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A NEW APPROACH TO ANTISYMMETRIC INFINITESIMAL BIALGEBRAS

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Abstract. We present a notion of an anti-covariant bialgebra extending the anti-symmetric infinitesimal bialgebra and also provide some equivalent characterizations of it. We also prove that an anti-associative Yang-Baxter pair can produce a special Rota-Baxter system.

Keywords: infinitesimal bialgebra; quasitriangular infinitesimal bialgebra

MSC 2020: 16T10, 17B38, 16T25

1. Introduction

Infinitesimal bialgebras first appeared in the work of Joni and Rota (see [8]) to give an algebraic framework for the calculus of divided differences. The anti-symmetric version of infinitesimal bialgebras was introduced in [21] by Zhelyabin by using the name associative D-bialgebra as an associative analog of Lie bialgebra defined by Drinfeld in [6]. Later this structure was studied systematically by Bai under the name anti-symmetric infinitesimal (for short ASI) bialgebra in [3]. Infinitesimal bialgebras and ASI bialgebras were concerned by many researchers, see [1], [2], [4], [5], [7], [9]–[20], etc. Interestingly, the latter can be characterized by the well-known matched pair of associative algebras and a double construction of a Frobenius algebra, from which the anti-symmetry appears, see [3]. In [5], Brzeziński introduced the notion of covariant bialgebras generalizing the infinitesimal bialgebra, which is related to Rota-Baxter systems and dendriform algebras. So it is very natural to consider the anti-version of covariant bialgebras and as an expectation, it can cover the ASI bialgebra. In this note, we provide the positive answer to the above question.

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Throughout this paper, K will be a field, and all vector spaces, tensor products, and homomorphisms are over K. We denote by id_M the identity map from M to M, $\sigma\colon M\otimes M\to M\otimes M$ by the flip map. If A is an algebra, then L and R denote the left and right multiplication maps, respectively. Let C be a coalgebra, we use the Sweedler's notation for the comultiplication: $\Delta(c)=c_1\otimes c_2$ for any $c\in C$. An element $r\in A\otimes A$ being anti-symmetric means $r=-\sigma(r)$. Given $r=r^1\otimes r^2\in A\otimes A$, we define $r_{12}=r\otimes 1$, $r_{13}=r^1\otimes 1\otimes r^2$, $r_{23}=1\otimes r$, where 1 means either the identity of A (if A is unital) or the identity in the extended unital algebra $K\oplus A$ (if A is nonunital).

2. Main definitions and results

Firstly, as a natural generalization of Zhelyabin's associative D-algebra or Bai's ASI bialgebra, we give the anti-version of Brzeziński's covariant bialgebra.

Definition 2.1. An *anti-covariant (for short AC) bialgebra* is a quadruple $(A,\delta_1,\delta_2,\Delta)$ such that

(a) A is an associative algebra.

(b) (A,Δ) is a coassociative coalgebra.

(c) Let $\delta_i\colon A\to A\otimes A$, $i=1,2$ (write $\delta_i(a)=a^i_{(1)}\otimes a^i_{(2)}$) be two *anti-derivations*, i.e., $\delta_i(ab)=a^i_{(1)}b\otimes a^i_{(2)}+b^i_{(1)}\otimes ab^i_{(2)}$, $i=1,2$, and

$$a^i_{(1)}\otimes a^i_{(2)}b+b^i_{(2)}a\otimes b^i_{(1)}=ba^i_{(1)}\otimes a^i_{(2)}+b^i_{(2)}\otimes ab^i_{(1)}. \tag{1}$$

Here Δ is an *AC derivation with respect to* (δ_1,δ_2), which means

$$\Delta(ab)=(R(b)\otimes\mathrm{id})\delta_2(a)+(\mathrm{id}\otimes L(a))\Delta(b)=a^2_{(1)}b\otimes a^2_{(2)}+b_1\otimes ab_2, \tag{2}$$

$$=(R(b)\otimes\mathrm{id})\Delta(a)+(\mathrm{id}\otimes L(a))\delta_1(b)=a_1b\otimes a_2+b^1_{(1)}\otimes ab^1_{(2)} \tag{3}$$

and

$$a_1\otimes a_2b+b_2a\otimes b_1=ba_1\otimes a_2+b_2\otimes ab_1. \tag{4}$$

If A has identity and $\Delta(1)=\lambda\, 1\otimes 1$, where $\lambda\in K$, then we call the AC bialgebra $(A,\delta_1,\delta_2,\Delta)$ λ-*unital.*

Remark 2.2.

