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ANNIHILATOR IDEALS OF FINITE DIMENSIONAL SIMPLE MODULES OF TWO-PARAMETER QUANTIZED ENVELOPING ALGEBRA $U_{r,s}(\mathfrak{sl}_2)$

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Abstract. Let U be the two-parameter quantized enveloping algebra $U_{r,s}(\mathfrak{sl}_2)$ and $F(U)$ the locally finite subalgebra of U under the adjoint action. The aim of this paper is to determine some ring-theoretical properties of $F(U)$ in the case when rs^{-1} is not a root of unity. Then we describe the annihilator ideals of finite dimensional simple modules of U by generators.

Keywords: two-parameter quantum group; locally finite subalgebra; adjoint action; annihilator ideal

MSC 2020: 16D25, 20G42

1. Introduction

Let H be a Hopf algebra over a field $\Bbbk$ with a comultiplication Δ, a counit ε and an antipode S. For any $h \in H$ we write $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$. The adjoint action of H is defined by

$$(\operatorname{ad} h)a = \sum h_{(1)} a S(h_{(2)}) \quad \text{for } h, a \in H,$$

which makes H a left H-module algebra. Let $F(H)$ be the set of all elements on which the adjoint action is locally finite, i.e., $F(H) = \{x \in H \colon \dim_{\Bbbk}(\operatorname{ad} H)x < \infty\}$. It is known that $F(H)$ is a subalgebra and a submodule of H under the adjoint action, see [9], Corollary 2.3. Hence, we call $F(H)$ the locally finite subalgebra of H. Kolb et al. in [12] proved that $F(H)$ is always a left coideal subalgebra

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of H. For virtually cocommutative H (i.e., H being finitely generated as module over a cocommutative Hopf subalgebra), Kolb et al. in [12] showed that $F(H)$ is a Hopf subalgebra of H. If H is finite dimensional, then it is clear that $F(H) = H$. So there has been significant attention to infinite dimensional Hopf algebras. When $H = \Bbbk G$ is a group algebra, it is known that $F(H) = \Bbbk\Delta$, the subgroup algebra of the FC-center $\Delta = \Delta(G) = \{g \in G\colon g \text{ has finitely many } G\text{-conjugates}\}$. The Δ-notation for FC-centers was introduced by Passman (see [16]) and should not be confused with the comultiplication notation. When $H = U(\mathfrak{g})$ is the enveloping algebra of a finite dimensional Lie algebra $\mathfrak{g}$, the adjoint action is locally finite. In particular, when $\mathfrak{g}$ is semisimple, the action plays a fundamental role in the study of all prime ideals of $U(\mathfrak{g})$, see [8], [9], [10]. Catoiu in [7] proved that $U(\mathfrak{sl}_2)$ is a principal ideal ring and gave all prime (primitive, maximal) ideals. Afterwards, Li and Zhang in [13], [14] studied the structure of $F(H)$, where H is the quantized enveloping algebra $U_q(\mathfrak{sl}_2)$ and q is not a root of unity, and they showed that $U_q(\mathfrak{sl}_2)$ is also a principal ideal ring. Burdík et al. in [6] found a new basis of $U_q(\mathfrak{sl}_2)$ and determined the decomposition of $U_q(\mathfrak{sl}_2)$ under the adjoint action in the case when q is not a root of unity. In recent years, we made use of the adjoint action to study an important class of Hopf algebras, called *finite dimensional pointed Hopf algebras of rank one*, see [18], [20]. We proved that this class of Hopf algebras are principal ideal rings and gave a complete list of all annihilator ideals of indecomposable modules. Particularly, we described all ideals of the Radford Hopf algebras (which are special examples of finite dimensional pointed Hopf algebras of rank one) and gave the classification of all ideals of 8-dimensional and 9-dimensional Radford Hopf algebra, see [19], [21].

In this paper we consider the two-parameter quantum group $U = U_{r,s}(\mathfrak{sl}_2)$ in the sense of [4], [5]. Two-parameter general linear and special linear quantum groups were introduced by Takeuchi, see [17]. As is shown in [5], $U_{r,s}(\mathfrak{sl}_n)$ is a Drinfeld double of a Borel-type subalgebra, and there is an R-matrix which comes from the double construction and which reduces to the standard R-matrix for the one-parameter quantum group $U_q(\mathfrak{sl}_n)$ (a quotient of $U_{q,q^{-1}}(\mathfrak{sl}_n)$). When rs^{-1} is not a root of unity, it is easy to see that the adjoint action is not locally finite. The aim of this paper is to study the structure of $F(U)$ under the adjoint action and give the classification of annihilator ideals of finite dimensional simple modules of U. Moreover, we characterize all ideals of finite codimension.

Throughout, we work over an algebraically closed field $\Bbbk$ of characteristic zero with $r, s \in \Bbbk \setminus \{0\}$. Unless otherwise stated, all modules are finite-dimensional left modules; all maps are $\Bbbk$-linear; $\otimes$ means $\otimes_{\Bbbk}$. Denote by $Z(U)$ the center of U and by $Z(F(U))$ the center of $F(U)$. We assume that the reader has a passing familiarity with the basics of Hopf algebras and representation theory; see [1], [2], [11], [15] for background.

2. Preliminaries

In this section, we recall the definition and some basic properties of $U = U_{r,s}(\mathfrak{sl}_2)$. Let $\Bbbk$ be an algebraically closed field of characteristic 0 and $\Bbbk^*$ the set of all nonzero elements of $\Bbbk$ viewed as a multiplicative group. In what follows, we assume that $r, s \in \Bbbk^*$ and rs^{-1} is not a root of unity. Recall that the two-parameter quantized enveloping algebra U is the associative algebra over $\Bbbk$ generated by elements $e, f, \omega^{\pm 1}, (\omega')^{\pm 1}$ associated with the following relations, see [4], [5]:
(R1) $\omega^{\pm 1}, (\omega')^{\pm 1}$ all commute with one another and $\omega\omega^{-1} = \omega'(\omega')^{-1} = 1$,
(R2) $\omega e = rs^{-1}e\omega$, $\omega f = r^{-1}sf\omega$, $\omega' e = r^{-1}se\omega'$, $\omega' f = rs^{-1}f\omega'$,
(R3) $[e, f] = (\omega - \omega')/(r - s)$.

