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ON OPTIMAL PARAMETERS INVOLVED WITH TWO-WEIGHTED ESTIMATES OF COMMUTATORS OF SINGULAR AND FRACTIONAL OPERATORS WITH LIPSCHITZ SYMBOLS

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Abstract. We prove two-weighted norm estimates for higher order commutator of singular integral and fractional type operators between weighted L^p and certain spaces that include Lipschitz, BMO and Morrey spaces. We also give the optimal parameters involved with these results, where the optimality is understood in the sense that the parameters defining the corresponding spaces belong to a certain region out of which the classes of weights are satisfied by trivial weights. We also exhibit pairs of nontrivial weights in the optimal region satisfying the conditions required.

Keywords: fractional operator; singular integral operator; commutator; weight

MSC 2020: 42B20, 42B25, 42B35

1. Introduction

A significant contribution is well known that represents the continuity properties of different operators from harmonic analysis in the study of regularity properties of the solutions of certain partial differential equations. There is a vast evidence of this fact and in this direction, the commutators of operators with symbol functions in certain adequate spaces play an important role, see, for example, [1] –[6] and [13]. Thus, their boundedness properties allow to derive regularity properties related with the solutions of such PDE's.

In [7] the authors proved one-weight boundedness results for the classical fractional integral operator I_α, $0 < \alpha < n$, between certain spaces including weighted L^p-Lipschitz(β) estimates, where the relation between p and β is standard, $\beta/n = \alpha/n - 1/p$. The Lipschitz spaces considered in that article are generalizations of some

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known integral version of classical Lipschitz(β) spaces. In [8] similar problems were studied for the Hilbert transform and certain generalizations of the Lipschitz spaces defined in [7].

On the other hand, in [10] a two-weighted problem for the boundedness of I_α of type L^p-Lipschitz(β) was studied. The parameters involved belong to a region out of which the weights are trivial, that is $v = 0$ or $w = \infty$ a.e., and, in this sense, this is an optimal estimate. Similar results in this spirit were proved in [12] for commutators of singular integral and fractional type operators by considering the Lipschitz spaces given in [11].

In this paper we prove two-weighted norm estimates for singular integral and fractional type operators and their higher order commutators between weighted L^p and certain spaces related to a parameter β, that include Lipschitz, BMO and Morrey spaces and that are wider than those considered in [12]. Moreover the classes of weights are quite different from those given there, including local and global conditions. We also give the optimal parameters involved with these results, where the optimality is understood in the sense that the parameters p and β belong to a certain region out of which the classes of weights are satisfied by trivial weights. Moreover, we exhibit concrete pairs of nontrivial weights in the optimal region satisfying the conditions required on the weights, where the boundedness results includes values of β describing Lipschitz(β), BMO and Morrey spaces, that is, $0 < \beta < 1$, $\beta = 0$ and $\beta < 0$, respectively. Our results extend those contained in [10] for the fractional integral operator, see also [7] for the one-weight case. We prove that a one-weight result can only holds whenever the relation between the parameters is standard. We also give the relation between our classes of weights and those given in [12], which are natural extensions of the A_1-Muckenhoupt class in one-weight estimates.

The paper is organized as follows. In Section 2 we give the preliminaries and state the main results. In Section 3 we prove the optimality of the classes of weights and give some other properties. Finally, in Section 4 we show the main results.

2. Preliminaries

We say that $A \lesssim B$ if there exists a positive constant c such that $A \leqslant c\,B$. In this section we give the definitions of operators we will be dealing with and the functional class of symbols in order to define the commutators.

We consider singular integral operators of convolution type T with kernel K, that is T is bounded on $L^2(\mathbb{R}^n)$ and if $x \notin \operatorname{supp} f$

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)\,\mathrm{d}y. \tag{2.1}$$

The kernel K is a measurable function defined away from 0, satisfying a certain smoothness condition to be described later. We also suppose that K satisfies the typical size condition given by

$$|K(x-y)| \leqslant \frac{C}{|x-y|^n},$$

which is called S_0^*.

Related with the singular integral operator T, we can formally define the commutator with symbol $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$[b,T]f = bTf - T(bf).$$

The commutator of order $m \in \mathbb{N} \cup \{0\}$ of T is defined by

$$T_b^0 = T, \quad T_b^m = [b, T_b^{m-1}].$$

We also consider fractional operators of convolution type T_α, $0 < \alpha < n$, defined by

$$T_\alpha f(x) = \int_{\mathbb{R}^n} K_\alpha(x-y) f(y)\,\mathrm{d}y, \tag{2.2}$$

where the kernel K_α is not identically zero and satisfies certain size and smoothness conditions.

Let $0 < \delta < 1$. We say that a function b belongs to the space $\Lambda(\delta)$ if there exists a positive constant C such that for every $x, y \in \mathbb{R}^n$,

$$|b(x) - b(y)| \leqslant C|x-y|^\delta.$$

The smallest of such constants is denoted by $\|b\|_{\Lambda(\delta)}$. The space $\Lambda(\delta)$ is the well known Lipschitz space in the classical literature. We will be dealing with commutators with symbols belonging to this class of functions.

Let $0 \leqslant \alpha < n$. We say that a kernel $K_\alpha \in S_\alpha^*$, if there exists a positive constant C such that

$$|K_\alpha(x)| \leqslant \frac{C}{|x|^{n-\alpha}}.$$

We say that a kernel K_α belongs to $K^*_{\alpha,\infty}$ if there exist a positive constant C and $0 < \eta \leqslant 1$ such that

$$|K_\alpha(x-y) - K_\alpha(x'-y)| + |K_\alpha(y-x) - K_\alpha(y-x')| \leqslant C\frac{|x-x'|^\eta}{|x-y|^{n-\alpha+\eta}},$$

whenever $|x-y| \geqslant 2|x-x'|$.

It is easy to check that the fractional integral operator I_α with the kernel $K_\alpha(x) = |x|^{\alpha-n}$ satisfies the conditions S_α^* and $K^*_{\alpha,\infty}$ for $0 < \alpha < n$.

Related with the fractional type integral operators T_α, we can formally define the higher order commutators with symbol $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$T^m_{\alpha,b} f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^m K_\alpha(x-y) f(y)\,\mathrm{d}y,$$

where $m \in \mathbb{N} \cup \{0\}$ is the order of the commutator. Clearly, $T^0_{\alpha,b} = T_\alpha$.

As we have said, we are interested in studying the boundedness properties of the commutators $T^m_{\alpha,b}$ with symbol $b \in \Lambda(\delta)$, from weighted Lebesgue spaces into a certain weighted version of Lipschitz spaces. For $\beta \in \mathbb{R}$ and a weight w, these spaces are denoted by $\mathcal{L}_w(\beta)$ and collect the functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ that satisfy

$$\frac{1}{w(B)|B|^\beta} \int_B |f(x) - m_B(f)|\,\mathrm{d}x \leqslant C$$

for some positive constant C. When $\beta = 0$, $\mathcal{L}_w(0)$ is a weighted version of the bounded mean oscilation space introduced by Muckenhoupt and Wheeden in [9]. Moreover, $\mathcal{L}_1(\beta)$ gives the known Lipschitz integral space for β in the range $0 < \beta < 1/n$ and the Morrey space for $-1 < \beta < 0$. This class of functions was defined in [7].

In [12] the authors proved two weighted boundedness results for commutators of a great variety of operators between Lebesgue and Lipschitz spaces $\mathbb{L}_w(\beta)$. These spaces collect the functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ that satisfy

$$\frac{\|(1/w)\chi_B\|_\infty}{|B|^{1+\beta}} \int_B |f(x) - m_B(f)|\,\mathrm{d}x \leqslant C \tag{2.3}$$

for some positive constant C. It is easy to check that, for a general weight w, $\mathbb{L}_w(\beta) \subset \mathcal{L}_w$ and, if w belongs to the A_1-Muckenhoupt class then both spaces coincide.

Related to the spaces $\mathcal{L}_w(\beta)$, we introduce the following class of weights.

Definition 2.1. Let $0 \leqslant \alpha < n$, $0 \leqslant \delta \leqslant 1$ and $1 < r \leqslant \infty$. Put $\widetilde{\alpha} = m\delta + \alpha$, $m \in \mathbb{N} \cup \{0\}$, and $\tilde{\delta} \leqslant \delta$. We say that a pair of weights (w, v) belongs to $\mathcal{H}(r, \widetilde{\alpha}, \tilde{\delta})$, if the inequality

$$|B|^{(\delta-\tilde{\delta})/n} \left(\int_{\mathbb{R}^n} \frac{v^{r'}(y)}{(|B|^{1/n} + |x_B - y|)^{r'(n-\widetilde{\alpha}+\delta)}}\,\mathrm{d}y \right)^{1/r'} \lesssim \frac{w(B)}{|B|} \tag{2.4}$$

holds for every ball $B \subset \mathbb{R}^n$, where x_B is the center of B. In the case $r = 1$ we say that (w, v) belongs to $\mathcal{H}(1, \widetilde{\alpha}, \tilde{\delta})$ if the inequality

$$|B|^{(\delta-\tilde{\delta})/n} \left\| \frac{v(\cdot)}{(|B|^{1/n} + |x_B - \cdot|)^{n-\widetilde{\alpha}+\delta}} \right\|_\infty \lesssim \frac{w(B)}{|B|} \tag{2.5}$$

holds for every ball $B \subset \mathbb{R}^n$, where x_B is the center of B.

