_0(\varphi(x),\varphi_i(\mu_j(y,z)))\\ &- \sum{}' \nu_i(\varphi_j(\mu_0(x,y)),\varphi_k(z)) + \sum_{\substack{i+j=n+1\\ i,j>0}} \varphi_i(\mu_j(\mu_0(x,y),z))\\ &+ \sum{}' \nu_i(\varphi_j(x),\varphi_k(\mu_0(y,z))) - \sum_{\substack{i+j=n+1\\ i,j>0}} \varphi_i(\mu_j(x,\mu_0(y,z)))\\ &- \sum{}' \nu_0(\nu_i(\varphi_j(x),\varphi_k(y)),\varphi(z)) + \sum_{\substack{i+j=n+1\\ i,j>0}} \nu_0(\varphi_i(\mu_j(x,y)),\varphi(z)).\end{aligned}$$

From (4.3), we have

$$\varphi_j\mu_0(x,y) = \sum_{\substack{\alpha+\beta+\gamma=j\\ \alpha,\beta,\gamma\geqslant 0}} \nu_\alpha(\varphi_\beta(x),\varphi_\gamma(y)) - \sum_{\substack{p+q=j\\ 1\leqslant q\leqslant j}} \varphi_p\mu_q(x,y). \tag{4.19}$$

Substituting the expression for $\varphi_j\mu_0$ from (4.19) into the third sum on the right hand side of (4.18) we can rewrite it as

$$\begin{aligned} -{\sum}' \nu_i(\varphi_j(\mu_0(x,y)),\varphi_k(z)) = &- \sum_{\substack{\alpha+\beta+\gamma=j\\ \alpha,\beta,\gamma\geqslant 0}}{}' \nu_i(\nu_\alpha(\varphi_\beta(x),\varphi_\gamma(y)),\varphi_k(z)) \\ &+ \sum_{\substack{p+q=j\\ 1\leqslant q\leqslant j}}{}' \nu_i(\varphi_p\mu_q(x,y),\varphi_k(z)). \end{aligned} \tag{4.20}$$

Here the first sum of (4.20) is given by

$$\sum_{\substack{\alpha+\beta+\gamma=j\\ \alpha,\beta,\gamma\geqslant 0}}{}' = \sum_{\substack{i+\alpha+\beta+\gamma=n+1\\ i,(\alpha+\beta+\gamma)>0\\ k=0}} + \sum_{\substack{\alpha+\beta+\gamma+k=n+1\\ (\alpha+\beta+\gamma),k>0\\ i=0}} + \sum_{\substack{k+i=n+1\\ k,i>0\\ \alpha=\beta=\gamma=0}} + \sum_{\substack{i+\alpha+\beta+\gamma+k=n+1\\ i,(\alpha+\beta+\gamma),k>0}}, \tag{4.21}$$

the second sum of (4.20) is given by

$$\sum_{\substack{p+q=j\\ 1\leqslant q\leqslant j}}{}' = \sum_{\substack{i+p+q=n+1\\ i,q>0,p\geqslant 0\\ k=0}} + \sum_{\substack{p+q+k=n+1\\ q,k>0,p\geqslant 0\\ i=0}} + \sum_{\substack{i+j+k=n+1\\ i,q,k>0,p\geqslant 0}}. \tag{4.22}$$

The first sum of (4.19) splits into four sums. The first one of these four sums splits as

$$\begin{aligned} &- \sum_{\substack{i+\alpha+\beta+\gamma=n+1\\ i,(\alpha+\beta+\gamma)>0\\ k=0}} \nu_i(\nu_\alpha(\varphi_\beta(x),\varphi_\gamma(y)),\varphi_k(z)) \\ &\qquad = - \sum_{\substack{i+\alpha=n+1\\ i,\alpha>0}} \nu_i(\nu_\alpha(\varphi(x),\varphi(y)),\varphi(z)) \\ &\qquad\quad - \sum_{\substack{i+\alpha+\beta+\gamma=n+1\\ i,(\beta+\gamma)>0\\ \alpha,\beta,\gamma\geqslant 0}} \nu_i(\nu_\alpha(\varphi_\beta(x),\varphi_\gamma(y)),\varphi(z)). \end{aligned} \tag{4.23}$$