It is well known that U has a Hopf algebra structure, where $\omega^{\pm 1}$ and $(\omega')^{\pm 1}$ are group-like elements, and the remaining coproducts are determined by

$$\Delta(e) = e \otimes 1 + \omega \otimes e, \quad \Delta(f) = 1 \otimes f + f \otimes \omega'. \tag{2.1}$$

This forces the counit and antipode maps to be

$$\varepsilon(\omega) = \varepsilon(\omega') = 1, \quad S(\omega) = \omega^{-1}, \qquad S(\omega') = (\omega')^{-1}, \tag{2.2}$$

$$\varepsilon(e) = \varepsilon(f) = 0, \quad S(e) = -\omega^{-1}e, \quad S(f) = -f(\omega')^{-1}. \tag{2.3}$$

It is clear that U is an iterated Ore extension, and hence, U is a Noetherian domain with a $\Bbbk$-basis $\{e^i f^j \omega^k (\omega')^m \colon i, j \in \mathbb{N}, k, m \in \mathbb{Z}\}$, see [3], Theorem 3.2. By Example 5.16 of [3], the center $Z(U)$ of U has a basis of monomials $\{(\omega\omega')^i \mathcal{C}^j \colon i \in \mathbb{Z}, j \in \mathbb{N}\}$, where $\mathcal{C}$ is the Casimir element

$$\mathcal{C} = ef + \frac{s\omega + r\omega'}{(r-s)^2} = fe + \frac{r\omega + s\omega'}{(r-s)^2}. \tag{2.4}$$

Thus, we have $Z(U) = \Bbbk[\omega\omega', (\omega\omega')^{-1}][\mathcal{C}]$. Let

$$[n] = \frac{r^n - s^n}{r - s}, \quad [n]! = [n][n-1]\dots[1], \quad \begin{bmatrix} n \\ i \end{bmatrix} = \frac{[n]!}{[i]!\,[n-i]!}.$$

By [4], Lemma 2.3, it follows that

$$e^n f = fe^n + [n]e^{n-1}\frac{s^{1-n}\omega - r^{1-n}\omega'}{r - s}. \tag{2.5}$$

Let V be a U-module and $\lambda, \mu \in \Bbbk$. Recall that a nonzero vector $v \in V$ is called a *weight vector* with weight (λ, μ) if $\omega v = \lambda v$ and $\omega' v = \mu v$. A weight vector v

is called a *highest* (or *lowest*) *weight vector* if $ev = 0$ (or $fv = 0$). Let U^0 be the subalgebra of U generated by the elements $\omega^{\pm 1}$, $(\omega')^{\pm 1}$ and $\psi\colon U^0 \to \Bbbk$ an algebra homomorphism with $\psi(\omega') = \psi(\omega) r^{-n} s^n$ for some $n \geqslant 0$. By [4], Proposition 2.8, there is an $(n+1)$-dimensional simple U-module $V(n, \psi(\omega))$ spanned by vectors $v_0, v_1, \ldots, v_n$ and having U-action given by

$$\begin{aligned} \omega v_j &= \psi(\omega) r^{-j} s^j v_j, & \omega' v_j &= \psi(\omega) r^{-n+j} s^{n-j} v_j, \\ e v_j &= \psi(\omega) r^{-n} [j][n+1-j] v_{j-1}, & f v_j &= v_{j+1} \end{aligned} \tag{2.6}$$

for $0 \leqslant j \leqslant n$ and $v_{-1} = v_{n+1} = 0$. Furthermore, any $(n+1)$-dimensional simple U-module is isomorphic to some $V(n, \psi(\omega))$. It is clear that v_0 (or v_n) is the highest (or lowest) weight vector with weight $(\psi(\omega), \psi(\omega) r^{-n} s^n)$ (or $(\psi(\omega) r^{-n} s^n, \psi(\omega))$).

By [9], Lemma 2.2, we know that $F(U)$ is a U-module algebra under the adjoint action:

$$\begin{aligned} (\operatorname{ad} \omega) u &= \omega u \omega^{-1}, & (\operatorname{ad} \omega') u &= \omega' u (\omega')^{-1}, \\ (\operatorname{ad} e) u &= e u - \omega u \omega^{-1} e, & (\operatorname{ad} f) u &= (f u - u f)(\omega')^{-1} \end{aligned}$$

for any $u \in U$. As is shown in [4], Remark 3.9, not all finite dimensional U-modules are completely reducible. But we should point out that all finite dimensional U-submodules of U under the adjoint action are completely reducible. We write it as the following proposition.

Proposition 2.1. *All finite dimensional U-submodules of U under the adjoint action are completely reducible.*

Proof. Note that U is generated by e, f, $\omega^{\pm 1}$, $(\omega\omega')^{\pm 1}$ with relations (R1), (R2), (R3) and $\omega\omega' \in Z(U)$. For any $u \in U$,

$$(\operatorname{ad}(\omega\omega')) u = (\omega\omega') u (\omega\omega')^{-1} = u, \quad (\operatorname{ad}(\omega\omega')^{-1}) u = (\omega\omega')^{-1} u (\omega\omega') = u.$$

So the eigenvalue of $\omega\omega'$ is always 1. We may take $\omega\omega'$ as 1. Hence, any U-submodule of U under the adjoint action can be viewed as a $U_q(\mathfrak{sl}_2)$-module, where $q = (rs^{-1})^{1/2}$. It is well known that all finite dimensional $U_q(\mathfrak{sl}_2)$-modules are completely reducible. Therefore, we finish the proof. $\square$

We end this section by giving the following lemmas which are useful in the sequel.

Lemma 2.2. *For $n \geqslant 1$ and $0 \leqslant i \leqslant n$ we have*

$$s^i \begin{bmatrix} n-1 \\ i \end{bmatrix} + r^{n-i} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} = \begin{bmatrix} n \\ i \end{bmatrix}.$$

Proof. It is clear by induction. $\square$

Lemma 2.3. *If $n \geqslant 1$, then*

$$\Delta(e^n) = \sum_{i=0}^{n} s^{-i(n-i)} \begin{bmatrix} n \\ i \end{bmatrix} e^{n-i}\omega^i \otimes e^i, \quad \Delta(f^n) = \sum_{i=0}^{n} s^{-i(n-i)} \begin{bmatrix} n \\ i \end{bmatrix} f^i \otimes f^{n-i}(\omega')^i.$$

Proof. For $n = 1$, the above equations are (2.1). Suppose that $n > 1$ and

$$\Delta(e^{n-1}) = \sum_{i=0}^{n-1} s^{-i(n-1-i)} \begin{bmatrix} n-1 \\ i \end{bmatrix} e^{n-1-i}\omega^i \otimes e^i.$$