When $0 < \alpha < n$, $m = 0$ and $\delta = 1$, the class in (2.4) was introduced in [10]. If, in addition, $w = v$ and $\tilde{\delta} = \widetilde{\alpha} - n/r$, then the class $\mathcal{H}(r, \widetilde{\alpha}, \tilde{\delta})$ was defined in [7]. When $\alpha = 0$, $\delta = 1$, $m = 0$ and $w = v$ the class $\mathcal{H}(\infty, 0, 0)$ is the class B_2 in [9].

Remark 2.2. Let $0 \leqslant \alpha < n$, $0 < \delta < \min\{\eta, (n-\alpha)/m\}$ and $1 < r \leqslant \infty$. Let $\tilde{\delta} \leqslant \delta$. In [12] the authors defined the classes $\mathcal{H}(r, \widetilde{\alpha}, \tilde{\delta})$ as the pair of weight (v, w) such that

$$\sup_B \Big\| \frac{\chi_B}{w} \Big\|_\infty |B|^{(\delta-\tilde{\delta})/n} \bigg(\int_{\mathbb{R}^n} \frac{v^{r'}(y)}{(|B|^{1/n} + |x_B - y|)^{r'(n-\widetilde{\alpha}+\delta)}} \, \mathrm{d}y \bigg)^{1/r'} < \infty,$$

where the supremun is taken over every ball B with center x_B. These classes characterize the boundedness of several operators between Lebesgue and Lipschtz spaces defined in (2.3), with two weights. Since

$$\|(1/w)\chi_B\|_\infty = \frac{1}{\inf_{x \in B} w} \geqslant \frac{|B|}{w(B)}$$

then the classes $\mathbb{H}(r, \widetilde{\alpha}, \tilde{\delta})$ are contained in the classes $\mathcal{H}(r, \widetilde{\alpha}, \tilde{\delta})$. We will prove later that this inclusion is strict.

Remark 2.3. We say $w \in \mathcal{H}(r, \widetilde{\alpha}, \tilde{\delta})$ if $w = v$ in (2.4) and (2.5).

In the one-weight case, we obtain the following lemma.

Lemma 2.4. *Let $0 \leqslant \alpha < n$, $0 < \delta < 1$ and $1 \leqslant r \leqslant \infty$. Put $\widetilde{\alpha} = m\delta + \alpha$ and $\tilde{\delta} \leqslant \min\{\delta, \widetilde{\alpha} - n/r\}$. If $w \in \mathcal{H}(r, \widetilde{\alpha}, \tilde{\delta})$, then $\tilde{\delta} = \widetilde{\alpha} - n/r$.*

Proof. Let $1 \leqslant r \leqslant \infty$ (if $r = 1$ we understand $\|\cdot\|_\infty$ instead $\|\cdot\|_{r'}$). Since $w \in \mathcal{H}(r, \widetilde{\alpha}, \tilde{\delta})$, we have

$$\begin{aligned}
\frac{w(B)}{|B|} &\gtrsim |B|^{(\delta-\tilde{\delta})/n} \bigg(\int_B \frac{w^{r'}(y)}{(|B|^{1/n} + |x_B - y|)^{r'(n-\widetilde{\alpha}+\delta)}} \, \mathrm{d}y \bigg)^{1/r'} \\
&\gtrsim |B|^{(\delta-\tilde{\delta})/n-(1/r-\widetilde{\alpha}/n+\delta/n)} \bigg(\frac{1}{|B|} \int_B w^{r'}(y) \, \mathrm{d}y \bigg)^{1/r'} \\
&\gtrsim |B|^{-\tilde{\delta}/n-1/r+\widetilde{\alpha}/n} \frac{w(B)}{|B|}.
\end{aligned}$$

Then, this inequality is true if $\tilde{\delta} = \widetilde{\alpha} - n/r$. □

3. Main results

We are now in a position to state our main results. We first state the results for singular integral operators with the corresponding weights belonging to $\mathcal{H}(r, m\delta, \tilde{\delta})$, that is $\alpha = 0$ in Definition 2.1.

Theorem 3.1. *Let $0 < \delta < \min\{\eta, n/m\}$ and $1 \leqslant r \leqslant \infty$. Let $\tilde{\delta} \leqslant \min\{\delta, m\delta - n/r\}$ and $b \in \Lambda(\delta)$. If $(w, v) \in \mathcal{H}(r, m\delta, \tilde{\delta})$ and $K \in S_0^* \cap K_{\alpha,\infty}^*$, then*

$$\|T_b^m f\|_{\mathcal{L}_w(\tilde{\delta}/n)} \lesssim \|b\|_{\Lambda(\delta)}^m \left\| \frac{f}{v} \right\|_{L^r(\mathbb{R}^n)}$$

holds for every f such that $f/v \in L^r(\mathbb{R}^n)$.

From the theorem above and Lemma 2.4 we obtain the following corollary.

Corollary 3.2. *Let $0 < \delta < \min\{\eta, n/m\}$ and $1 \leqslant r < \infty$. Let $\tilde{\delta} = m\delta - n/r$ and $b \in \Lambda(\delta)$. If $w \in \mathcal{H}(r, m\delta, \tilde{\delta})$ and $K \in S_0^* \cap K_{\alpha,\infty}^*$, then*

$$\|T_b^m f\|_{\mathcal{L}_w(\tilde{\delta}/n)} \lesssim \|b\|_{\Lambda(\delta)}^m \left\| \frac{f}{v} \right\|_{L^r(\mathbb{R}^n)}$$

holds for every f such that $f/w \in L^r(\mathbb{R}^n)$.

For the Hilbert transform, $m = 0$ and $r = \infty$, this corollary was proved in [9].

We now state the main results for the boundedness of fractional integral operators, that is $0 < \alpha < n$.

Theorem 3.3. *Let $0 < \alpha < n$, $0 < \delta < \min\{\eta, (n-\alpha)/m\}$ and $1 \leqslant r \leqslant \infty$. Let $\tilde{\delta} \leqslant \min\{\delta, \widetilde{\alpha} - n/r\}$ and $b \in \Lambda(\delta)$. If $(w, v) \in \mathcal{H}(r, \widetilde{\alpha}, \tilde{\delta})$ and $K \in S_0^* \cap K_{\alpha,\infty}^*$, then*

$$\|T_{\alpha,b}^m f\|_{\mathcal{L}_w(\tilde{\delta}/n)} \lesssim \|b\|_{\Lambda(\delta)}^m \left\| \frac{f}{v} \right\|_{L^r(\mathbb{R}^n)}$$

holds for every f such that $f/v \in L^r(\mathbb{R}^n)$.

From Theorem 3.3 and Lemma 2.4, we obtain the following result.

Corollary 3.4. *Let $0 < \alpha < n$, $0 < \delta < \min\{\eta, (n-\alpha)/m\}$ and $1 \leqslant r \leqslant \infty$. Let $\tilde{\delta} = \widetilde{\alpha} - n/r$ and $b \in \Lambda(\delta)$. If $w \in \mathcal{H}(r, \widetilde{\alpha}, \tilde{\delta})$ and $K \in S_0^* \cap K_{\alpha,\infty}^*$, then*

$$\|T_{\alpha,b}^m f\|_{\mathcal{L}_w(\tilde{\delta}/n)} \lesssim \|b\|_{\Lambda(\delta)}^m \left\| \frac{f}{v} \right\|_{L^r(\mathbb{R}^n)}$$

holds for every f such that $f/w \in L^r(\mathbb{R}^n)$.

4. Properties of classes of weights

In this section we give some properties of the classes of weights $\mathcal{H}(r,\widetilde{\alpha},\tilde{\delta})$ given in Definition 2.1. Recall that $\tilde{\delta}\leqslant\min\{\widetilde{\alpha}-n/r,\delta\}$ and $\widetilde{\alpha}=m\delta+\alpha$, where $0\leqslant\alpha<n$, $1\leqslant r\leqslant\infty$.

We prove that the range of the parameters involved in the classes $\mathcal{H}(r,\widetilde{\alpha},\tilde{\delta})$ lies in the shaded region of Figure 1.

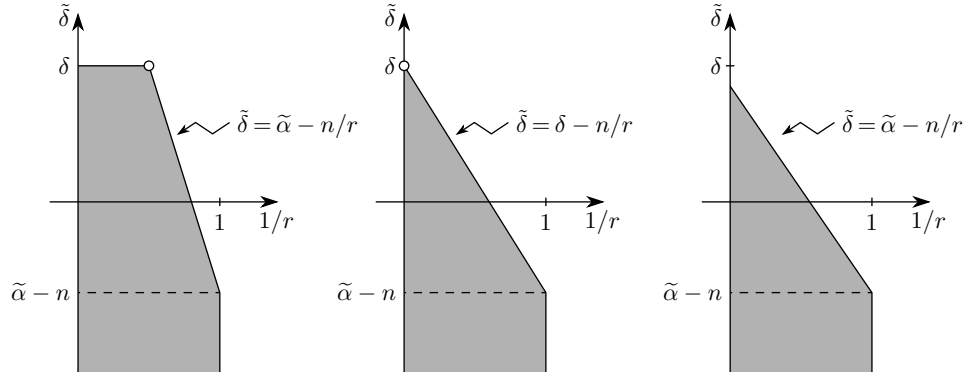

Figure 1. Permissible range of the parameters r and $\tilde{\delta}$ for different values of $\widetilde{\alpha}$.