The first sum on the rhs of (4.23) appears as the third sum on the rhs of (4.17). By applying a similar argument to the fifth sum on the rhs of (4.18) and using (4.3) on $\varphi_k\mu_0(y,z)$, one can rewrite it as

$$\sum{}' \nu_i(\varphi_j(x), \varphi_k\mu_0(y,z)) = \sum_{\substack{\alpha+\beta+\gamma=j\\ \alpha,\beta,\gamma\geqslant 0}}{}' \nu_i(\varphi_j(x), \nu_\alpha(\varphi_\beta(y), \varphi_\gamma(z))) - \sum_{\substack{p+q=k\\ 1\leqslant q\leqslant j}}{}' \nu_i(\varphi_j(x), \varphi_p\mu_q(y,z)). \tag{4.24}$$

As above the first sum on the rhs of (4.24) is a sum of four sums similar to (4.21) except that the roles of j and k are interchanged. One of these four terms splits as

$$\begin{aligned}\sum_{\substack{i+\alpha+\beta+\gamma=n+1\\ i,(\alpha+\beta+\gamma)>0\\ k=0}} &\nu_i(\varphi_j(x), \nu_\alpha(\varphi_\beta(y), \varphi_\gamma(z)))\\ &= \sum_{\substack{i+\alpha=n+1\\ i,\alpha>0}} \nu_i(\varphi(x), \nu_\alpha(\varphi(y), \varphi(z)))\\ &\quad + \sum_{\substack{i+\alpha+\beta+\gamma=n+1\\ i,(\beta+\gamma)>0\\ \alpha,\beta,\gamma\geqslant 0}} \nu_i(\varphi(x), \nu_\alpha(\varphi_\beta(y), \varphi_\gamma(z))).\end{aligned} \tag{4.25}$$

The first sum on the rhs of (4.25) appears as the fourth sum on the rhs of (4.17). In the fourth sum on the rhs of (4.18), we use (4.1) to substitute for $\mu_j(\mu_0(x,y),z)$ to obtain

(4.26)

$$\begin{aligned}\sum_{\substack{i+j=n+1\\ i,j>0}} \varphi_i\mu_j(\mu_0(x,y),z) &= \sum_{\substack{i+j=n+1\\ i,j>0}} \varphi_i\mu_j(x, \mu_0(y,z) + \sum_{\substack{i+j+k=n+1\\ i,k>0,j\geqslant 0}} \varphi_i\mu_j(x, \mu_k(y,z))\\ &\quad - \sum_{\substack{i+j+k=n+1\\ i,k>0,j\geqslant 0}} \varphi_i\mu_j(\mu_k(x,y),z).\end{aligned}$$

The first sum on the rhs of (4.26) cancels with the sixth sum on the rhs of (4.18). The Second sum on the rhs of (4.26) splits as

$$\begin{aligned}\sum_{\substack{i+j+k=n+1\\ i,k>0,j\geqslant 0}} \varphi_i\mu_j(x, \mu_k(y,z)) &= \sum_{k=1}^{n} \sum_{\substack{i+j+k=n+1\\ i,j\geqslant 0}} \varphi_i\mu_j(x, \mu_k(y,z)\\ &\quad - \varphi \sum_{\substack{j+k=n+1\\ j,k>0}} \mu_j(x, mu_k(y,z)).\end{aligned} \tag{4.27}$$