Then by Lemma 2.2, it follows that

$$\begin{aligned}
\Delta(e^n) &= (e \otimes 1 + \omega \otimes e)\Bigl(\sum_{i=0}^{n-1} s^{-i(n-1-i)} \begin{bmatrix} n-1 \\ i \end{bmatrix} e^{n-1-i}\omega^i \otimes e^i\Bigr) \\
&= \sum_{i=0}^{n-1} s^{-i(n-1-i)} \begin{bmatrix} n-1 \\ i \end{bmatrix} e^{n-i}\omega^i \otimes e^i \\
&\quad + \sum_{i=0}^{n-1} s^{-i(n-1-i)} \begin{bmatrix} n-1 \\ i \end{bmatrix} (rs^{-1})^{n-1-i} e^{n-1-i}\omega^{i+1} \otimes e^{i+1} \\
&= e^n \otimes 1 + \omega^n \otimes e^n \\
&\quad + \sum_{i=1}^{n-1} \Bigl(s^{-i(n-1-i)} \begin{bmatrix} n-1 \\ i \end{bmatrix} + s^{-i(n-i)} r^{n-i} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}\Bigr) e^{n-i}\omega^i \otimes e^i \\
&= e^n \otimes 1 + \omega^n \otimes e^n + \sum_{i=1}^{n-1} s^{-i(n-i)} \begin{bmatrix} n \\ i \end{bmatrix} e^{n-i}\omega^i \otimes e^i \\
&= \sum_{i=0}^{n} s^{-i(n-i)} \begin{bmatrix} n \\ i \end{bmatrix} e^{n-i}\omega^i \otimes e^i.
\end{aligned}$$

The argument for the second equation can be done similarly. □

3. Structure of $F(U)$

It is clear that $Z(U) \subseteq F(U)$. We claim that the adjoint action is not locally finite. For $n \geqslant 0$ and $m \in \mathbb{Z}$ we have

$$(\operatorname{ad} e)(e^n\omega^{-m}) = e(e^n\omega^{-m}) - \omega(e^n\omega^{-m})\omega^{-1}e = (1 - (rs^{-1})^{n-m})e^{n+1}\omega^{-m}.$$

If $n > m$, then $e^n\omega^{-m} \notin F(U)$. Let $[u] = (\operatorname{ad} U)u$ be the U-submodule generated by $u \in U$. Then we have the following proposition.

Proposition 3.1. *For $n \geqslant 0$ we have*

$$[e^n\omega^{-n}] = [f^n(\omega\omega')^{-n}] \cong [f^n] \cong V(2n, r^n s^{-n}).$$

Proof. Since the $(2n+1)$-dimensional U-module with the highest weight $(r^n s^{-n}, r^{-n}s^n)$ is isomorphic to $V(2n, r^n s^{-n})$, we only need to show that $(\omega^{-1}e)^n$ is the highest weight vector with weight $(r^n s^{-n}, r^{-n}s^n)$ and the action of f on $(\omega^{-1}e)^n$ is nilpotent. It is clear that $(\mathrm{ad}\,\omega)(\omega^{-1}e)^n = r^n s^{-n}(\omega^{-1}e)^n$, $(\mathrm{ad}\,\omega')(\omega^{-1}e)^n = r^{-n}s^n(\omega^{-1}e)^n$ and $(\mathrm{ad}\,e)(\omega^{-1}e)^n = 0$. Hence, $(\omega^{-1}e)^n$ is the highest weight vector with weight $(r^n s^{-n}, r^{-n}s^n)$. Note that $(\mathrm{ad}\,f)(\omega^{-1}e) = (r^{-1}sfe - ef)(\omega\omega')^{-1}$. Hence,

$$\begin{aligned}(\mathrm{ad}\,f)^2(\omega^{-1}e) &= (f(r^{-1}sfe - ef)(\omega\omega')^{-1} - (r^{-1}sfe - ef)(\omega\omega')^{-1}f)(\omega')^{-1}\\ &= -(rs)^{-1}[2]!\,f(\omega\omega')^{-1}\end{aligned}$$

and $(\mathrm{ad}\,f)^3(\omega^{-1}e) = 0$. By Lemma 2.3 and induction, we have

$$\begin{aligned}&(\mathrm{ad}\,f)^{2n+1}(\omega^{-1}e)^n = (\mathrm{ad}\,f)^{2n+1}((\omega^{-1}e)^{n-1}(\omega^{-1}e))\\ &\quad= \sum_{i=0}^{2n+1} s^{-i(2n+1-i)} \begin{bmatrix} 2n+1 \\ i \end{bmatrix} ((\mathrm{ad}\,f)^i(\omega^{-1}e)^{n-1})((\mathrm{ad}\,f)^{2n+1-i}(\mathrm{ad}\,\omega')^i(\omega^{-1}e))\\ &\quad= \sum_{i=0}^{2n+1} r^{-i}s^{-i(2n-i)} \begin{bmatrix} 2n+1 \\ i \end{bmatrix} ((\mathrm{ad}\,f)^i(\omega^{-1}e)^{n-1})((\mathrm{ad}\,f)^{2n+1-i}(\omega^{-1}e)) = 0.\end{aligned}$$

Hence, the action of f on $(\omega^{-1}e)^n$ is nilpotent and $[e^n\omega^{-n}] = [(\omega^{-1}e)^n] \cong V(2n, r^n s^{-n})$. Noticing that for $n \geqslant 2$,

$$\begin{aligned}(\mathrm{ad}\,f)^{2n}(\omega^{-1}e)^n &= \sum_{i=0}^{2n} s^{-i(2n-i)} \begin{bmatrix} 2n \\ i \end{bmatrix} ((\mathrm{ad}\,f)^i(\omega^{-1}e)^{n-1})((\mathrm{ad}\,f)^{2n-i}(\mathrm{ad}\,\omega')^i(\omega^{-1}e))\\ &= \sum_{i=0}^{2n} r^{-i}s^{-i(2n-1-i)} \begin{bmatrix} 2n \\ i \end{bmatrix} ((\mathrm{ad}\,f)^i(\omega^{-1}e)^{n-1})((\mathrm{ad}\,f)^{2n-i}(\omega^{-1}e)),\end{aligned}$$

we assume

$$(\mathrm{ad}\,f)^{2n-2}(\omega^{-1}e)^{n-1} = (-1)^{n-1}(rs)^{-(n-1)^2}[2n-2]!\,f^{n-1}(\omega\omega')^{1-n}.$$