Lemma 4.1. *Let $0\leqslant\alpha<n$, $0<\delta<\min\{\eta,(n-\alpha)/m\}$ and $1\leqslant r\leqslant\infty$. If $(w,v)\in\mathcal{H}(r,\widetilde{\alpha},\tilde{\delta})$ then*

$$\|v\chi_{2B}\|_{r'}\lesssim|B|^{(\tilde{\delta}-\widetilde{\alpha})/n}w(B).$$

Proof. Since $(w,v)\in\mathcal{H}(r,\widetilde{\alpha},\tilde{\delta})$, we know that

$$|B|^{(\delta-\tilde{\delta})/n}\biggl(\int_{\mathbb{R}^n}\frac{v^{r'}(y)}{(|B|^{1/n}+|x_B-y|)^{r'(n-\widetilde{\alpha}+\delta)}}\,\mathrm{d}y\biggr)^{1/r'}\leqslant C\frac{w(B)}{|B|}$$

for every ball $B\subset\mathbb{R}^n$, where x_B is the center of B. Then, we have

$$\begin{aligned}v^{r'}(2B)&=\frac{|B|^{(n-\widetilde{\alpha}+\delta)r'/n}}{|B|^{(n-\widetilde{\alpha}+\delta)r'/n}}\int_{2B}v^{r'}(y)\,\mathrm{d}y\\&\lesssim|B|^{(n-\widetilde{\alpha}+\delta)r'/n}\int_{2B}\frac{v^{r'}(y)}{(|B|^{1/n}+|x_B-y|)^{r'(n-\widetilde{\alpha}+\delta)}}\,\mathrm{d}y\\&\lesssim|B|^{(n-\widetilde{\alpha}+\delta)r'/n}\int_{\mathbb{R}^n}\frac{v^{r'}(y)}{(|B|^{1/n}+|x_B-y|)^{r'(n-\widetilde{\alpha}+\delta)}}\,\mathrm{d}y\\&\lesssim|B|^{(n-\widetilde{\alpha}+\delta)r'/n}\Bigl(\frac{w(B)}{|B|^{(\delta-\tilde{\delta})/n+1}}\Bigr)^{r'}\lesssim|B|^{(\tilde{\delta}-\widetilde{\alpha})r'/n}(w(B))^{r'}.\end{aligned}$$

□

As a consequence of the lemma above and Lemma 2.4 we obtain the following result.

Corollary 4.2. *Let $0 \leqslant \alpha < n$, $0 < \delta < \min\{\eta, (n-\alpha)/m\}$ and $1 \leqslant r \leqslant \infty$. If $w \in \mathcal{H}(r, \widetilde{\alpha}, \tilde{\delta})$ then $w^{r'}$ satisfies a doubling condition.*

When $m = 0$ this corollary was proved in [7].

We say that a weight w belongs to $RH(s)$ if there exists a positive constant C such that

$$\left(\frac{1}{|B|}\int_B w^s(x)\,\mathrm{d}x\right)^{1/s} \leqslant C\frac{w(B)}{|B|}.$$

As a consequence of Lemmas 2.4 and 4.1, we get the following result.

Corollary 4.3. *Let $0 \leqslant \alpha < n$, $0 < \delta < \min\{\eta, (n-\alpha)/m\}$ and $1 \leqslant r \leqslant \infty$. If w is a weight in $\mathcal{H}(r, \widetilde{\alpha}, \tilde{\delta})$, then w belongs to $RH(r')$.*

Our next lemma shows the equivalence between the class $\mathcal{H}(r, \widetilde{\alpha}, \tilde{\delta})$ and a pair of local and global conditions. The proof is straightforward and we omit it.

Lemma 4.4. *Let $0 \leqslant \alpha < n$, $0 < \delta < \min\{\eta, (n-\alpha)/m\}$ and $1 \leqslant r \leqslant \infty$. The condition $\mathcal{H}(r, \widetilde{\alpha}, \tilde{\delta})$ is equivalent to the following two inequalities:*

$$|B|^{(\widetilde{\alpha}-\tilde{\delta})/n-1/r}\left(\frac{1}{|B|}\int_B v^{r'}(y)\,\mathrm{d}y\right)^{1/r'} \lesssim \frac{w(B)}{|B|} \tag{4.1}$$

and

$$|B|^{(\delta-\tilde{\delta})/n}\left(\int_{\mathbb{R}^n - B}\frac{v^{r'}(y)}{|x_B - y|^{r'(n-\widetilde{\alpha}+\delta)}}\,\mathrm{d}y\right)^{1/r'} \lesssim \frac{w(B)}{|B|} \tag{4.2}$$

hold simultaneously for every ball $B \subset \mathbb{R}^n$, where x_B is the center of B.

It is important to note that both the conditions (4.1) and (4.2) cannot be reduced to (4.2) as in [7] for the one-weighted case. However, under certain additional hypothesis on v then $\mathcal{H}(r, \widetilde{\alpha}, \tilde{\delta})$ is the condition (4.2). This fact is established in the following lemma.

Lemma 4.5. *Let $0 \leqslant \alpha < n$, $0 < \delta < \min\{\eta, (n-\alpha)/m\}$ and $1 \leqslant r \leqslant \infty$. Let v be a weight such that $v^{r'}$ satisfies a doubling condition. Then, the global condition* (4.2) *implies the local condition* (4.1).

P r o o f. Since $v^{r'}$ is a doubling weight, we get

$$\begin{aligned}\left(\frac{1}{|B|}\int_B v^{r'}(y)\,\mathrm{d}y\right)^{1/r'} &\lesssim \left(\frac{1}{|B|}\int_{2B-B} v^{r'}(y)\,\mathrm{d}y\right)^{1/r'}\\ &= \frac{|B|^{(n-\widetilde{\alpha}+\delta)/n-1/r'}}{|B|^{(n-\widetilde{\alpha}+\delta)/n}}\left(\int_{2B-B} v^{r'}(y)\,\mathrm{d}y\right)^{1/r'}\\ &\lesssim |B|^{(n-\widetilde{\alpha}+\delta)/n-1/r'}\left(\int_{2B-B}\frac{v^{r'}(y)}{|x_B-y|^{r'(n-\widetilde{\alpha}+\delta)}}\,\mathrm{d}y\right)^{1/r'}.\end{aligned}$$

Then, using the global condition (4.2), we have that

$$\left(\frac{1}{|B|}\int_B v^{r'}(y)\,\mathrm{d}y\right)^{1/r'} \lesssim |B|^{(\tilde{\delta}-\widetilde{\alpha})/n+1/r}\frac{w(B)}{|B|},$$

which is (4.1). □

Even though $v^{r'}$ satisfies a doubling condition, both the inequalities (4.1) and (4.2) are not equivalent. This fact is showed in Lemma 4.7. We first give well known estimates in order to prove it.

Lemma 4.6. *Let $B = B(x_B, R) \subset \mathbb{R}^n$ and $\alpha > -n$. Then, the following statements hold.*
(1) *If $|x_B| \leqslant R$, $\int_B |x|^\alpha\,\mathrm{d}x \approx R^{\alpha+n}$.*
(2) *If $|x_B| > R$, $\int_B |x|^\alpha\,\mathrm{d}x \approx |x_B|^\alpha R^n$.*

Lemma 4.7. *Let $0 \leqslant \alpha < n$, $0 < \delta < \min\{\eta, (n-\alpha)/m\}$ and $1 \leqslant r \leqslant \infty$. There exist nontrivial pairs of weights (w, v) that satisfy the local condition (4.1) but not the global condition (4.2) for $\tilde{\delta}$ in the range*

$$\tilde{\delta} \leqslant \min\Big\{\delta, \widetilde{\alpha} - \frac{n}{r}\Big\},$$

excluding the case $\tilde{\delta} = \delta$ when $\widetilde{\alpha} - n/r = \delta$.

P r o o f. Let us first consider $\tilde{\delta} = \delta < \widetilde{\alpha} - n/r$. Let $w = 1$ and $v(x) = |x|^{n/r-\widetilde{\alpha}+\delta}$, we prove that (w, v) satisfies (4.1) but not (4.2). Indeed, let $B = B(x_B, R)$. By Lemma 4.6, if $|x_B| \leqslant R$ we get

$$\frac{|B|^{(\widetilde{\alpha}-\delta)/n}}{w(B)}\left(\int_B v^{r'}(y)\,\mathrm{d}y\right)^{1/r'} \lesssim R^{\widetilde{\alpha}-\delta-n}R^{n/r-\widetilde{\alpha}+\delta+n/r'} \lesssim C$$

and if $|x_B| \geqslant R$, by Lemma 4.6, we have

$$\frac{|B|^{(\widetilde{\alpha}-\delta)/n}}{w(B)}\left(\int_B v^{r'}(y)\,\mathrm{d}y\right)^{1/r'} \lesssim R^{\widetilde{\alpha}-\delta-n}|x_B|^{n/r-\widetilde{\alpha}+\delta}R^{n/r'} \lesssim C.$$

On the other hand, if we now take $B = B(0,R)$, we get

$$\frac{|B|}{w(B)}\left(\int_{\mathbb{R}^n\setminus B}\frac{v^{r'}(y)}{|y|^{(n-\widetilde{\alpha}+\delta)r'}}\,\mathrm{d}y\right)^{1/r'} \gtrsim \left(\int_{\{|y|>R\}}\frac{|y|^{(n/r-\widetilde{\alpha}+\delta)r'}}{|y|^{(n-\widetilde{\alpha}+\delta)r'}}\,\mathrm{d}y\right)^{1/r'} \gtrsim \left(\int_{\{|y|>R\}}\frac{1}{|y|^n}\,\mathrm{d}y\right)^{1/r'}$$

and the last integral is infinite. Thus, (w,v) does not satisfy (4.2).