(1) If $\delta_i=\Delta$, $i=1,2$ in Definition 2.1, then we obtain the ASI bialgebra (A,Δ,Δ,Δ) introduced in [21] under the name "associative D-algebra" and studied in [3], [4].

(2) Equation (4) is exactly the second identity in [21], Theorem 1 or [3], Theorem 2.2.3.

Now we give an equivalent characterization of AC bialgebras.

Theorem 2.3. *Let A be a unital associative algebra, $\delta_i\colon A \to A\otimes A$, $i=1,2$ be two anti-derivations. There exists a coassociative AC derivation with respect to (δ_1,δ_2) if and only if there exists an element $u=u^1\otimes u^2\in A\otimes A$ such that for all $a,b\in A$,*

$$\delta_1(a)-\delta_2(a)=u^1\otimes au^2-u^1a\otimes u^2, \tag{5}$$

$$(\delta_1\otimes \mathrm{id}-\mathrm{id}\otimes\delta_1)(u)=u_{23}u_{12}-u_{12}u_{13}, \tag{6}$$

$$(\delta_1\otimes \mathrm{id}-\mathrm{id}\otimes\delta_1)\circ\delta_1(a)=-u_{12}\delta_1(a)_{13}, \tag{7}$$

$$u^1a\otimes u^2b+u^2a\otimes u^1b=bu^1a\otimes u^2+u^2\otimes au^1b. \tag{8}$$

In this case,

$$\Delta(a)=u^1a\otimes u^2+\delta_1(a)=u^1\otimes au^2+\delta_2(a). \tag{9}$$

P r o o f. By $\Delta(a)=u^1a\otimes u^2+\delta_1(a)$ in (9) and (1) for $i=1$, it is obvious that (8) is equivalent to (4).

($\Rightarrow$) Assume that Δ is a coassociative AC derivation, set $u=\Delta(1)$. Let $b=1$ in (2), we have $\Delta(a)=a^2_{(1)}\otimes a^2_{(2)}+1_1\otimes a1_2$. Similarly let $a=1$ in (3), we have $\Delta(b)=1_1b\otimes 1_2+b^1_{(1)}\otimes b^1_{(2)}$. So

$$\Delta(a)=1_1\otimes a1_2+a^2_{(1)}\otimes a^2_{(2)}=1_1a\otimes 1_2+a^1_{(1)}\otimes a^1_{(2)}.$$

Then equations (9) and (5) hold.

By (9), we have

$$\begin{aligned}(\mathrm{id}\otimes\Delta)\Delta(a)&=1_1a\otimes\bar 1_1 1_2\otimes\bar 1_2+1_1a\otimes(1_2)^1_{(1)}\otimes(1_2)^1_{(2)}\\&\quad+a^1_{(1)}\otimes\bar 1_1a^1_{(2)}\otimes\bar 1_2+a^1_{(1)}\otimes(a^1_{(2)})^1_{(1)}\otimes(a^1_{(2)})^1_{(2)}\end{aligned}$$

and

$$\begin{aligned}(\Delta\otimes\mathrm{id})\Delta(a)&=\bar 1_1 1_1a\otimes\bar 1_2\otimes 1_2+(1_1)^1_{(1)}a\otimes(1_1)^1_{(2)}\otimes 1_2+a^1_{(1)}\otimes 1_1a^1_{(2)}\otimes 1_2\\&\quad+1_1a^1_{(1)}\otimes 1_2\otimes a^1_{(2)}+(a^1_{(1)})^1_{(1)}\otimes(a^1_{(1)})^1_{(2)}\otimes a^1_{(2)}.\end{aligned}$$