By induction, we have

$$\begin{aligned}(\mathrm{ad}\,f)^{2n}(\omega^{-1}e)^n &= (rs)^{-(2n-2)} \begin{bmatrix} 2n \\ 2n-2 \end{bmatrix} (-1)^{n-1}(rs)^{-(n-1)^2}[2n-2]!\\ &\qquad \times f^{n-1}(\omega\omega')^{1-n}(-(rs)^{-1}[2]!\,f(\omega\omega')^{-1})\\ &= (-1)^n(rs)^{-n^2}[2n]!\,f^n(\omega\omega')^{-n}.\end{aligned}$$

It follows that $f^n(\omega\omega')^{-n}$ is the lowest weight vector with weight $(r^{-n}s^n, r^n s^{-n})$. Note that $\omega\omega' \in Z(U)$. It follows that f^n is the lowest weight vector with weight $(r^{-n}s^n, r^n s^{-n})$ and $[e^n\omega^{-n}] = [f^n(\omega\omega')^{-n}] \cong [f^n]$. $\square$

We use the following proposition to describe all simple submodules of $F(U)$ under the adjoint action.

Proposition 3.2. *Let V be any simple U-submodule of $F(U)$ under the adjoint action. Then $V = [he^n\omega^{-n}] \cong [e^n\omega^{-n}] \cong V(2n, r^n s^{-n})$ for some $0 \neq h \in Z(U)$ and $n \geqslant 1$.*

P r o o f. Suppose that V is an $(l+1)$-dimensional simple U-submodule of $F(U)$. Note that any $(l+1)$-dimensional simple U-module is isomorphic to some $V(l, \psi(\omega))$, where $\psi\colon U^0 \to \Bbbk$ is an algebra homomorphism with $\psi(\omega') = \psi(\omega)r^{-l}s^l$. We may assume that $u = \sum\limits_{i,j,k,m} a_{ijkm}e^i f^j \omega^k(\omega')^m \in V$ is the highest weight vector with weight $(\psi(\omega), \psi(\omega)r^{-l}s^l)$ for the action of $\operatorname{ad}\omega$ and $\operatorname{ad}\omega'$. It follows that

$$\sum_{i,j,k,m} a_{ijkm}(rs^{-1})^{i-j}e^i f^j \omega^k(\omega')^m = \psi(\omega)\sum_{i,j,k,m} a_{ijkm}e^i f^j \omega^k(\omega')^m,$$
$$\sum_{i,j,k,m} a_{ijkm}(r^{-1}s)^{i-j}e^i f^j \omega^k(\omega')^m = \psi(\omega)r^{-l}s^l\sum_{i,j,k,m} a_{ijkm}e^i f^j \omega^k(\omega')^m.$$

This implies $(rs^{-1})^{i-j} = \psi(\omega)$ and $(r^{-1}s)^{i-j} = \psi(\omega)r^{-l}s^l$. Hence, $(rs^{-1})^{2i-2j-l} = 1$. Since rs^{-1} is not a root of unity, we have $2i - 2j = l$, which implies that $i \geqslant j$ and l is even. So we can rewrite u in the form $\sum\limits_j e^{n+j}f^j g_j$ for some polynomials $g_j \in \Bbbk[\omega^{\pm 1}, (\omega')^{\pm 1}]$ and $l = 2n$. According to (2.4), we can replace each factor ef by the Casimir element $\mathcal{C}$ modulo a polynomial in $\Bbbk[\omega^{\pm 1}, (\omega')^{\pm 1}]$. Therefore, we rewrite u in the form $e^n g(\mathcal{C})$ for some polynomial $g(\mathcal{C}) \in \Bbbk[\omega^{\pm 1}, (\omega')^{\pm 1}][\mathcal{C}]$. Noticing that $(\operatorname{ad} e)u = 0$, we have

$$ee^n g(\mathcal{C}) = \omega e^n g(\mathcal{C})\omega^{-1}e = r^n s^{-n}e^n g(\mathcal{C})e.$$

Since U has no zero divisors, it implies

$$eg(\mathcal{C}) = r^n s^{-n} g(\mathcal{C})e.$$

Noting that $\mathcal{C} \in Z(U)$, we obtain that $g(\mathcal{C})$ has the form $\omega^{-n}h$ for some $h \in Z(U)$. Hence, $u = he^n\omega^{-n}$ and $V = [he^n\omega^{-n}]$. Since $he^n\omega^{-n}$ is also the highest weight vector with weight $(r^n s^{-n}, r^{-n}s^n)$, we have $[he^n\omega^{-n}] \cong [e^n\omega^{-n}] \cong V(2n, r^n s^{-n})$. $\square$

Corollary 3.3. *Let $u, v \in F(U)$ be two weight vectors. Then u and v are both the highest weight vectors if and only if uv is the highest weight vector.*

Proof. Let u, v be the highest weight vectors. Then by Proposition 3.2, it is clear that uv is the highest weight vector. Conversely, assume that u and v are weight vectors with weight $(r^c s^{-c}, r^{-c} s^c)$ and $(r^d s^{-d}, r^{-d} s^d)$, respectively. Note that $u, v \in F(U)$. There exist $a, b \in \mathbb{N}$ such that $(\operatorname{ad} e)^a u \neq 0$, $(\operatorname{ad} e)^{a+1} u = 0$, $(\operatorname{ad} e)^b v \neq 0$ and $(\operatorname{ad} e)^{b+1} v = 0$. By Lemma 2.3 we have

$$\begin{aligned}(\operatorname{ad} e)^{a+b}(uv) &= \sum_{i=0}^{a+b} s^{-i(a+b-i)} \begin{bmatrix} a+b \\ i \end{bmatrix} (\operatorname{ad} e^{a+b-i}\omega^i) u (\operatorname{ad} e^i) v \\ &= r^{bc} s^{-ab-bc} \begin{bmatrix} a+b \\ b \end{bmatrix} (\operatorname{ad} e)^a u (\operatorname{ad} e)^b v \\ &\quad + r^{ac} s^{-ab-ac} \begin{bmatrix} a+b \\ a \end{bmatrix} (\operatorname{ad} e)^b u (\operatorname{ad} e)^a v.\end{aligned}$$

This implies that $(\operatorname{ad} e)^{a+b}(uv) \neq 0$. Since $(\operatorname{ad} e)(uv) = 0$, we have $a + b = 0$, which implies $a = b = 0$. Hence, u and v are both the highest weight vectors. $\square$

Now we characterize the unit group of $F(U)$.