Similar estimates can be obtained for the case $\tilde{\delta} < \delta \leqslant \widetilde{\alpha} - n/r$ by considering $(|x|^{\widetilde{\alpha}-\tilde{\delta}-n/r}, 1)$. For the case $\tilde{\delta} \leqslant \widetilde{\alpha} - n/r \leqslant \delta$ the same is true for $(|x|^{\beta}, |x|^{\theta})$ with $\theta > n/r - \widetilde{\alpha} + \delta$ and $\beta = \theta + \widetilde{\alpha} - \tilde{\delta} - n/r$. □

Proposition 4.8. *Let $0 \leqslant \alpha < n$, $0 < \delta < \min\{\eta, (n-\alpha)/m\}$ and $1 \leqslant r \leqslant \infty$. Then:*

(i) *If $\tilde{\delta} > \delta$ or $\tilde{\delta} > \widetilde{\alpha} - n/r$, then $(w,v) \in \mathcal{H}(r,\widetilde{\alpha},\tilde{\delta})$ if and only if $v = 0$ almost everywhere in $\mathbb{R}^n$.*

(ii) *If $\tilde{\delta} = \widetilde{\alpha} - n/r = \delta$ then the same conclusion as in* (i) *holds.*

P r o o f. Let us first see (i) and let $\tilde{\delta} > \delta$. Let $B = B(x,R)$, where x is a Lebesgue point of w. Suppose that r is finite, since $(w,v) \in \mathcal{H}(r,\widetilde{\alpha},\tilde{\delta})$ we get

$$\left(\int_{\mathbb{R}^n}\frac{v^{r'}(y)}{(|B|^{1/n}+|x-y|)^{r'(n-\widetilde{\alpha}+\delta)}}\,\mathrm{d}y\right)^{1/r'} \lesssim \frac{w(B)}{|B|}|B|^{(\tilde{\delta}-\delta)/n}.$$

From the inequality above, by letting $R \to 0$, we obtain that

$$\left(\int_{\mathbb{R}^n}\frac{v^{r'}(y)}{(|B|^{1/n}+|x-y|)^{r'(n-\widetilde{\alpha}+\delta)}}\,\mathrm{d}y\right)^{1/r'} = 0$$

and so $v = 0$ for a.e. $x \in \mathbb{R}^n$.

Now, if $\tilde{\delta} > \widetilde{\alpha} - n/r$, since $(w,v) \in \mathcal{H}(r,\widetilde{\alpha},\tilde{\delta})$ and by Lemma 4.4, we have

$$\Big(\frac{v^{r'}(B)}{|B|}\Big)^{1/r'} \lesssim \frac{w(B)}{|B|}|B|^{1/r-(\widetilde{\alpha}-\tilde{\delta})/n}.$$

If we choose $B(x,R)$ as before, since x is a Lebesgue point of w, we get

$$\lim_{R\to 0}\frac{w(B)}{|B|}|B|^{1/r-(\widetilde{\alpha}-\tilde{\delta})/n} = 0,$$

from which it follows that

$$\limsup_{R\to 0}\frac{v^{r'}(B(x,R))}{|B(x,R)|} = 0.$$

Clearly we can get the same conclusion for a.e. $x \in \mathbb{R}^n$. By standard arguments we can deduce that $v(x) = 0$ at a.e. $x \in \mathbb{R}^n$. If $r = \infty$ we have to consider $1/r = 0$ and $r' = 1$ in the previous argument.

We now proceed with the proof of (ii), since $\tilde{\delta} = \delta = \widetilde{\alpha} - n/r$ we are going to see that $(w, v) \in \mathcal{H}(r, \widetilde{\alpha}, \tilde{\delta})$ with $r = n/(\widetilde{\alpha} - \delta)$ if $\widetilde{\alpha} > \delta$ and $r = \infty$ if $\widetilde{\alpha} = \delta$, if $v(x) = 0$ at a.e. $x \in \mathbb{R}^n$.

Let $B = B(x_0, R) \subset \mathbb{R}^n$, since $(w, v) \in \mathcal{H}(r, \widetilde{\alpha}, \tilde{\delta})$ we get

$$\left(\int_{\mathbb{R}^n} \frac{v^{r'}(y)}{(|B|^{1/n} + |x_0 - y|)^{r'(n-\widetilde{\alpha}+\delta)}}\,\mathrm{d}y\right)^{1/r'} \lesssim \frac{w(B)}{|B|}.$$

Since $n - \widetilde{\alpha} + \delta = n/r'$, we have

$$\left(\int_{\mathbb{R}^n} \frac{v^{r'}(y)}{(|B|^{1/n} + |x_0 - y|)^{n}}\,\mathrm{d}y\right)^{1/r'} \lesssim \frac{w(B)}{|B|}. \tag{4.3}$$

We now proceed as in the proof of Theorem 5.6 in [11] in order to obtain that $v(x) = 0$ for a.e. $x \in \mathbb{R}^n$. □

Remark 4.9. Let $0 \leqslant \alpha < n$, $0 < \delta < \min\{\eta, (n - \alpha/m)\}$ and $n/\widetilde{\alpha} < r < n/(\widetilde{\alpha} - \delta)$. Let

$$\Big(2\Big(\widetilde{\alpha} - \frac{n}{r}\Big) - \delta\Big)^{+} \leqslant \theta \leqslant \widetilde{\alpha} - \frac{n}{r}, \quad \widetilde{\alpha} - \frac{n}{r} - \theta < \tilde{\delta} < \min\Big\{\widetilde{\alpha} - \frac{n}{r}, \frac{n}{r} - \widetilde{\alpha} + \delta\Big\},$$

we now exhibit a pair of weights (w, v) such that $(w, v) \in \mathcal{H}(r, \widetilde{\alpha}, \tilde{\delta})$ but $(w, v) \notin \mathbb{H}(r, \widetilde{\alpha}, \tilde{\delta})$.

Let $w(x) = |x|^{\theta}\chi_{\{|x|\leqslant 1\}} + |x|^{\theta+\tilde{\delta}}\chi_{\{|x|>1\}}$ and $v(x) = |x|^{\tilde{\delta}}$. It is easy to check that (w, v) does not belong to $\mathbb{H}(r, \widetilde{\alpha}, \tilde{\delta})$. However, we see that $(w, v) \in \mathcal{H}(r, \widetilde{\alpha}, \tilde{\delta})$. Since $v^{r'}$ is a doubling weight, by Lemma 4.5, we only prove (4.2).

Let $B = B(x_B, R)$ and $B_i = 2^i B$. If $|x_B| \leqslant R$, by Lemma 4.6 we obtain that

$$\begin{aligned}
\frac{|B|^{1+(\delta-\tilde{\delta})/n}}{w(B)}\left(\int_{\mathbb{R}^n\setminus B} \frac{v^{r'}(y)}{|x_B - y|^{(n-\widetilde{\alpha}+\delta)r'}}\,\mathrm{d}y\right)^{1/r'} &\lesssim \frac{R^{\widetilde{\alpha}-\tilde{\delta}}}{w(B)}\sum_{i=1}^{\infty} 2^{-i(n-\widetilde{\alpha}+\delta)}(v^{r'}(B_i))^{1/r'} \\
&\lesssim \frac{R^{\widetilde{\alpha}+n/r'}}{w(B)}\sum_{i=1}^{\infty} 2^{-i(n/r-\tilde{\delta}-\widetilde{\alpha}+\delta)} \\
&\lesssim \frac{R^{\widetilde{\alpha}+n/r'}}{w(B)}.
\end{aligned}$$

Thus, since $w(B) \gtrsim \max\{R^{\theta+n}, R^{\theta+\tilde{\delta}+n}\}$ we obtain that (4.2) holds for this case.

Now, if $|x_B| > R$, then there exists N_1 such that $2^{N_1} R \leqslant |x_B| \leqslant 2^{N_1+1} R$.

$$\begin{aligned}
&\frac{|B|^{1+(\delta-\tilde\delta)/n}}{w(B)} \biggl(\int_{\mathbb{R}^n \setminus B} \frac{v^{r'}(y)}{|x_B - y|^{(n-\widetilde\alpha+\delta)r'}} \,\mathrm{d}y \biggr)^{1/r'} \\
&\quad \lesssim \frac{R^{\widetilde\alpha-\tilde\delta}}{w(B)} \sum_{i=1}^{\infty} 2^{-i(n-\widetilde\alpha+\delta)} (v^{r'}(B_i))^{1/r'} \\
&\quad = \frac{R^{\widetilde\alpha-\tilde\delta}}{w(B)} \sum_{i=1}^{N_1} 2^{-i(n-\widetilde\alpha+\delta)} (v^{r'}(B_i))^{1/r'} + \frac{R^{\widetilde\alpha-\tilde\delta}}{w(B)} \sum_{i=N_1+1}^{\infty} 2^{-i(n-\widetilde\alpha+\delta)} (v^{r'}(B_i))^{1/r'} \\
&\quad = S_1 + S_2.
\end{aligned}$$

Let us first estimate S_1. Since $i \leqslant N_1$, $n/r - \widetilde\alpha + \delta > 0$ and

$$w(B) \gtrsim \max\{|x_B|^{\widetilde\alpha} R^n, |x_B|^{\theta+\tilde\delta} R^n\}$$

we have

$$S_1 \lesssim \frac{R^{\widetilde\alpha-\tilde\delta+n/r'}}{w(B)} |x_B|^{\tilde\delta} \lesssim C.$$

In order to estimate S_2, we first observe that

$$S_2 \lesssim \frac{R^{\widetilde\alpha+n/r'}}{w(B)}$$

and then we proceed as in the estimate of S_1 to obtain that $S_2 \lesssim C$.