Then by the coassociativity of Δ at $a=1$, we obtain

$$\begin{aligned}&1_1\otimes\bar 1_1 1_2\otimes\bar 1_2+1_1\otimes(1_2)^1_{(1)}\otimes(1_2)^1_{(2)}+1^1_{(1)}\otimes\bar 1_1 1^1_{(2)}\otimes\bar 1_2+1^1_{(1)}\otimes(1^1_{(2)})^1_{(1)}\otimes(1^1_{(2)})^1_{(2)}\\&\quad=\bar 1_1 1_1\otimes\bar 1_2\otimes 1_2+(1_1)^1_{(1)}\otimes(1_1)^1_{(2)}\otimes 1_2+1^1_{(1)}\otimes\bar 1_1 1^1_{(2)}\otimes\bar 1_2\\&\qquad+1_1\bar 1^1_{(1)}\otimes 1_2\otimes\bar 1^1_{(2)}+(1^1_{(1)})^1_{(1)}\otimes(1^1_{(1)})^1_{(2)}\otimes 1^1_{(2)}.\end{aligned}$$

Based on the properties of anti-derivations and AC derivations, we can get (6).

Apply $R(a)\otimes \mathrm{id}\otimes \mathrm{id}$ to (6) and by the coassociativity for all $a\in A$, we have

$$a^1_{(1)}\otimes (a^1_{(2)})^1_{(1)}\otimes (a^1_{(2)})^1_{(2)} = 1_1 a^1_{(1)}\otimes 1_2\otimes a^1_{(2)} + (a^1_{(1)})^1_{(1)}\otimes (a^1_{(1)})^1_{(2)}\otimes a^1_{(2)},$$

i.e., $(\delta_1\otimes \mathrm{id} - \mathrm{id}\otimes \delta_1)\delta_1(a) = -1_1 a^1_{(1)}\otimes 1_2\otimes a^1_{(2)} = -u_{12}\delta_1(a)_{13}$. Thus, equation (7) holds.

($\Leftarrow$) By (9) for all $a,b\in A$ we have

$$a^2_{(1)}b\otimes a^2_{(2)} + b_1\otimes ab_2 = a_1 b\otimes a_2 + b^1_{(1)}\otimes ab^1_{(2)},$$

$$\begin{aligned}\Delta(ab) &\overset{(9)}{=} u^1 ab\otimes u^2 + \delta_1(ab) = u^1 ab\otimes u^2 + a^1_{(1)}b\otimes a^1_{(2)} + b^1_{(1)}\otimes ab^1_{(2)}\\ &\overset{(9)}{=} a_1 b\otimes a_2 + b^1_{(1)}\otimes ab^1_{(2)}\end{aligned}$$

and

$$\begin{aligned}\Delta(ab) &\overset{(9)}{=} u^1\otimes abu^2 + \delta_2(ab) = u^1\otimes abu^2 + a^2_{(1)}b\otimes a^2_{(2)} + b^2_{(1)}\otimes ab^2_{(2)}\\ &\overset{(9)}{=} a^2_{(1)}b\otimes a^2_{(2)} + b_1\otimes ab_2.\end{aligned}$$

Then we can obtain equations (2) and (3). By (6) and (7) and the properties of anti-derivation, one can get the coassociativity of Δ. □

Remark 2.4. By $\Delta(a) = u^1\otimes au^2 + \delta_2(a)$ in (9) and (1) for $i=2$, we also obtain that (4) is equivalent to

$$bu^1\otimes au^2 + bu^2\otimes au^1 = u^1\otimes au^2 b + bu^2 a\otimes u^1. \tag{10}$$

Corollary 2.5. *Let A be a unital associative algebra and $\Delta\colon A\to A\otimes A$ an anti-derivation. Then Δ is coassociative if and only if*

$$(\Delta\otimes \mathrm{id} - \mathrm{id}\otimes \Delta)\circ \Delta(a) = 0. \tag{11}$$

Proof. Let $\delta_1 = \delta_2 = \Delta$ in Theorem 2.3, then by (9), $u = \Delta(1) = 0$. Thus, in this case, equations (5), (6) and (8) hold automatically, and (7) is exactly (11). The proof is finished. □

Remark 2.6. Corollary 2.5 is just the characterization of ASI bialgebras.