Proposition 3.4. *The unit group of $F(U)$ is equal to $\{a(\omega\omega')^n \colon a \in \Bbbk^*, n \in \mathbb{Z}\}$.*

Proof. Let u be any invertible element in $F(U)$. Since 1 is the highest weight vector with weight $(1, 1)$ and ω and ω' are group-like elements, it follows that u and u^{-1} are weight vectors. According to Corollary 3.3, both u and u^{-1} are the highest weight vectors with weight $(1, 1)$. By Proposition 3.2, we have $u \in Z(U)$. Since u is invertible, $u \in \{a(\omega\omega')^n \colon a \in \Bbbk^*, n \in \mathbb{Z}\}$. It is obvious that each element of $\{a(\omega\omega')^n \colon a \in \Bbbk^*, n \in \mathbb{Z}\}$ is invertible in $F(U)$. $\square$

Proposition 3.5. *Suppose $u = \sum\limits_{i=1}^{m} h_i e^{n_i} \omega^{-n_i}$, where $0 \neq h_i \in Z(U)$, $1 \leqslant i \leqslant m$ and $n_1 > n_2 > \ldots > n_m$. Then*

$$[u] = \bigoplus_{i=1}^{m} [h_i e^{n_i} \omega^{-n_i}] \cong \bigoplus_{i=1}^{m} [e^{n_i} \omega^{-n_i}].$$

Proof. Obviously, we have

$$[u] \subseteq \sum_{i=1}^{m} [h_i e^{n_i} \omega^{-n_i}]. \tag{3.1}$$

Since $h_i \in Z(U)$ for $1 \leqslant i \leqslant m$, by Proposition 3.1, it follows that

$$[h_i e^{n_i} \omega^{-n_i}] \cong [e^{n_i} \omega^{-n_i}] \cong V(2n_i, r^{n_i} s^{-n_i}).$$

Note that the integers n_i are pairwise distinct. It follows that the right-hand side of (3.1) is a sum of non-isomorphic simple modules, and hence it is direct. For the converse inclusion, since $n_1 > n_2 > \ldots > n_m$, we have

$$(\operatorname{ad} f)^{2n_1} u = (\operatorname{ad} f)^{2n_1}(h_1 e^{n_1} \omega^{-n_1}) \in [u].$$

According to Proposition 3.1, $(\operatorname{ad} f)^{2n_1}(h_1 e^{n_1} \omega^{-n_1})$ is the lowest weight vector with weight $(r^{-n_1} s^{n_1}, r^{n_1} s^{-n_1})$ in the simple module $[h_1 e^{n_1} \omega^{-n_1}]$. This implies that

$$h_1 e^{n_1} \omega^{-n_1} \in [u].$$

Similarly, one can show that $h_i e^{n_i} \omega^{-n_i} \in [u]$ for $1 < i \leqslant m$. Thus,

$$\sum_{i=1}^{m} [h_i e^{n_i} \omega^{-n_i}] \subseteq [u].$$

$\square$

In order to depict the elements in $F(U)$, we need the following lemma.

Lemma 3.6. *We have* $[\omega^{-1}] = [e\omega^{-1}] \oplus [\mathcal{C}(\omega\omega')^{-1}]$ *and* $[\omega'] = [e\omega'] \oplus [\mathcal{C}]$.

Proof. From the proof of Proposition 3.1, we know that $e\omega^{-1}$ is the highest weight vector and

$$\begin{aligned}(\operatorname{ad} f) e\omega^{-1} &= (1 - rs^{-1})\Big(ef + \frac{s(\omega - \omega')}{(r-s)^2}\Big)(\omega\omega')^{-1},\\ (\operatorname{ad} f)^2 e\omega^{-1} &= -s^{-2}[2]!\, f(\omega\omega')^{-1}, \quad (\operatorname{ad} f)^3 e\omega^{-1} = 0.\end{aligned}$$

Hence, $[e\omega^{-1}]$ is spanned by

$$e\omega^{-1}, \quad \Big(ef + \frac{s(\omega - \omega')}{(r-s)^2}\Big)(\omega\omega')^{-1}, \quad f(\omega\omega')^{-1}.$$

By direct calculations, we have

$$\begin{aligned}(\operatorname{ad} e)\omega^{-1} = (1 - r^{-1}s)e\omega^{-1}, &\quad (\operatorname{ad} f)\omega^{-1} = (1 - rs^{-1}) f(\omega\omega')^{-1},\\ (\operatorname{ad} e) f(\omega\omega')^{-1} &= (1 - r^{-1}s)\Big(ef + \frac{s(\omega - \omega')}{(r-s)^2}\Big)(\omega\omega')^{-1}.\end{aligned}$$

Then $[\omega^{-1}]$ is spanned by ω^{-1}, $e\omega^{-1}$, $f(\omega\omega')^{-1}$ and $(ef + s(\omega - \omega')/(r-s)^2)(\omega\omega')^{-1}$. Since

$$\omega^{-1} = -\frac{(r-s)^2}{r+s}\Big(ef + \frac{s(\omega - \omega')}{(r-s)^2}\Big)(\omega\omega')^{-1} + \frac{(r-s)^2}{r+s}\mathcal{C}(\omega\omega')^{-1},$$

it follows that $[\omega^{-1}] \subseteq [e\omega^{-1}] \oplus [\mathcal{C}(\omega\omega')^{-1}]$. Since

$$\mathcal{C}(\omega\omega')^{-1} = \frac{r+s}{(r-s)^2}\omega^{-1} + \Big(ef + \frac{s(\omega-\omega')}{(r-s)^2}\Big)(\omega\omega')^{-1} \in [\omega^{-1}] + [e\omega^{-1}] = [\omega^{-1}],$$

we have that $[\omega^{-1}] = [e\omega^{-1}] \oplus [\mathcal{C}(\omega\omega')^{-1}]$. Note that $\omega' = \omega^{-1}(\omega\omega')$ and $\omega\omega' \in Z(U)$. The second equation holds. $\square$

Proposition 3.7. *$F(U)$ contains the elements $e^i\omega^{-j}(\omega')^k$ and $f^a\omega^{-b}(\omega')^c$ for $a \geqslant 0$, $b+c \geqslant 0$ and $j+k \geqslant i \geqslant 0$.*

Proof. By Proposition 3.1 and Lemma 3.6, we have

$$e\omega^{-1}, e\omega', f, \omega^{-1}, \omega', \omega\omega' \in F(U).$$

Since $F(U)$ is a subalgebra of U, it is clear that the assertion holds. $\square$