Theorem 4.10. *Let $0 \leqslant \alpha < n$ and $0 < \delta < \min\{\eta, (n-\alpha)/m\}$. There exist pairs of weights with v not identically equal to zero, that satisfy the condition $\mathcal{H}(r, \widetilde\alpha, \tilde\delta)$ in the range of r and $\tilde\delta$ given by*

$$1 \leqslant r \leqslant \infty \quad \text{and} \quad \tilde\delta \leqslant \min\Bigl\{\delta, \widetilde\alpha - \frac{n}{r}\Bigr\}$$

excluding the case $\tilde\delta = \delta$ when $\widetilde\alpha - n/r = \delta$.

P r o o f. By Remark 2.2, the pair of weights given in [12] belongs to $\mathcal{H}(r, \widetilde\alpha, \tilde\delta)$ for $1 \leqslant r \leqslant \infty$ and $\widetilde\alpha - n \leqslant \tilde\delta \leqslant \min\{\delta, \widetilde\alpha - n/r\}$ excluding the case $\tilde\delta = \delta$ when $\widetilde\alpha - n/r = \delta$.

So, we exhibit examples of pairs of weights for the case $\tilde\delta < \widetilde\alpha - n$. We first consider $1 < r \leqslant \infty$. We split the range $\tilde\delta < \widetilde\alpha - n$ in two regions:

(i) $\widetilde\alpha - n - k\delta < \tilde\delta \leqslant \min\{\widetilde\alpha - n/r - k\delta, \widetilde\alpha - n - (k-1)\delta\}$, $k \in \mathbb{N}$, see Figure 2.

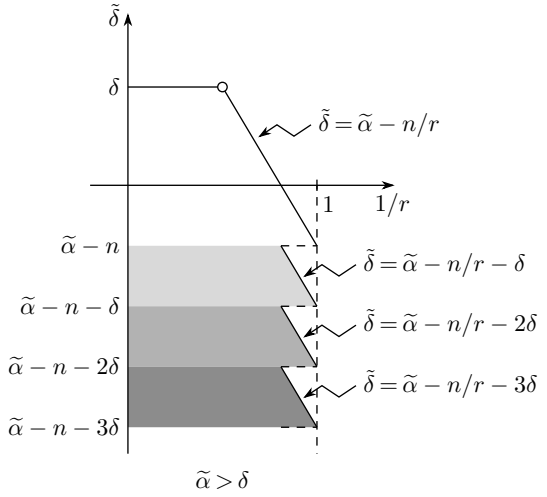

Figure 2. Case (i) in the proof of Theorem 4.10.

(ii) $\widetilde\alpha - n/r - k\delta - \delta < \tilde\delta \leqslant \widetilde\alpha - n - k\delta$, $k \in \mathbb{N} \cup \{0\}$, see Figure 3.

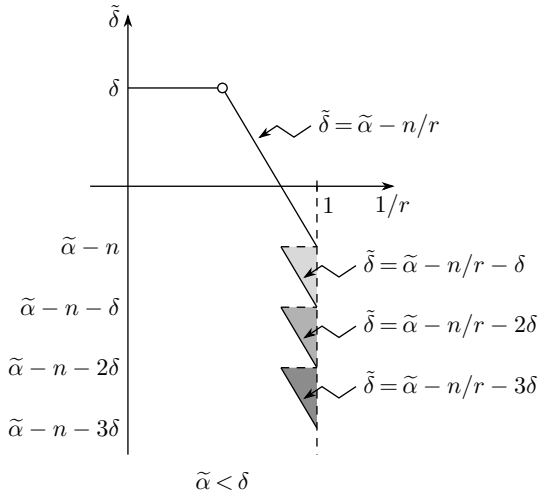

Figure 3. Case (ii) in the proof of Theorem 4.10.

For the case (i) we consider the pairs (w, v) given by $w(x) = |x|^{k\delta}$ and $v(x) = |x|^{n/r - \widetilde\alpha + \tilde\delta + k\delta}$ with

$$\widetilde\alpha - n - k\delta < \tilde\delta \leqslant \min\Big\{\widetilde\alpha - \frac{n}{r} - k\delta,\, \widetilde\alpha - n - k\delta + \delta\Big\}, \quad k \in \mathbb{N}.$$

Since $v^{r'}$ satisfies the doubling condition, we use Lemma 4.5 to estimate only the global condition (4.2). Let $B = B(x_B, R)$, we have two cases, $|x_B| \leqslant R$ or $|x_B| > R$.

If $|x_B| \leqslant R$, by Proposition 4.6 and since $(n/r - \widetilde{\alpha} + \tilde{\delta} + k\delta)r' > -n$ (because $\widetilde{\alpha} - n - k\delta < \tilde{\delta}$),

$$\int_B v^{r'}(x)\,\mathrm{d}x \approx R^{(n/r-\widetilde{\alpha}+\tilde{\delta}+k\delta)r'+n} \quad \text{and} \quad w(B) = \int_B |x|^{k\delta}\,\mathrm{d}x \approx R^{k\delta+n}.$$

Then,

$$\begin{aligned}
&\frac{|B|^{(\delta-\tilde{\delta})/n+1}}{w(B)}\bigg(\int_{\mathbb{R}^n\setminus B}\frac{v^{r'}(y)}{|x_B-y|^{r'(n-\widetilde{\alpha}+\delta)}}\,\mathrm{d}y\bigg)^{1/r'}\\
&\quad\lesssim \frac{|B|^{(\delta-\tilde{\delta})/n+1}}{w(B)}\sum_{i=1}^{\infty}\bigg(\int_{2^iB\setminus 2^{i-1}B}\frac{v^{r'}(y)}{|x_B-y|^{r'(n-\widetilde{\alpha}+\delta)}}\,\mathrm{d}y\bigg)^{1/r'}\\
&\quad\lesssim \frac{R^{\delta-\tilde{\delta}+n}}{R^{\delta k+n}}\sum_{i=1}^{\infty}\frac{(2^iR)^{n/r-\widetilde{\alpha}+\tilde{\delta}+k\delta+n/r'}}{(2^iR)^{n-\widetilde{\alpha}+\delta}} \lesssim \sum_{i=1}^{\infty}\Big(\frac{1}{2^i}\Big)^{\delta-\tilde{\delta}-k\delta} \approx C,
\end{aligned}$$

where the last sum is finite because $\tilde{\delta} + k\delta < \widetilde{\alpha} - n + \delta < \delta$.

Now let $|x_B| > R$. Then there exists N_1 such that $|x_B|/R \approx 2^{N_1}$. On the other hand we have

$$\begin{aligned}
&\frac{|B|^{(\delta-\tilde{\delta})/n+1}}{w(B)}\bigg(\int_{\mathbb{R}^n\setminus B}\frac{v^{r'}(y)}{|x_B-y|^{r'(n-\widetilde{\alpha}+\delta)}}\,\mathrm{d}y\bigg)^{1/r'}\\
&\qquad\lesssim \frac{R^{\widetilde{\alpha}-\tilde{\delta}-n}}{|x_B|^{\delta k}}\sum_{i=1}^{\infty}\frac{1}{2^{i(n-\widetilde{\alpha}+\delta)}}\bigg(\int_{2^iB}v^{r'}\bigg)^{1/r'}.
\end{aligned} \tag{4.4}$$

The last term in (4.4) can be divided into S_1 and S_2, where S_1 is the sum up to the N_1th term and S_2 is the sum of the remaining terms. We first estimate S_1,

$$S_1 \lesssim \frac{R^{\widetilde{\alpha}-\tilde{\delta}-n}}{|x_B|^{\delta k}}\sum_{i=1}^{N_1}\frac{|x_B|^{n/r-\widetilde{\alpha}+\tilde{\delta}+k\delta}(2^iR)^{n/r'}}{2^{i(n-\widetilde{\alpha}+\delta)}} \lesssim \Big(\frac{R}{|x_B|}\Big)^{\widetilde{\alpha}-\tilde{\delta}-n/r}\sum_{i=1}^{N_1}\Big(\frac{1}{2^i}\Big)^{\delta-\tilde{\delta}} \lesssim C,$$

and the last sum is finite because $\tilde{\delta} < \delta$.

For S_2 we have

$$S_2 \lesssim \frac{R^{\widetilde{\alpha}-\tilde{\delta}-n}}{|x_B|^{\delta k}}\sum_{i=N_1+1}^{\infty}\frac{(2^iR)^{n-\widetilde{\alpha}+\tilde{\delta}+\delta k}}{2^{i(n-\widetilde{\alpha}+\delta)}} \approx \frac{R^{\delta k}}{|x_B|^{\delta k}}\sum_{i=N_1+1}^{\infty}\frac{1}{2^{i(\delta-\tilde{\delta}-\delta k)}}. \tag{4.5}$$

Since $\tilde{\delta} + \delta k < \delta$, the last term of (4.5) is less than or equal to $(R/|x_B|)^{\delta k}$, which is bounded by a constant.