Corollary 2.7. *Let A be a unital associative algebra, $\delta_i\colon A\to A\otimes A$, $i=1,2$ be two anti-derivations. Then $(A,\delta_1,\delta_2,\Delta)$ is a λ-unital AC bialgebra if and only if for all $a,b\in A$,*

$$\delta_1(a) - \delta_2(a) = \lambda 1\otimes a - \lambda a\otimes 1, \tag{12}$$

$$(\delta_1\otimes \mathrm{id} - \mathrm{id}\otimes \delta_1)\circ \delta_1(a) = -\lambda\delta_1(a)_{13}, \tag{13}$$

$$2\lambda a\otimes b = \lambda ba\otimes 1 + \lambda 1\otimes ab. \tag{14}$$

In this case,

$$\Delta(a) = \lambda a \otimes 1 + \delta_1(a) = \lambda 1 \otimes a + \delta_2(a). \tag{15}$$

P r o o f. It can be proved by letting $u = \Delta(1) = \lambda 1 \otimes 1$ in Theorem 2.3. □

Proposition 2.8.

(1) 0-*unital AC bialgebra is exactly the ASI bialgebra* $(A, \Delta, \Delta, \Delta)$.

(2) *If* char(K)=0, *then* λ-*unital (for* $\lambda \neq 0$*) AC bialgebra is trivial, that is to say, it is one dimensional.*

P r o o f. If $(A, \delta_1, \delta_2, \Delta)$ is a λ-unital AC bialgebra, then (8) and (10) hold. So we have

$$2\lambda(a \otimes b - b \otimes a) = 0. \tag{16}$$

Thus, if $\lambda = 0$, then $\Delta(1) = 0$. So by Corollary 2.5, we get the first conclusion. If $\lambda \neq 0$ and char(K)=0, then by (16), we have $a \otimes b = b \otimes a$ for all $a, b \in A$. So in this case, A is trivial. We finish the proof. □

Remark 2.9.

(1) Corollary 2.7 is different from Corollary 3.11 of [5] even if $\lambda = 1$ since here 1-unital AC bialgebra is trivial when char$(K) \neq 2$.

(2) By Corollary 3.11 of [5] one gets that a 1-unital covariant bialgebra $(A, \Delta, \delta, \delta)$ is trivial.

The following characterization of AC bialgebras induces the notion of associative Yang-Baxter equation.

Proposition 2.10. *Let* A *be an algebra and* $r, s \in A \otimes A$ *two anti-symmetric elements. Define the linear maps*

$$\delta_r\colon\ A \to A \otimes A, \quad \delta_r(a) = r^1 \otimes ar^2 - r^1 a \otimes r^2, \tag{17}$$

$$\delta_s\colon\ A \to A \otimes A, \quad \delta_s(a) = s^1 \otimes as^2 - s^1 a \otimes s^2, \tag{18}$$

$$\Delta\colon\ A \to A \otimes A, \quad \Delta(a) = r^1 \otimes ar^2 - s^1 a \otimes s^2. \tag{19}$$

Then $(A, \delta_r, \delta_s, \Delta)$ *is an AC bialgebra if and only if for all* $a \in A$,

$$(R(a)\otimes\mathrm{id}\otimes\mathrm{id})(s_{13}r_{23}-s_{23}s_{12}+s_{12}s_{13}) = (\mathrm{id}\otimes\mathrm{id}\otimes L(a))(s_{12}r_{13}-r_{23}r_{12}+r_{13}r_{23}) \tag{20}$$

and

$$(\mathrm{id} \otimes L(a) \circ R(b) - L(b) \circ R(a) \otimes \mathrm{id})(r - s) = 0. \tag{21}$$