Corollary 3.8. *We have $Z(F(U)) = Z(U)$.*

Proof. Since $Z(U) \subseteq F(U)$, we have $Z(U) \subseteq Z(F(U))$. Take any $u \in Z(F(U))$. By Proposition 3.7, $e\omega^{-1}$, f, ω^{-1} and ω' are contained in $F(U)$. Hence, $ue\omega^{-1} = e\omega^{-1}u$ and $u\omega^{-1} = \omega^{-1}u$. So we have $ue\omega^{-1} = eu\omega^{-1}$. Since U has no zero divisors, it follows that $ue = eu$. Thus, $u \in Z(U)$. Therefore, $Z(F(U)) \subseteq Z(U)$. $\square$

4. Annihilator ideals of finite dimensional simple modules

The aim of this section is to determine all annihilator ideals of finite dimensional simple modules of U. We first describe all ideals of U by using the structure of $F(U)$. Then we classify the annihilator ideals of finite dimensional simple modules by generators. Moreover, we characterize all ideals of finite codimension.

Proposition 4.1. *Let I be any nonzero ideal of U. Then there exist $t > 0$, $n_1, n_2, \ldots, n_t \in \mathbb{N}$ and $h_1, h_2, \ldots, h_t \in Z(U)$ such that*

$$I = (h_1e^{n_1}, h_2e^{n_2}, \ldots, h_te^{n_t}).$$

Proof. Since U is a Noetherian ring, each ideal of U is finitely generated. It suffices to show that the assertion holds for $I = (u_1, u_2, \ldots, u_s)$ for $u_i \in U$, $1 \leqslant i \leqslant s$. By Proposition 3.7, $e\omega^{-1}, e\omega', f, \omega^{-1}, \omega' \in F(U)$. It follows that $e^af^b\omega^{-c}(\omega')^d \in F(U)$ for $a, b \in \mathbb{N}$, $c, d \in \mathbb{Z}$ and $0 \leqslant a \leqslant c+d$. Thus, there exists $l \in \mathbb{N}$ such that

$u_i\omega^{-l}(\omega')^l \in F(U)$ for $1 \leqslant i \leqslant s$. It follows that we may write $I = (u_1, u_2, \ldots, u_s)$ for $u_i \in F(U)$, $1 \leqslant i \leqslant s$. By Propositions 2.1 and 3.2, we have that $[u_i]$ is completely reducible and

$$[u_i] = \sum_{j=1}^{k_i} [h_{ij} e^{n_{ij}} \omega^{-n_{ij}}],$$

where $h_{ij} \in Z(U)$, $k_i > 0$, $1 \leqslant j \leqslant k_i$, $1 \leqslant i \leqslant s$. It is clear $[u_i] \subseteq (u_i)$. Hence,

$$(h_{i1}e^{n_{i1}}, h_{i2}e^{n_{i2}}, \ldots, h_{ik_i}e^{n_{ik_i}}) \subseteq (u_i).$$

Conversely,

$$u_i \in [u_i] = \sum_{j=1}^{k_i} [h_{ij} e^{n_{ij}} \omega^{-n_{ij}}] \subseteq (h_{i1}e^{n_{i1}}, h_{i2}e^{n_{i2}}, \ldots, h_{ik_i}e^{n_{ik_i}}).$$

So we have

$$I = (h_{ij_i} e^{n_{ij_i}} \colon\ h_{ij_i} \in Z(U),\ 1 \leqslant j_i \leqslant k_i,\ 1 \leqslant i \leqslant s).$$

Renumber h_{ij_i} and $e^{n_{ij_i}}$, $1 \leqslant j_i \leqslant k_i, 1 \leqslant i \leqslant s$, as $h_1, h_2, \ldots h_t$ and $e^{n_1}, e^{n_2}, \ldots, e^{n_t}$. Thus, we finish the proof. □

For $h \in Z(U)$, denote by $\langle h \rangle$ the ideal of $Z(U)$ generated by h. To describe the ideals of U more precisely, we give the following theorem.

Theorem 4.2. *Let I be any nonzero ideal of U. Then there exist $t > 0$, $n_1 > n_2 > \ldots > n_t \geqslant 0$, $m_i \geqslant 1$, $h_{ij_i} \in Z(U)$ for $1 \leqslant j_i \leqslant m_i$, $1 \leqslant i \leqslant t$ such that*

$$I = (h_{ij_i} e^{n_i} \colon\ 1 \leqslant j_i \leqslant m_i, 1 \leqslant i \leqslant t),$$

where

$$\langle h_{11}, \ldots, h_{1m_1} \rangle \supsetneq \langle h_{21}, \ldots, h_{2m_2} \rangle \supsetneq \ldots \supsetneq \langle h_{t1}, \ldots, h_{tm_t} \rangle.$$

Proof. According to Proposition 4.1, there exist $t > 0$, $n_i \geqslant 0$, $m_i \geqslant 1$, $h_{ij_i} \in Z(U)$ for $1 \leqslant j_i \leqslant m_i$, $1 \leqslant i \leqslant t$ such that

$$I = (h_{ij_i} e^{n_i} \colon\ h_{ij_i} \in Z(U),\ 1 \leqslant j_i \leqslant m_i,\ 1 \leqslant i \leqslant t). \tag{4.1}$$

After rearranging the generators, we may assume that $n_1 > n_2 > \ldots > n_t \geqslant 0$. Note that

$$\begin{aligned} h_{t1}e^{n_{t-1}}, \ldots, h_{tm_t}e^{n_{t-1}} &\in (h_{t1}e^{n_t}, \ldots, h_{tm_t}e^{n_t}) \\ &\subseteq (h_{t-1,1}e^{n_{t-1}}, \ldots, h_{t-1,m_{t-1}}e^{n_{t-1}}, h_{t1}e^{n_t}, \ldots, h_{tm_t}e^{n_t}). \end{aligned}$$

We have

$$
\begin{aligned}
&(h_{t-1,1}e^{n_{t-1}}, \dots, h_{t-1,m_{t-1}}e^{n_{t-1}}, h_{t1}e^{n_t}, \dots, h_{tm_t}e^{n_t}) \\
&\quad = (h_{t-1,1}e^{n_{t-1}}, \dots, h_{t-1,m_{t-1}}e^{n_{t-1}}, h_{t1}e^{n_{t-1}}, \dots, h_{tm_t}e^{n_{t-1}}, h_{t1}e^{n_t}, \dots, h_{tm_t}e^{n_t}).
\end{aligned}
$$