We now deal with (ii). Let $\widetilde{\alpha} - n/r - (k-1)\delta < \tilde{\delta} \leqslant \widetilde{\alpha} - n - k\delta$, $k \in \mathbb{N}\cup\{0\}$. We consider the pair (w, v) defined by $w(x) = |x|^{\theta}$ and $v(x) = |x|^{\beta}$ with

$$\theta = \widetilde{\alpha} - \frac{n}{r} - k\delta - 2\tilde{\delta} \quad \text{and} \quad \beta = -k\delta - \tilde{\delta}.$$

Since $v^{r'}$ satisfies a doubling condition, by Lemma 4.5 it suffices to study the global condition (4.2). Let $B = B(x_B, R)$. If $|x_B| \leqslant R$, by Proposition 4.6 we have

$$\begin{aligned}(4.6)\qquad &\frac{|B|^{(\delta-\tilde\delta)/n+1}}{w(B)}\bigg(\int_{\mathbb{R}^n\setminus B}\frac{v^{r'}(y)}{|x_B-y|^{r'(n-\widetilde\alpha+\delta)}}\,\mathrm{d}y\bigg)^{1/r'}\\ &\qquad\lesssim \frac{|B|^{(\delta-\tilde\delta)/n+1}}{w(B)}\sum_{i=1}^{\infty}\frac{1}{(2^iR)^{n-\widetilde\alpha+\delta}}\bigg(\int_{2^iB}v^{r'}\bigg)^{1/r'}\\ &\qquad\approx R^{-\tilde\delta-\theta+\beta-n/r+\widetilde\alpha}\sum_{i=1}^{\infty}\frac{1}{2^{i(n/r-\widetilde\alpha+\delta-\beta)}}.\end{aligned}$$

Noting that

$$-\tilde\delta-\theta+\beta-\frac{n}{r}+\widetilde\alpha=0\quad\text{and}\quad\frac{n}{r}-\widetilde\alpha+\delta-\beta>0,$$

it is immediate that the last sum in (4.6) is bounded by a constant.

Let us now consider $|x_B| > R$. As in the case (i), we obtain

$$\begin{aligned}&\frac{|B|^{(\delta-\tilde\delta)/n+1}}{w(B)}\bigg(\int_{\mathbb{R}^n\setminus B}\frac{v^{r'}(y)}{|x_B-y|^{r'(n-\widetilde\alpha+\delta)}}\,\mathrm{d}y\bigg)^{1/r'}\\ &\qquad\lesssim\frac{R^{\widetilde\alpha-\tilde\delta}}{|x_B|^\theta R^n}\sum_{i=1}^{\infty}\frac{1}{2^{i(n-\widetilde\alpha+\delta)}}\bigg(\int_{2^iB}v^{r'}\bigg)^{1/r'}.\end{aligned}$$

Then, we take S_1 and S_2 like in case (i).

Observe that $\theta = \widetilde\alpha - \tilde\delta + \beta - n/r$. Since $i \leqslant N_1$, we have that $|x_B| > 2^iR$. So we can apply Lemma 4.6 to obtain

$$\begin{aligned}S_1&\lesssim\frac{R^{\widetilde\alpha-\tilde\delta}}{R^n}\sum_{i=0}^{N_1}\frac{|x_B|^{\beta-\theta}(2^iR)^{n/r'}}{2^{i(n-\widetilde\alpha+\delta)}}\\ &\lesssim R^{\widetilde\alpha-\tilde\delta-n/r-\theta+\beta}\sum_{i=0}^{N_1}2^{-i(n-\widetilde\alpha+\delta-n/r'+\theta-\beta)}\lesssim\sum_{i=1}^{N_1}2^{i(\tilde\delta-\delta)},\end{aligned}$$

which is finite since $\tilde\delta < \delta$.

For S_2 we get

$$\begin{aligned}S_2&\lesssim\frac{R^{\widetilde\alpha-\tilde\delta+n/r'}}{|x_B|^\theta R^n}\sum_{i=N_1+1}^{\infty}\frac{|x_B|^\beta}{2^{i(n/r-\widetilde\alpha+\delta)}}\lesssim\frac{R^{\widetilde\alpha-\tilde\delta+\beta-n/r}}{|x_B|^\theta}\sum_{i=N_1+1}^{\infty}2^{-i(n/r-\widetilde\alpha+\delta-\beta)}\\ &\lesssim\Big(\frac{R}{|x_B|}\Big)^\theta\sum_{i=N_1+1}^{\infty}2^{-i(n/r-\widetilde\alpha+\delta-\beta)}.\end{aligned}$$

Now, since $|x_B| > R$, $\widetilde{\alpha} - \tilde{\delta} + \beta - n/r = \theta > 0$ and $n/r - \widetilde{\alpha} + \delta - \beta > 0$, we obtain

$$S_2 \lesssim C.$$

This concludes the proof of (ii).

For the case $r = 1$ and $\tilde{\delta} < \widetilde{\alpha} - n$ we set $w(x) = |x|^{-\tilde{\delta}}$ and $v(x) = |x|^{n-\widetilde{\alpha}}$. By Lemma 4.5, we estimate (4.2). Let $B = B(x_B, R)$, if $|x_B| \leqslant R$, we then have

$$\begin{aligned}\frac{|B|^{(\delta-\tilde{\delta})/n+1}}{w(B)}\Big\|\frac{\chi_{\mathbb{R}^n\setminus B}v}{(|B|^{1/n}+|x_B-\cdot|)^{n-\widetilde{\alpha}+\delta}}\Big\|_\infty &\lesssim \frac{|B|^{(\delta-\tilde{\delta})/n+1}}{w(B)}\sum_{i=1}^{\infty}\frac{1}{(2^iR)^{n-\widetilde{\alpha}+\delta}}\|\chi_{B_i}v\|_\infty \\ &\lesssim \frac{R^{\delta-\tilde{\delta}+n}}{R^{n-\tilde{\delta}}}\sum_{i=1}^{\infty}\frac{(2^iR)^{n-\widetilde{\alpha}}}{(2^iR)^{n-\widetilde{\alpha}+\delta}} \\ &\approx C\sum_{i=1}^{\infty}\frac{1}{2^{i\delta}} \lesssim C.\end{aligned}$$

If $|x_B| > R$, we proceed as in the case $p > 1$ to obtain that the first term of the above inequality is bounded by S_1 and S_2, where

$$S_1 \approx \frac{R^{\delta-\tilde{\delta}+n}}{|x_B|^{-\tilde{\delta}}R^n}\sum_{i=1}^{N_1}\frac{\|\chi_{B_i}v\|_\infty}{(2^iR)^{n-\widetilde{\alpha}+\delta}}, \quad S_2 \approx \frac{R^{\delta-\tilde{\delta}+n}}{|x_B|^{-\tilde{\delta}}R^n}\sum_{i=N_1+1}^{\infty}\frac{\|\chi_{B_i}v\|_\infty}{(2^iR)^{n-\widetilde{\alpha}+\delta}}.$$

In order to estimate S_1, since $|x_B| > 2^iR$ for $i \leqslant N_1$, we have

$$S_1 \lesssim \frac{R^{\delta-\tilde{\delta}}}{|x_B|^{-\tilde{\delta}}}\sum_{i=1}^{N_1}\frac{|x_B|^{n-\widetilde{\alpha}}}{(2^iR)^{n-\widetilde{\alpha}+\delta}} \lesssim R^{\widetilde{\alpha}-\tilde{\delta}-n}|x_B|^{\tilde{\delta}+n-\widetilde{\alpha}} \lesssim C.$$

On the other hand,

$$S_2 \lesssim \frac{R^{\delta-\tilde{\delta}}}{|x_B|^{-\tilde{\delta}}}\sum_{i=N_1+1}^{\infty}\frac{(2^iR)^{n-\widetilde{\alpha}}}{(2^iR)^{n-\widetilde{\alpha}+\delta}} \lesssim \Big(\frac{R}{|x_B|}\Big)^{-\tilde{\delta}}\sum_{i=2}^{\infty}\frac{1}{2^{i\delta}},$$

and since $\tilde{\delta} < \widetilde{\alpha} - n < 0$ and $|x_B| > R$, the last term is bounded by a constant. □

Proposition 4.11. *Let $0 \leqslant \alpha < n$, $0 < \delta < \min\{\eta, (n-\alpha)/m\}$. Let $1 \leqslant r < \infty$ and $\tilde{\delta} \leqslant \min\{\delta, \widetilde{\alpha} - n/r\}$, excluding the case $\tilde{\delta} = \delta$ when $\widetilde{\alpha} - n/r = \delta$. Then there exist pairs of weights (w, v) belonging to $\mathcal{H}(r, \widetilde{\alpha}, \tilde{\delta})$ such that (w, v) does not belong to $\mathcal{H}((r't)', \widetilde{\alpha}, \tilde{\delta})$ for any $t > 0$, where $t \neq 1$ and $(r't)' > 1$.*

P r o o f. We exhibit examples of pairs of weights (w,v) such that $(w,v) \in \mathcal{H}(r,\widetilde{\alpha},\tilde{\delta})$ but $(w,v) \notin \mathcal{H}((r't)',\widetilde{\alpha},\tilde{\delta})$ for all $t>0$, $t\neq 1$.

Let $1<r<\infty$ and $\widetilde{\alpha}-n<\tilde{\delta}\leqslant \widetilde{\alpha}-n/r<\delta$. Let us consider the pair of weights $(w,v)=(1,|x|^{-\theta})$ with $\theta=\widetilde{\alpha}-n/r-\tilde{\delta}$. By Theorem 3.6 of [12] we have that $(w,v)\in \mathbb{H}(r,\widetilde{\alpha},\tilde{\delta})\subset \mathcal{H}(r,\widetilde{\alpha},\tilde{\delta})$. Let us see that (w,v) does not belong to $\mathcal{H}((r't)',\widetilde{\alpha},\tilde{\delta})$ for any $t>0$ if $t\neq 1$. By Lemma 4.4, it is enough to show that the pair (w,v) does not satisfy the local condition

$$|B|^{(\widetilde{\alpha}-\tilde{\delta})/n-1}\|\chi_B v\|_{r't}\lesssim \frac{w(B)}{|B|}. \tag{4.7}$$

Let $0<t<n/(\theta r')$ (otherwise $\|\chi_B v\|_{r't}=\infty$). Let $B(0,R)$, then the left hand side of the inequality (4.7) is bounded below by

$$|B|^{-1+(\widetilde{\alpha}-\tilde{\delta})/n}\|v\chi_B\|_{r't}\gtrsim R^{-n+\widetilde{\alpha}-\tilde{\delta}}R^{-\theta+n/(r't)}\gtrsim R^{-n/(r't')}$$

and the last term tends to infinity when R tends to zero or infinity if $t>1$ or $t<1$, respectively.