The second sum on the rhs of (4.27) appears as the second sum on the rhs of (4.17). Also, by using (4.3) the first sum on the rhs of (4.27) splits as

$$
\begin{aligned}
(4.28)\qquad \sum_{k=1}^{n}\sum_{\substack{i+j+k=n+1\\ i,j\geqslant 0}}\varphi_i\mu_j(x,\mu_k(y,z)) &= \sum_{k=1}^{n}\sum_{\substack{\alpha+\beta+\gamma+k=n+1\\ \alpha,\beta,\gamma\geqslant 0}}\nu_\alpha(\varphi_\beta,\varphi_\gamma\mu_k(y,z))\\
&= \sum_{\substack{i+j=n+1\\ i,j>0}}\nu_0(\varphi(x),\varphi_i\mu_j(y,z))\\
&\quad+\sum_{\substack{p+q=k\\ i\leqslant q\leqslant k}}{}' \nu_i(\varphi_j(x),\varphi_p\mu_q(y,z)).
\end{aligned}
$$

In the last line the two terms cancel with the second terms on the rhs of (4.18) and (4.24), respectively. The third term on the rhs of (4.27) splits as

$$
\begin{aligned}
(4.29)\qquad -\sum_{\substack{i+j+k=n+1\\ i,k>0,j\geqslant 0}}\varphi_i\mu_j(\mu_k(x,y),z) &= -\sum_{\substack{i+j=n+1\\ i,j>0}}\nu_0(\varphi_i\mu_j(x,y),\varphi(z))\\
&\quad-\sum_{\substack{p+q=j\\ 1\leqslant q\leqslant j}}{}' \nu_i(\varphi_p\mu_q(x,y),\varphi_k(z))\\
&\quad+\sum_{\substack{j+k=n+1\\ j,k>0}}\varphi\mu_j(\mu_k(x,y),z).
\end{aligned}
$$

On the rhs of (4.29), the last term cancels with the first sum on the rhs of (4.17), the first sum cancels with the last sum on the rhs of (4.18) and the second term cancels with the second sum on the rhs of (4.20). From our previous arguments we have

(4.30)

$$
\begin{aligned}
&\varphi O_1 - O_2\varphi - \delta^2(O_3)(x,y,z)\\
&= \sum{}'\{\nu_0(\varphi(x),\nu_i(\varphi_j(y),\varphi_k(z))) - \nu_0(\nu_i(\varphi_j(x),\varphi_k(y),\varphi(z)))\}\\
&\quad+\sum_{\substack{i+\alpha+\beta+\gamma=n+1\\ i,(\beta+\gamma)>0\\ \alpha,\beta,\gamma\geqslant 0}}\{\nu_i(\varphi(x),\nu_\alpha(\varphi_\beta(y),\varphi_\gamma(z)) - \nu_i(\nu_\alpha(\varphi_\beta(x),\varphi_\gamma(y),\varphi(z))\}\\
&\quad-\Bigg\{\sum_{\substack{\alpha+\beta+\gamma+k=n+1\\ (\alpha+\beta+\gamma),k>0\\ i=0}}+\sum_{\substack{k+i=n+1\\ k,i>0\\ \alpha=\beta=\gamma=0}}+\sum_{\substack{i+\alpha+\beta+\gamma+k=n+1\\ i,(\alpha+\beta+\gamma),k>0}}\Bigg\}\nu_i(\nu_\alpha(\varphi_\beta(x),\varphi_\gamma(y)),\varphi_k(z))\\
&\quad+\Bigg\{\sum_{\substack{\alpha+\beta+\gamma+j=n+1\\ (\alpha+\beta+\gamma),j>0\\ i=0}}+\sum_{\substack{i+j=n+1\\ i,j>0\\ \alpha=\beta=\gamma=0}}+\sum_{\substack{i+j+\alpha+\beta+\gamma=n+1\\ i,j,(\alpha+\beta+\gamma)>0}}\Bigg\}\nu_i(\varphi_j(x),\nu_\alpha(\varphi_\beta(y),\varphi_\gamma(z))).
\end{aligned}
$$