P r o o f. The functions δ_r, δ_s are anti-derivations: For all $a, b \in A$,

$$\begin{aligned}
a^r_{(1)}b \otimes a^r_{(2)} + b^r_{(1)} \otimes ab^r_{(2)} &= r^1 b \otimes ar^2 - r^1 ab \otimes r^2 + r^1 \otimes abr^2 - r^1 b \otimes ar^2 \\
&= r^1 \otimes abr^2 - r^1 ab \otimes r^2 = \delta_r(ab), \\
a^r_{(1)} \otimes a^r_{(2)}b + b^r_{(2)}a \otimes b^r_{(1)} &= r^1 \otimes ar^2 b - r^1 a \otimes r^2 b + br^2 a \otimes r^1 - r^2 a \otimes r^1 b \\
&= -r^2 \otimes ar^1 b + r^2 a \otimes r^1 b - br^1 a \otimes r^2 - r^2 a \otimes r^1 b \\
&= -r^2 \otimes ar^1 b - br^1 a \otimes r^2 \\
&= -br^2 \otimes ar^1 - br^1 a \otimes r^2 + br^2 \otimes ar^1 - r^2 \otimes ar^1 b \\
&= br^1 \otimes ar^2 - br^1 a \otimes r^2 + br^2 \otimes ar^1 - r^2 \otimes ar^1 b \\
&= ba^r_{(1)} \otimes a^r_{(2)} + b^r_{(2)} \otimes ab^r_{(1)}.
\end{aligned}$$

The proof for δ_s is similar.

Equations (2) and (3) can be checked as follows. For all $a, b \in A$, we have

$$\begin{aligned}
a_1 b \otimes a_2 + b^r_{(1)} \otimes ab^r_{(2)} &= r^1 b \otimes ar^2 - s^1 ab \otimes s^2 + r^1 \otimes abr^2 - r^1 b \otimes ar^2 \\
&= r^1 \otimes abr^2 - s^1 ab \otimes s^2 = \Delta(ab)
\end{aligned}$$

and

$$\begin{aligned}
a^s_{(1)}b \otimes a^s_{(2)} + b_1 \otimes ab_2 &= s^1 b \otimes as^2 - s^1 ab \otimes s^2 + r^1 \otimes abr^2 - s^1 b \otimes as^2 \\
&= r^1 \otimes abr^2 - s^1 ab \otimes s^2 = \Delta(ab).
\end{aligned}$$

By the anti-symmetry of r and s, (4) is equivalent to (21).

For all $a \in A$ and $r = R, s = S$, one can compute

$$\begin{aligned}
(\mathrm{id} \otimes \Delta) \circ \Delta(a) &= r^1 \otimes \Delta(ar^2) - s^1 a \otimes \Delta(s^2) \\
&= r^1 \otimes (R^1 \otimes ar^2 R^2 - s^1 ar^2 \otimes s^2) - s^1 a \otimes (r^1 \otimes s^2 r^2 - S^1 s^2 \otimes S^2) \\
&= r^1 \otimes R^1 \otimes ar^2 R^2 - r^1 \otimes s^1 ar^2 \otimes s^2 - s^1 a \otimes r^1 \otimes s^2 r^2 \\
&\quad + s^1 a \otimes S^1 s^2 \otimes S^2
\end{aligned}$$

and

$$\begin{aligned}
(\Delta \otimes \mathrm{id}) \circ \Delta(a) &= \Delta(r^1) \otimes ar^2 - \Delta(s^1 a) \otimes s^2 \\
&= (R^1 \otimes r^1 R^2 - s^1 r^1 \otimes s^2) \otimes ar^2 - (r^1 \otimes s^1 ar^2 - S^1 s^1 a \otimes S^2) \otimes s^2 \\
&= R^1 \otimes r^1 R^2 \otimes ar^2 - s^1 r^1 \otimes s^2 \otimes ar^2 - r^1 \otimes s^1 ar^2 \otimes s^2 \\
&\quad + S^1 s^1 a \otimes S^2 \otimes s^2.
\end{aligned}$$

Then Δ is coassociative if and only if

$$
\begin{aligned}
& s^1 a \otimes r^1 \otimes s^2 r^2 - s^1 a \otimes S^1 s^2 \otimes S^2 + S^1 s^1 a \otimes S^2 \otimes s^2 \\
& \qquad = s^1 r^1 \otimes s^2 \otimes a r^2 - R^1 \otimes r^1 R^2 \otimes a r^2 + r^1 \otimes R^1 \otimes a r^2 R^2 .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& (R(a) \otimes \mathrm{id} \otimes \mathrm{id})(s^1 \otimes r^1 \otimes s^2 r^2 - s^1 \otimes S^1 s^2 \otimes S^2 + S^1 s^1 \otimes S^2 \otimes s^2) \\
& \qquad = (\mathrm{id} \otimes \mathrm{id} \otimes \text{Ł}(a))(s^1 r^1 \otimes s^2 \otimes r^2 - R^1 \otimes r^1 R^2 \otimes r^2 + r^1 \otimes R^1 \otimes r^2 R^2),
\end{aligned}
$$