So one can replace the generators $h_{t-1,1}e^{n_{t-1}}, \dots, h_{t-1,m_{t-1}}e^{n_{t-1}}$ by

$$h_{t-1,1}e^{n_{t-1}}, \dots, h_{t-1,m_{t-1}}e^{n_{t-1}}, h_{t1}e^{n_{t-1}}, \dots, h_{tm_t}e^{n_{t-1}}.$$

It is clear that

$$\langle h_{t-1,1}, \dots, h_{t-1,m_{t-1}}, h_{t1}, \dots, h_{tm_t}\rangle \supseteq \langle h_{t1}, \dots, h_{tm_t}\rangle.$$

Therefore, without loss of generality, we may assume that

$$\langle h_{t-1,1}, \dots, h_{t-1,m_{t-1}}\rangle \supseteq \langle h_{t1}, \dots, h_{tm_t}\rangle.$$

Repeating this process with $t-1$ instead of t, we obtain

$$\langle h_{t-2,1}, \dots, h_{t-2,m_{t-2}}\rangle \supseteq \langle h_{t-1,1}, \dots, h_{t-1,m_{t-1}}\rangle$$

and so on. So we can rewrite (4.1) as

$$I = (h_{ij_i}e^{n_i}\colon\ h_{ij_i} \in Z(U),\ 1 \leqslant j_i \leqslant m_i,\ 1 \leqslant i \leqslant t),$$

where

$$\langle h_{11}, \dots, h_{1m_1}\rangle \supseteq \langle h_{21}, \dots, h_{2m_2}\rangle \supseteq \dots \supseteq \langle h_{t1}, \dots, h_{tm_t}\rangle.$$

If

$$\langle h_{k-1,1}, \dots, h_{k-1,m_{k-1}}\rangle = \langle h_{k1}, \dots, h_{km_k}\rangle$$

for some k, then we can suppose

$$h_{k-1,i} = g_{i1}h_{k1} + \dots + g_{im_k}h_{km_k}$$

for $g_{ij} \in Z(U)$, $1 \leqslant i \leqslant m_{k-1}$, $1 \leqslant j \leqslant m_k$. It follows that

$$h_{k-1,i}e^{n_{k-1}} = (g_{i1}h_{k1}e^{n_k} + \dots + g_{im_k}h_{km_k}e^{n_k})e^{n_{k-1}-n_k} \in (h_{k1}e^{n_k}, \dots, h_{km_k}e^{n_k}).$$

Thus, we can eliminate the generators $h_{k-1,1}e^{n_{k-1}}, \dots, h_{k-1,m_{k-1}}e^{n_{k-1}}$ of I. Since I is finitely generated, the process will end in finitely many steps. Hence, we finish the proof. □

To characterize the ideals of finite codimension, we introduce a family of polynomials. For $n \geqslant 1$, let

$$\varphi_n(\mathcal{C}) = \mathcal{C}^2 - \frac{r^{n+1}s^{1-n} + r^{1-n}s^{n+1} + 2rs}{(r-s)^4}\omega\omega' \in Z(U).$$

Then we have the following lemma.

Lemma 4.3. *For $n \geqslant 1$, we have*
(1) $e^{n-1}\varphi_n(\mathcal{C}) \in (e^n)$;
(2) *if* $0 \leqslant m < n$, *then* $e^m \prod_{i=m+1}^{n} \varphi_i(\mathcal{C}) \in (e^n)$.

Proof. By (2.5) we have

$$s^{1-n}e^{n-1}\omega - r^{1-n}e^{n-1}\omega' \in (e^n).$$

It follows that

$$s^2e^{n-1}\omega^2 - s^{n+1}r^{1-n}e^{n-1}\omega\omega' \in (e^n), \quad s^{1-n}r^{n+1}e^{n-1}\omega\omega' - r^2e^{n-1}(\omega')^2 \in (e^n).$$

Hence

$$\begin{aligned} e^{n-1}\mathcal{C}^2 &= e^{n-1}\Big(ef + \frac{s\omega + r\omega'}{(r-s)^2}\Big)^2 \equiv e^{n-1}\frac{s^2\omega^2 + r^2(\omega')^2 + 2sr\omega\omega'}{(r-s)^4} \mod(e^n) \\ &\equiv e^{n-1}\frac{s^{n+1}r^{1-n} + s^{1-n}r^{1+n} + 2sr}{(r-s)^4}\omega\omega' \mod(e^n). \end{aligned}$$

Thus, assertion (1) holds. In order to show (2), we use induction on $n - m$. If $m = n - 1$, then it holds by (1). Assume that the result is true for $m + 1$, namely,

$$e^{m+1}\prod_{i=m+2}^{n}\varphi_i(\mathcal{C}) \in (e^n).$$

By (1) we have $e^m\varphi_{m+1}(\mathcal{C}) \in (e^{m+1})$. Hence,

$$e^m\prod_{i=m+1}^{n}\varphi_i(\mathcal{C}) = e^m\varphi_{m+1}(\mathcal{C})\prod_{i=m+2}^{n}\varphi_i(\mathcal{C}) \in (e^{m+1})\prod_{i=m+2}^{n}\varphi_i(\mathcal{C}).$$

Note that $\prod_{i=m+2}^{n}\varphi_i(\mathcal{C}) \in Z(U)$, so the assertion holds for m by induction hypothesis. $\square$

Proposition 4.4. *If I is an ideal of U of finite codimension, then there exist positive integers $n, m_1, m_2, \ldots, m_n$ and $h_{ij_i} \in Z(U)$ for $1 \leqslant j_i \leqslant m_i$, $1 \leqslant i \leqslant n$ such that*

$$I = (e^n, h_{ij_i} e^{n-i}\colon\ h_{ij_i} \in Z(U),\ 1 \leqslant j_i \leqslant m_i,\ 1 \leqslant i \leqslant n),$$

where

$$Z(U) \supsetneqq \langle h_{11}, \ldots, h_{1m_1} \rangle \supsetneqq \langle h_{21}, \ldots, h_{2m_2} \rangle \supsetneqq \ldots \supsetneqq \langle h_{n1}, \ldots, h_{nm_n} \rangle,$$

and

$$\langle h_{i1}, \ldots, h_{im_i} \rangle \supseteq \left\langle \prod_{t=n-i+1}^{n} \varphi_t(\mathcal{C}) \right\rangle.$$