We can write the above equation more compactly as

$$
\begin{aligned}
(4.31)\quad & \varphi O_1 - O_2\varphi - \delta^2(O_3)(x,y,z) \\
&\qquad = \overline{\sum}\{\nu_i(\varphi_\alpha(x), \nu_j(\varphi_\beta(y), \varphi_\gamma(z))) - \nu_i(\nu_j(\varphi_\alpha(x), \varphi_\beta(y)), \varphi_\gamma(z))\},
\end{aligned}
$$

where

$$
(4.32)\qquad \overline{\sum} = \sum_{\substack{i+j+\alpha+\beta+\gamma=n+1 \\ 1\leqslant\alpha+\beta+\gamma\leqslant n \\ i,j,\alpha,\beta,\gamma\geqslant 0}} + \sum_{\substack{\alpha+\beta=n+1 \\ \alpha,\beta>0 \\ i,j,\gamma=0}} + \sum_{\substack{\beta+\gamma=n+1 \\ \beta,\gamma>0 \\ i,j,\alpha=0}} + \sum_{\substack{\alpha+\gamma=n+1 \\ \alpha,\gamma>0 \\ i,j,\beta=0}} + \sum_{\substack{\alpha+\beta+\gamma=n+1 \\ \alpha,\beta,\gamma>0 \\ i,j=0}} .
$$

It follows from (4.31) and (4.2) that the sum on the rhs of (4.31) is 0 and hence $\varphi O_1 - O_2\varphi - \delta^2(O_3) = 0$. This finishes the proof of the theorem. $\square$

Theorem 4.2. *Let $(\mu_t, \nu_t, \varphi_t)$ be an equivariant deformation of φ of order n. Then $(\mu_t, \nu_t, \varphi_t)$ extends to an equivariant deformation of order $n+1$ if and only if the cohomology class of the $(n+1)$st obstruction $Ob_{n+1}(\varphi_t)$ vanishes.*

Proof. Suppose that an equivariant deformation $(\mu_t, \nu_t, \varphi_t)$ of φ of order n extends to an equivariant deformation of order $n+1$. This implies that (4.1), (4.2) and (4.3) are satisfied for $r = n+1$. Observe that this implies

$$Ob_{n+1}(\varphi_t) = d_2(\mu_{n+1}, \nu_{n+1}, \varphi_{n+1}).$$

So the cohomology class of $Ob_{n+1}(\varphi_t)$ vanishes. Conversely, suppose that the cohomology class of $Ob_{n+1}(\varphi_t)$ vanishes, that is $Ob_{n+1}(\varphi_t)$ is a coboundary. Let

$$Ob_{n+1}(\varphi_t) = d_2(\mu_{n+1}, \nu_{n+1}, \varphi_{n+1})$$

for some 2-cochain $(\mu_{n+1}, \nu_{n+1}, \varphi_{n+1}) \in C^2_G(\varphi, \varphi)$. Take

$$(\widetilde{\mu}_t, \widetilde{\nu}_t, \widetilde{\varphi}_t) = (\mu_t + \mu_{n+1}t^{n+1}, \nu_t + \nu_{n+1}t^{n+1}, \varphi_t + \varphi_{n+1}t^{n+1}).$$

Observe that $(\widetilde{\mu}_t, \widetilde{\nu}_t, \widetilde{\varphi}_t)$ satisfies (4.1), (4.2) and (4.3) for $0 \leqslant r \leqslant n+1$. So $(\widetilde{\mu}_t, \widetilde{\nu}_t, \widetilde{\varphi}_t)$ is an equivariant extension of $(\mu_t, \nu_t, \varphi_t)$ of order $n+1$. $\square$

Corollary 4.1. *If $H^3_G(\varphi, \varphi) = 0$, then every 2-cocycle in $C^2_G(\varphi, \varphi)$ is an infinitesimal of some equivariant deformation of φ.*

5. Equivalence of equivariant deformations

Let $(\mu_t, \nu_t, \varphi_t)$ and $(\widetilde{\mu}_t, \widetilde{\nu}_t, \widetilde{\varphi}_t)$ be two equivariant deformations of φ. Recall from [14] that an equivariant formal isomorphism between the equivariant deformations μ_t and $\widetilde{\mu}_t$ of an associative algebra A is a $k[[t]]$-linear G-automorphism $\Psi_t\colon A[[t]] \to A[[t]]$ of the form $\Psi_t = \sum\limits_{i\geqslant 0} \psi_i t^i$, where each ψ_i is an equivariant k-linear map $A \to A$, $\psi_0(a) = a$ for all $a \in A$ and $\widetilde{\mu}_t(\Psi_t(a), \Psi_t(b)) = \Psi_t\mu_t(a, b)$ for all $a, b \in A$.