We now consider the case $1<r<\infty$ and $\tilde{\delta}\leqslant\delta<\widetilde{\alpha}-n/r$, and the pair $(|x|^{-\beta},|x|^{-\theta})$, where

$$\widetilde{\alpha}-\frac{n}{r}-\delta<\theta<\frac{n}{r'}\quad\text{and}\quad 0<\beta=\theta+\delta-\widetilde{\alpha}+\frac{n}{r}.$$

By Theorem 3.6 of [12] we have that $(w,v)\in\mathbb{H}(r,\widetilde{\alpha},\tilde{\delta})\subset\mathcal{H}(r,\widetilde{\alpha},\tilde{\delta})$. Let us now see that it does not belong to $\mathcal{H}((r't')',\widetilde{\alpha},\delta)$ for any $t>0$ if $t\neq 1$. By Lemma 4.4 it is enough to see that there exists a ball B such that the local condition (4.7) does not hold. In fact, if $B=B(0,R)$ then

$$\frac{|B|^{(\widetilde{\alpha}-\tilde{\delta})/n}\|v\chi_B\|_{r't}}{w(B)}\gtrsim R^{\beta-n+\widetilde{\alpha}-\delta}R^{-\theta+n/(r't)}\simeq R^{-n/r'+n/(r't)}\simeq R^{-n/(r't')}.$$

Consequently, the last term tends to infinity when R tends to zero or infinity if $t>1$ or $t<1$, respectively.

Let $1<r<\infty$, $\widetilde{\alpha}-n-k\delta<\tilde{\delta}\leqslant\min\{\widetilde{\alpha}-n/r-k\delta,\widetilde{\alpha}-n-(k-1)\delta\}$, $k\in\mathbb{N}$ and $(w,v)=(|x|^{k\delta},|x|^{\theta})$, where

$$\theta=\frac{n}{r}-\widetilde{\alpha}+\tilde{\delta}+k\delta.$$

By Theorem 4.10, we have $(w,v)\in\mathcal{H}(r,\widetilde{\alpha},\tilde{\delta})$. However, $(w,v)\notin\mathcal{H}((r't')',\widetilde{\alpha},\delta)$ for any $t>0$, with $t\neq 1$ because the local condition (4.7) does not hold. In fact,

if $B = B(0,R)$,

$$\frac{|B|^{(\widetilde{\alpha}-\tilde{\delta})/n}\|v\chi_B\|_{r't}}{w(B)} \gtrsim R^{\widetilde{\alpha}-\tilde{\delta}-k\delta-n}R^{n/r-\widetilde{\alpha}+\tilde{\delta}+k\delta+n/r't} \simeq R^{-n/r't'},$$

where the last term tends to infinity when R tends to zero or infinity if $t>1$ or $t<1$, respectively.

Similar arguments show that, if $1<r<\infty$ and $\widetilde{\alpha}-n/r-k\delta-\delta<\tilde{\delta}\leqslant\widetilde{\alpha}-n-k\delta$, $k\in\mathbb{N}$, then the pair $(w,v)=(|x|^\theta,|x|^\beta)$, where

$$\theta=\widetilde{\alpha}-\frac{n}{r}-k\delta-2\tilde{\delta} \quad\text{and}\quad \beta=-k\delta-\tilde{\delta},$$

belongs to $\mathcal{H}(r,\widetilde{\alpha},\tilde{\delta})$. However, $(w,v)\notin\mathcal{H}((r't')',\widetilde{\alpha},\delta)$ for any $t>0$ if $t\neq1$ because the local condition (4.7) does not hold.

For the case $r=1$ and $\tilde{\delta}=\widetilde{\alpha}-n$, it is immediate that the pair (w,v) given by $w=v=1$ belongs to $\mathcal{H}(1,\widetilde{\alpha},\widetilde{\alpha}-n)$. However, it is easy to check that $(w,v)\notin\mathcal{H}(1+\varepsilon,\widetilde{\alpha},\widetilde{\alpha}-n)$ for every $\varepsilon>0$.

Finally, if $r=1$ and $\tilde{\delta}<\widetilde{\alpha}-n$, let us consider the pair $(|x|^{-\tilde{\delta}},|x|^{n-\widetilde{\alpha}})$. It was proved in Theorem 4.10 that (w,v) belongs to $\mathcal{H}(1,\widetilde{\alpha},\tilde{\delta})$. Let us see that (w,v) does not belong to $\mathcal{H}(1+\varepsilon,\widetilde{\alpha},\tilde{\delta})$ for any $\varepsilon>0$. By Lemma 4.4 it is enough to show that (w,v) does not satisfy the condition (4.1) with $r=1+\varepsilon$. In fact, if $B=B(0,R)$, we get

$$\frac{|B|^{(\widetilde{\alpha}-\tilde{\delta})/n}}{w(B)}\|v\chi_B\|_{(1+\varepsilon)'}\gtrsim R^{n/(1+\varepsilon)'}$$

and the last expression tends to ∞ when R tends to ∞. □

5. Proof of the main results

We now give some previous lemmas that we will use in the proofs of the main results. We are considering $m\in\mathbb{N}\cup\{0\}$.

Lemma 5.1. *Let $0\leqslant\alpha<n$, $0<\delta<\min(\eta,(n-\alpha)/m)$ and $1\leqslant r\leqslant\infty$. Let $K_\alpha\in K^*_{\alpha,\infty}$ and $b\in\Lambda(\delta)$. If $(w,v)\in\mathcal{H}(r,\widetilde{\alpha},\tilde{\delta})$, then*

$$\int_{(2B)^c}|b(x)-b(z)|^m|K_\alpha(x-z)-K_\alpha(y-z)||f(z)|\,\mathrm{d}z\lesssim\|b\|^m_{\Lambda(\delta)}w(B)|B|^{\tilde{\delta}/n-1}\Big\|\frac{f}{v}\Big\|_r$$

for all $x,y\in B$.

P r o o f. If $x, y \in B$, by using that $b \in \Lambda(\delta)$ and $K_\alpha \in K^*_{\alpha,\infty}$, we have that

$$\begin{aligned}
&\int_{(2B)^c} |b(x)-b(z)|^m |K_\alpha(x-z)-K_\alpha(y-z)||f(z)|\,\mathrm{d}z \\
&\qquad \lesssim \|b\|^m_{\Lambda(\delta)} \int_{(2B)^c} |x-z|^{\delta m} |K_\alpha(x-z)-K_\alpha(y-z)||f(z)|\,\mathrm{d}z \\
&\qquad \lesssim \|b\|^m_{\Lambda(\delta)} \sum_{j=1}^{\infty} \frac{2^{j\delta m}|B|^{\delta m/n+\eta/n}}{2^{j(n-\alpha+\eta)}|B|^{(n-\alpha+\eta)/n}} \int_{2^{j+1}B\setminus 2^j B} |f(z)|\,\mathrm{d}z \\
&\qquad \lesssim \|b\|^m_{\Lambda(\delta)} |B|^{(\delta m-n+\alpha)/n} \sum_{j=1}^{\infty} (2^j)^{\delta m-n+\alpha-\eta} \int_{2^{j+1}B\setminus 2^j B} |f(z)|vv^{-1}\,\mathrm{d}z.
\end{aligned}$$

Now, we can apply Hölder's inequality to get

$$\begin{aligned}
&\int_{(2B)^c} |b(x)-b(z)|^m |K_\alpha(x-z)-K_\alpha(y-z)||f(z)|\,\mathrm{d}z \\
&\qquad \lesssim \|b\|^m_{\Lambda(\delta)} \Big\|\frac{f}{v}\Big\|_r |B|^{(\delta m-n+\alpha)/n} \sum_{j=1}^{\infty} 2^{j(\delta m-n+\alpha-\eta)} \bigg(\int_{2^{j+1}B\setminus 2^j B} v^{r'}(z)\,\mathrm{d}z\bigg)^{1/r'} \\
&\qquad \lesssim \|b\|^m_{\Lambda(\delta)} \Big\|\frac{f}{v}\Big\|_r |B|^{(\delta m-n+\alpha)/n} \\
&\qquad\quad \times \sum_{j=1}^{\infty} 2^{j(\delta m-n+\alpha-\eta)} |2^j B|^{(n-\widetilde{\alpha}+\delta)/n} \bigg(\int_{2^{j+1}B\setminus 2^j B} \frac{v^{r'}(z)}{|x_B-z|^{r'(n-\widetilde{\alpha}+\delta)}}\,\mathrm{d}z\bigg)^{1/r'} \\
&\qquad \lesssim \|b\|^m_{\Lambda(\delta)} \Big\|\frac{f}{v}\Big\|_r |B|^{\delta/n} \sum_{j=1}^{\infty} 2^{j(\delta-\eta)} \bigg(\int_{2^{j+1}B\setminus 2^j B} \frac{v^{r'}(z)}{|x_B-z|^{r'(n-\widetilde{\alpha}+\delta)}}\,\mathrm{d}z\bigg)^{1/r'} \\
&\qquad \lesssim \|b\|^m_{\Lambda(\delta)} \Big\|\frac{f}{v}\Big\|_r |B|^{\delta/n} \sum_{j=1}^{\infty} 2^{j(\delta-\eta)} \bigg(\int_{\mathbb{R}^n\setminus B} \frac{v^{r'}(z)}{|x_B-z|^{r'(n-\widetilde{\alpha}+\delta)}}\,\mathrm{d}z\bigg)^{1/r'}
\end{aligned}$$