Proof. Assume that $e^i\omega^{-i} \notin I$ for some $i \in \mathbb{N}$. Since $[e^i\omega^{-i}]$ is a simple U-module, it follows that $[e^i\omega^{-i}] \cap I = 0$. Thus, $\dim_{\Bbbk} U/I \geqslant \dim_{\Bbbk}[e^i\omega^{-i}] \geqslant i$. Since I is of finite codimension, the set $\{i \in \mathbb{N}\colon\ e^i\omega^{-i} \notin I\}$ is finite. Let $n = \min\{i \in \mathbb{N}\colon\ e^i\omega^{-i} \in I\}$. Then $e^n\omega^{-n} \in I$. Hence, $e^n \in I$. By Lemma 4.3, we have

$$e^{n-i} \prod_{t=n-i+1}^{n} \varphi_t(\mathcal{C}) \in (e^n) \subseteq I$$

for $1 \leqslant i \leqslant n$. By Theorem 4.2, the assertion holds. □

In order to depict the annihilator ideal of simple module $V(n, \psi(\omega))$, we need the following lemma.

Lemma 4.5. *Set $R = \Bbbk[x, y^{\pm 1}]$. Suppose that $h(x, y, y^{-1}) \in R$ satisfies $h(a, b, b^{-1}) = 0$ for $a \in \Bbbk$ and $b \in \Bbbk^*$. Then it follows that*

$$h(x, y, y^{-1}) \in (x - a)R + (y - b)R.$$

Proof. If $h(x, y, y^{-1}) = 0$, then it is clear that the assertion holds. If $h(x, y, y^{-1}) \neq 0$, then there exists $m \in \mathbb{N}$ such that $y^m h(x, y, y^{-1})$ is a polynomial of variables x and y. Suppose that $\deg_x y^m h(x, y, y^{-1}) = n \geqslant 0$. We may assume that

$$y^m h(x, y, y^{-1}) = (x - a)^n f_1(y) + (x - a)^{n-1} f_2(y) + \ldots + f_{n+1}(y)$$

for $f_i(y) \in \Bbbk[y]$, $1 \leqslant i \leqslant n + 1$. Noting that $h(a, b, b^{-1}) = 0$, we have

$$0 = b^m h(a, b, b^{-1}) = f_{n+1}(b).$$

Thus, $y - b \mid f_{n+1}(y)$. Hence, it follows that

$$y^m h(x, y, y^{-1}) \in (x - a)R + (y - b)R.$$

Since y is invertible in R, the assertion holds. □

Theorem 4.6. *Let* $\psi\colon U^0 \to \Bbbk$ *be an algebra homomorphism with* $\psi(\omega') = \psi(\omega)r^{-n}s^n$ *and* $\psi_n = \psi(\omega)r^{-n}(r^{n+1}+s^{n+1})/(r-s)^2$. *Then the annihilator ideal of simple module* $V(n,\psi(\omega))$ *is*

$$(e^{n+1}, \mathcal{C}-\psi_n, \omega\omega' - \psi(\omega)^2 r^{-n}s^n).$$

Proof. By Proposition 4.1, let

$$I = (h_1e^{m_1}, h_2e^{m_2}, \ldots, h_te^{m_t})$$

be the annihilator ideal of $V(n,\psi(\omega))$, where $h_i \in Z(U)$ and $m_i \geqslant 0$ for $1 \leqslant i \leqslant t$. Since $V(n,\psi(\omega))$ has a $\Bbbk$-basis $v_0, v_1, \ldots, v_n$ with U-action given by (2.6), it follows that

$$e^{n+1}v_j = 0, \quad (\omega\omega')v_j = \omega(\psi(\omega)r^{-n+j}s^{n-j})v_j = \psi(\omega)^2r^{-n}s^nv_j$$

and

$$\begin{aligned}\mathcal{C}v_j &= \Big(fe + \frac{r\omega+s\omega'}{(r-s)^2}\Big)v_j \\ &= \psi(\omega)r^{-n}[j][n+1-j]v_j + \frac{\psi(\omega)r^{-j+1}s^j + \psi(\omega)r^{-n+j}s^{n+1-j}}{(r-s)^2}v_j = \psi_nv_j\end{aligned}$$

for $0 \leqslant j \leqslant n$. Therefore,

$$(e^{n+1}, \mathcal{C}-\psi_n, \omega\omega' - \psi(\omega)^2r^{-n}s^n) \subseteq I.$$

For the converse inclusion, since I is the annihilator ideal of $V(n,\psi(\omega))$, we have

$$h_ie^{m_i}v_j = 0$$

for $1 \leqslant i \leqslant t$ and $0 \leqslant j \leqslant n$. If $m_i \geqslant n+1$ for some i, then it is clear that

$$h_ie^{m_i} \in (e^{n+1}) \subseteq (e^{n+1}, \mathcal{C}-\psi_n, \omega\omega' - \psi(\omega)^2r^{-n}s^n).$$

If $1 \leqslant m_i \leqslant n$, noting that $e^{m_i}v_j = 0$ for $0 \leqslant j \leqslant m_i - 1$, then we have $h_iv_{j-m_i} = 0$ for $m_i \leqslant j \leqslant n$. Since $h_i \in Z(U) = \Bbbk[\omega\omega', (\omega\omega')^{-1}][\mathcal{C}]$, we may rewrite h_i in the form $h_i(\mathcal{C}, \omega\omega', (\omega\omega')^{-1})$. It follows that

$$h_i(\psi_n, \psi(\omega)^2r^{-n}s^n, \psi(\omega)^{-2}r^ns^{-n}) = 0.$$

Thus, by Lemma 4.5 we have

$$h_i \in \langle \mathcal{C}-\psi_n, \omega\omega' - \psi(\omega)^2r^{-n}s^n\rangle.$$

Therefore,

$$h_i e^{m_i} \in (\mathcal{C} - \psi_n, \omega\omega' - \psi(\omega)^2 r^{-n} s^n) \subseteq (e^{n+1}, \mathcal{C} - \psi_n, \omega\omega' - \psi(\omega)^2 r^{-n} s^n).$$

If $m_i = 0$, then we can prove

$$h_i e^{m_i} \in (e^{n+1}, \mathcal{C} - \psi_n, \omega\omega' - \psi(\omega)^2 r^{-n} s^n)$$

in a similar way. Hence, $I \subseteq (e^{n+1}, \mathcal{C} - \psi_n, \omega\omega' - \psi(\omega)^2 r^{-n} s^n)$. $\square$

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