Definition 5.1. An equivariant formal isomorphism $(\mu_t, \nu_t, \varphi_t) \to (\widetilde{\mu}_t, \widetilde{\nu}_t, \widetilde{\varphi}_t)$ is a pair (Ψ_t, Θ_t), where $\Psi_t\colon A[[t]] \to A[[t]]$ and $\Theta_t\colon B[[t]] \to B[[t]]$ are equivariant formal isomorphisms from μ_t to $\widetilde{\mu}_t$ and from ν_t to $\widetilde{\nu}_t$, respectively, such that

$$\widetilde{\varphi}_t \circ \Psi_t = \Theta_t \circ \varphi_t.$$

Two equivariant deformations $(\mu_t, \nu_t, \varphi_t)$ and $(\widetilde{\mu}_t, \widetilde{\nu}_t, \widetilde{\varphi}_t)$ are said to be equivalent if there exists an equivariant formal isomorphism (Ψ_t, Θ_t) from $(\mu_t, \nu_t, \varphi_t)$ to $(\widetilde{\mu}_t, \widetilde{\nu}_t, \widetilde{\varphi}_t)$.

Definition 5.2. Any equivariant deformation of $\varphi\colon A \to B$ that is equivalent to the deformation (μ_0, ν_0, φ) is said to be the trivial deformation.

Theorem 5.1. *The cohomology class of the infinitesimal of an equivariant deformation $(\mu_t, \nu_t, \varphi_t)$ of $\varphi\colon A \to B$ is determined by the equivalence class of $(\mu_t, \nu_t, \varphi_t)$.*

P r o o f. Let (Ψ_t, Θ_t) from $(\mu_t, \nu_t, \varphi_t)$ to $(\widetilde{\mu}_t, \widetilde{\nu}_t, \widetilde{\varphi}_t)$ be an equivariant formal isomorphism. So, we have $\widetilde{\mu}_t\Psi_t = \Psi_t \circ \mu_t$, $\widetilde{\nu}_t\Theta_t = \Theta_t \circ \nu_t$, and $\widetilde{\varphi}_t \circ \Psi_t = \Theta_t \circ \varphi_t$. This implies that $\mu_1 - \widetilde{\mu}_1 = \delta\psi_1$, $\nu_1 - \widetilde{\nu}_1 = \delta\theta_1$, and $\varphi_1 - \widetilde{\varphi}_1 = \varphi\psi_1 - \theta_1\varphi$. So we have $d^1(\psi_1, \theta_1, 0) = (\mu_1, \nu_1, \varphi_1) - (\widetilde{\mu}_1, \widetilde{\nu}_1, \widetilde{\varphi}_1)$. This finishes the proof. $\square$

6. Maurer-Cartan equation and geometric interpretation of deformations

We recall the following definition:

Definition 6.1. A finite group G is said to act on a vector space A from the left if there exists a function

$$\varphi\colon G \times A \to A, \quad (g, a) \mapsto \varphi(g, a) = ga$$

satisfying the following conditions:

(1) $ex = x$ for all $x \in A$, where $e \in G$ is the group identity.

(2) $g_1(g_2x) = (g_1g_2)x$ for all $g_1, g_2 \in G$ and $x \in A$.

(3) For every $g \in G$, the left translation $\varphi_g = \varphi(g,)\colon A \to A$, $a \to ga$ is a linear map.