Since $\sum\limits_{j=1}^{\infty} (2^j)^{\delta-\eta}$ is finite, by using the global condition (4.2), we get

$$\int_{(2B)^c} |b(x)-b(z)|^m |K_\alpha(x-z)-K_\alpha(y-z)||f(z)|\,\mathrm{d}z \lesssim \|b\|^m_{\Lambda(\delta)} |B|^{\tilde{\delta}/n} \Big\|\frac{f}{v}\Big\|_r \frac{w(B)}{|B|}.$$

□

Lemma 5.2. *Let $0 \leqslant \alpha < n$, $0 < \delta < \min(\eta, (n-\alpha)/m)$ and $1 \leqslant r \leqslant \infty$. Let $K_\alpha \in S^*_\alpha$ and $b \in \Lambda(\delta)$. Let $(w, v) \in \mathcal{H}(r, \widetilde{\alpha}, \tilde{\delta})$, then*

$$\frac{1}{w(B)} \int_B |T^m_{\alpha,b} f \chi_{2B}(x)|\,\mathrm{d}x \lesssim \|b\|^m_{\Lambda(\delta)} |B|^{\tilde{\delta}/n} \Big\|\frac{f}{v}\Big\|_{L^r}$$

for every ball $B \in \mathbb{R}^n$.

P r o o f. By using Tonelli's theorem and the fact that $b \in \Lambda(\delta)$, we obtain that

$$\begin{aligned}\frac{1}{w(B)}\int_B |T^m_{\alpha,b} f\chi_{2B}(x)|\,\mathrm{d}x &\leqslant \frac{1}{w(B)}\int_B\int_{2B} |b(x)-b(y)|^m |K_\alpha(x-y)||f(y)|\,\mathrm{d}y\,\mathrm{d}x\\ &\lesssim \frac{1}{w(B)}\int_{2B}|f(y)|\bigg(\int_B |b(x)-b(y)|^m|K_\alpha(x-y)|\,\mathrm{d}x\bigg)\,\mathrm{d}y\\ &\lesssim \|b\|^m_{\Lambda(\delta)}\frac{1}{w(B)}\int_{2B}|f(y)|\bigg(\int_{2B}|x-y|^{\delta m}|K_\alpha(x-y)|\,\mathrm{d}x\bigg)\,\mathrm{d}y.\end{aligned}$$

Now, since $K_\alpha \in S^*_\alpha$ and $\delta m < n-\alpha$, it is easy to see that

$$\int_{2B}|x-y|^{\delta m}|K_\alpha(x-y)|\,\mathrm{d}x \lesssim |B|^{(\delta m+\alpha)/n}.$$

Then, by Hölder's inequality, using Lemma 4.1, we have

$$\begin{aligned}\frac{1}{w(B)}\int_B |T^m_{\alpha,b} f\chi_{2B}(x)|\,\mathrm{d}x &\lesssim \|b\|^m_{\Lambda(\delta)}|B|^{(\delta m+\alpha)/n}\|v\chi_{2B}\|_{r'}\Big\|\frac{f}{v}\Big\|_r (w(B))^{-1}\\ &\lesssim \|b\|^m_{\Lambda(\delta)}|B|^{(\delta m+\alpha)/n}\Big\|\frac{f}{v}\Big\|_r |B|^{(\tilde\delta-\tilde\alpha)/n}\\ &\lesssim \|b\|^m_{\Lambda(\delta)}|B|^{\tilde\delta/n}\Big\|\frac{f}{v}\Big\|_r.\end{aligned}$$

□

We now proceed with the proof of Theorems 3.1 and 3.3.

P r o o f of Theorem 3.3. Let $f/v \in L^r(\mathbb{R}^n)$. Let $B \subset \mathbb{R}^n$ be a ball and $x \in B$. We split $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and put $a_B = 1/|B|\int_B T^m_{\alpha,b} f_2$. Then,

$$\begin{aligned}\frac{1}{w(B)}\int_B |T^m_{\alpha,b}f(x) - a_B|\,\mathrm{d}x &\lesssim \frac{1}{w(B)}\int_B |T^m_{\alpha,b}f_1(x)|\,\mathrm{d}x\\ &\quad + \frac{1}{w(B)}\int_B |T^m_{\alpha,b}f_2(x) - a_B|\,\mathrm{d}x\\ &= I_1 + I_2.\end{aligned}$$

By Lemma 5.2, we have

$$I_1 \lesssim \|b\|^m_{\Lambda(\delta)}|B|^{\tilde\delta/n}\Big\|\frac{f}{v}\Big\|_r. \tag{5.1}$$

Since

$$|T^m_{\alpha,b}f_2(x) - a_B| = |T^m_{\alpha,b}f_2(x) - (T^m_{\alpha,b}f_2)_B| \lesssim \frac{1}{|B|}\int_B |T^m_{\alpha,b}f_2(x) - T^m_{\alpha,b}f_2(y)|\,\mathrm{d}y,$$

then,

$$I_2 \lesssim \frac{1}{w(B)} \int_B \frac{1}{|B|} \int_B |T^m_{\alpha,b} f_2(x) - T^m_{\alpha,b} f_2(y)| \, \mathrm{d}y \, \mathrm{d}x. \tag{5.2}$$

Let $A = |T^m_{\alpha,b} f_2(x) - T^m_{\alpha,b} f_2(y)|$. If $x, y \in B$,

$$\begin{aligned} A &\lesssim \int_{(2B)^c} |(b(x)-b(z))^m K_\alpha(x-z) - (b(y)-b(z))^m K_\alpha(y-z)||f(z)| \, \mathrm{d}z \\ &\lesssim \int_{(2B)^c} |b(x)-b(z)|^m |K_\alpha(x-z) - K_\alpha(y-z)||f(z)| \, \mathrm{d}z \\ &\quad + \int_{(2B)^c} |(b(x)-b(z))^m - (b(y)-b(z))^m||K_\alpha(y-z)||f(z)| \, \mathrm{d}z \\ &= I_3 + I_4. \end{aligned}$$

By Lemma 5.1, we have

$$I_3 \lesssim \|b\|^m_{\Lambda(\delta)} w(B) |B|^{\tilde{\delta}/n - 1} \left\| \frac{f}{v} \right\|_r.$$

In order to estimate I_4, we use that $b \in \Lambda(\delta)$. If $A_j = 2^{j+1}B \setminus 2^j B$, then

$$\begin{aligned} I_4 &\lesssim |b(x)-b(y)| \sum_{k=0}^{m-1} \int_{(2B)^c} |b(x)-b(z)|^{m-1-k} |b(y)-b(z)|^k |K_\alpha(y-z)||f(z)| \, \mathrm{d}z \\ &\lesssim \|b\|^m_{\Lambda(\delta)} |B|^{\delta/n} \sum_{j=1}^{\infty} |2^{j+1}B|^{\delta(m-1)/n} \int_{A_j} |K_\alpha(x-z)||f(z)| \, \mathrm{d}z \\ &\lesssim \|b\|^m_{\Lambda(\delta)} |B|^{\delta/n} \sum_{j=1}^{\infty} \int_{A_j} \frac{|f(z)|}{|x_B - z|^{(n-\widetilde{\alpha}+\delta)}} \, \mathrm{d}z \\ &\lesssim \|b\|^m_{\Lambda(\delta)} |B|^{\delta/n} \int_{\mathbb{R}^n \setminus B} \frac{|f(z)|}{|x_B - z|^{(n-\widetilde{\alpha}+\delta)}} \, \mathrm{d}z. \end{aligned}$$

Then, by Hölder's inequality and the global condition (4.2), we have

$$I_4 \lesssim \|b\|^m_{\Lambda(\delta)} |B|^{\delta/n} \left\| \frac{f}{v} \right\|_r \left(\int_{\mathbb{R}^n \setminus B} \frac{v^{r'}(z)}{|x_B - z|^{r'(n-\widetilde{\alpha}+\delta)}} \, \mathrm{d}z \right)^{1/r'} \lesssim \|b\|^m_{\Lambda(\delta)} |B|^{\tilde{\delta}/n} \left\| \frac{f}{v} \right\|_r \frac{w(B)}{|B|}$$

We now get

$$I_2 \lesssim \frac{1}{w(B)} \int_B \frac{1}{|B|} \int_B (I_3 + I_4) \, \mathrm{d}y \, \mathrm{d}x \lesssim \|b\|^m_{\Lambda(\delta)} |B|^{\tilde{\delta}/n} \left\| \frac{f}{v} \right\|_r$$

and then, we obtain that

$$\frac{1}{w(B)} \int_B |T^m_{\alpha,b} f(x) - a_B| \, \mathrm{d}x \lesssim \|b\|^m_{\Lambda(\delta)} |B|^{\tilde{\delta}/n} \left\| \frac{f}{v} \right\|_r,$$

so it remains to take supremum over all balls B to get the desired result. □

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