Let A be vector spaces with an action of a group G. Define a linear map

$$\odot\colon\ C_G^{m+1}(A;B)\otimes C_G^{n+1}(A;B)\to C_G^{m+n+1}(A;B)$$

by

$$\eta\odot\psi(x_1,\ldots,x_{m+n+1})=\sum_{i=1}^{n}(-1)^{(i-1)n}\eta(x_1,\ldots,x_{i-1},\psi(x_i,\ldots,x_{i+n}),\ldots,x_{m+n+1}).$$

One can easily verify that $C_G^*(A;B)=\bigoplus_{i=0}^{\infty}C_G^i(A;B)$ is a graded associative algebra with respect to the operation $\odot$. Put

$$[\eta,\psi]_{MG}=\eta\odot\psi-(-1)^{mn}\psi\odot\eta$$

for $\eta\in C_G^{m+1}(A;B)$, $\psi\in C_G^{n+1}(A;B)$. Write $L_i=C_G^{i+1}(A;A)$ and $L=\bigoplus_{i=0}^{\infty}L_i$. Then $(L,[\,,\,]_{MG},0)$ is a differential graded Lie algebra with an action of G.

Definition 6.2. A Maurer-Cartan element in a differential graded Lie algebra $\left(L=\bigoplus_{i\in\mathbb{Z}}L_i,[\,,\,],\delta\right)$ is an element in $\psi\in L_1$ such that

$$\delta\psi+\frac{1}{2}[\psi,\psi]=0. \tag{6.1}$$

The equation (6.1) is called the *Maurer-Cartan equation.*

We have the following characterization of G-equivariant associative algebra structure on a vector space A in terms of Maurer-Cartan elements.

Theorem 6.1. *An element $\mu\in C_G^2(A;A)$ is a G-equivariant associative algebra structure on A if and only if it is a Maurer-Cartan element, that is*

$$[\mu,\mu]_{MG}=0.$$

Assume that A is a finite dimensional vector space with an action of a finite group G. Observe that introducing coordinates (structure constants) the equation $\mu(a,\mu(b,c))=\mu(\mu(a,b),c)$ becomes a set of quadratic polynomial equations with variables as structure constants and $\mu(ga,gb)=g\mu(a,b)$ corresponds to a set of linear polynomial equations with variables as structure constants. Hence, the solution set of the equations $\mu(a,\mu(b,c))=\mu(\mu(a,b),c)$ and $\mu(ga,gb)=g\mu(a,b)$ forms an algebraic variety $\mathcal{L}_{G,A}$ of $C_G^2(A;A)$.

Let A and B be associative algebras with an action of a group G. Let $\varphi\colon A \to B$ be a G-equivariant associative algebra morphism. Define a linear map

$$\diamond\colon C_G^m(A;B) \otimes C_G^n(A;B) \to C_G^{m+n}(A;B)$$

by

$$\eta \diamond \psi(x_1, \dots, x_{m+n}) = \eta(x_1, \dots, x_m)\psi(x_{m+1}, \dots, x_n).$$

Clearly, $C_G^*(A;B) = \bigoplus\limits_{i=0}^{\infty} C_G^i(A;B)$ is a graded associative algebra with respect to the operation $\diamond$. Now, put $[\eta, \psi] = \eta \diamond \psi - (-1)^{mn}\psi \diamond \eta$ for $\eta \in C_G^m(A;B)$, $\psi \in C_G^n(A;B)$. Then $(C_G^*(A;B), [\ ,\ ])$ is a graded Lie algebra. Define a linear map $D\colon C_G^n(A;B) \to C_G^{n+1}(A;B)$ by

$$D\psi(x_1, \dots, x_{n+1}) = \sum_{i=1}^{n} (-1)^i \psi(x_1, \dots, x_i x_{i+1}, \dots, x_{n+1}).$$

By direct computation one can verify that $(C_G^*(A;B), [\ ,\ ], D)$ is a differential graded Lie algebra with an action of G.

We have the following characterization of a G-equivariant associative algebra morphism from A to B in terms of Maurer-Cartan elements.

Theorem 6.2. *Let A and B be associative algebras. An element $\psi \in C_G^1(A;B)$ is a G-equivariant associative algebra morphism from A to B if and only if it is a Maurer-Cartan element, that is*

$$D\psi + \frac{1}{2}[\psi, \psi] = 0.$$

Let (A, μ) and (B, ν) be associative k-algebras with actions (G, A) and (G, B), respectively. Let $\varphi\colon A \to B$ be a G-equivariant associative algebra morphism. Let $\mu_t = \sum\limits_{i=0}^{\infty} \mu_i t^i$, $\nu_t = \sum\limits_{i=0}^{\infty} \nu_i t^i$, $\varphi_t = \sum\limits_{i=0}^{\infty} \varphi_i t^i$, where $\mu_i \in C_G^2(A;A)$, $\nu_i \in C_G^2(B;B)$, $\varphi_i \in C_G^1(A;B)$ for $i \geqslant 0$ and $\mu_0 = \mu$, $\nu_0 = \nu$, $\varphi_0 = \varphi$. Write

$$\widetilde{\mu}_t = \mu_t - \mu_0, \quad \widetilde{\nu}_t = \nu_t - \nu_0, \quad \widetilde{\varphi}_t = \varphi_t - \varphi_0.$$

Using Theorems 6.1, 6.2 and the definition of equivariant deformation we obtain the following result.

Theorem 6.3. *Let $(\mu_t, \nu_t, \varphi_t)$ be an equivariant formal deformation if and only if μ_t and ν_t satisfy the Maurer-Cartan equation* (6.1) *and φ_t satisfies the equation*

$$\varphi_t(\mu(a,b)) - \nu(\varphi_t(a), \varphi_t(b)) - \{\widetilde{\nu}_t(\varphi_t(a), \varphi_t(b)) - \varphi_t(\widetilde{\mu}_t(a,b))\} = 0. \tag{6.2}$$

Let A and B be two finite dimensional vector spaces with an action of a finite group G. Denote by $L_G(A;B)$ the space of all G-equivariant linear maps from A to B. Let μ and ν be G-equivariant associative algebra structures on vector spaces A and B, respectively. Put

$$M_{G,\mu,\nu} = \{\psi \in L_G(A,B)\colon\ \psi(\mu(a,b)) = \nu(\psi(a),\psi(b))\}.$$

Consider a distribution M_G of the trivial vector bundle $\mathcal{L}_{G,A} \times \mathcal{L}_{G,B} \times L_G(A;B)$ defined as

$$M_G = \{(\mu,\nu,\varphi) \in \mathcal{L}_{G,A} \times \mathcal{L}_{G,B} \times L_G(A;B)\colon\ \varphi \in M_{G,\mu,\nu}\}.$$

Remark 6.1. In the equation (6.2), $\varphi_t(\mu(a,b)) - \nu(\varphi_t(a),\varphi_t(b))$ is the left hand side of the Maurer-Cartan equation (6.1) for the differential graded Lie algebra $(C^*_G(A;B),[\,,\,],D)$. It is not surprising because the deformation given by the Maurer-Cartan equation of the differential graded Lie algebra $(C^*_G(A;B),[\,,\,],D)$ corresponds to a deformation along a fiber of M_G. Since we have considered a deformation which is not necessarily confined in a particular fiber of M_G the term $\{\widetilde{\nu}_t(\varphi_t(a),\varphi_t(b)) - \varphi_t(\widetilde{\mu}_t(a,b))\}$ appears in the equation (6.2).

Next example gives a geometric interpretation of deformations and formal deformations.

Example 6.1. Let G be a finite group and A, B be finite dimensional vector spaces. Let $\varphi\colon (A,\mu) \to (B,\nu)$ be a G-equivariant associative algebra morphism. Clearly, any curve $\gamma(t) = (\mu_t,\nu_t,\varphi_t)$ in the distribution M_G of $\mathcal{L}_{G,A} \times \mathcal{L}_{G,B} \times L_G(A;B)$ such that $\gamma(0) = (\mu,\nu,\varphi)$ is an equivariant deformation of φ. If the curve $\gamma(t)$ is analytic, then the deformation comes out to be equivariant formal deformation.

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