i.e.,

$$
(R(a) \otimes \mathrm{id} \otimes \mathrm{id})(s_{13} r_{23} - s_{23} s_{12} + s_{12} s_{13}) = (\mathrm{id} \otimes \mathrm{id} \otimes L(a))(s_{12} r_{13} - r_{23} r_{12} + r_{13} r_{23}),
$$

finishing the proof. □

Definition 2.11. An *anti-associative Yang-Baxter pair* in A is a pair of elements $r, s \in A \otimes A$ satisfying

$$
r_{13} r_{23} - r_{23} r_{12} + s_{12} r_{13} = 0, \tag{22}
$$

$$
s_{13} r_{23} - s_{23} s_{12} + s_{12} s_{13} = 0 \tag{23}
$$

and equation (21).

Specially, if $r = s$, then we call

$$
r_{13} r_{23} - r_{23} r_{12} + r_{12} r_{13} = 0
$$

the *anti-associative Yang-Baxter equation in A*.

Remark 2.12. The anti-associative Yang-Baxter pair in Definition 2.11 is exactly the associative Yang-Baxter pair in A^{op} (the opposite algebra) in [11] satisfying (21).

Proposition 2.13. *If (r, s) is an anti-symmetric solution of the anti-associative Yang-Baxter pair in A, δ_r, δ_s, Δ are defined by (17)–(19). Then $(A, \delta_r, \delta_s, \Delta)$ is an AC bialgebra. In this case $(A, \delta_r, \delta_s, \Delta)$ is called an anti-quasitriangular AC bialgebra.*

Theorem 2.14. *Let A be an associative algebra, $r, s \in A \otimes A$ anti-symmetric satisfying (21) and δ_r, δ_s, Δ are defined by (17)–(19). Then $(A, \delta_r, \delta_s, \Delta)$ is an anti-quasitriangular AC bialgebra if and only if*

$$
(\mathrm{id} \otimes \Delta)(r) = r_{23} r_{12} - s_{12} r_{13} - s_{23} r_{12}, \tag{24}
$$

$$
(\Delta \otimes \mathrm{id})(s) = s_{23} r_{12} + s_{13} r_{23} - s_{23} s_{12}. \tag{25}
$$

P r o o f. One easily checks that

$$\begin{aligned}(\mathrm{id}\otimes\Delta)(r) &= r^1\otimes(R^1\otimes r^2R^2 - s^1r^2\otimes s^2) = r^1\otimes R^1\otimes r^2R^2 - r^1\otimes s^1r^2\otimes s^2\\ &= r_{13}r_{23} - s_{23}r_{12}, (\Delta\otimes\mathrm{id})(s) = (r^1\otimes s^1r^2 - S^1s^1\otimes S^2)\otimes s^2\\ &= r^1\otimes s^1r^2\otimes s^2 - S^1s^1\otimes S^2\otimes s^2 = s_{23}r_{12} - s_{12}s_{13},\end{aligned}$$

and the rest is direct. □

Corollary 2.15. *Let A be a unital associative algebra, $r\in A\otimes A$ anti-symmetric. Then a 0-unital anti-quasitriangular AC bialgebra (A,Δ,Δ,Δ) is exactly a quasitriangular ASI bialgebra studied in* [3], *Corollary* 2.4.1. *In this case, $\Delta(a) = r^1\otimes ar^2 - r^1a\otimes r^2$.*

P r o o f. Let $a = 1$ in (19), we have $\Delta(1) = r^1\otimes r^2 - s^1\otimes s^2$. An anti-quasitriangular AC bialgebra $(A,\delta_r,\delta_s,\Delta)$ is λ-unital if and only if $r^1\otimes r^2 - s^1\otimes s^2 = \lambda 1\otimes 1$. Since $\lambda = 0$, $r = s$. The rest is obvious. □

Remark 2.16. By Proposition 2.10, a λ-unital (for $\lambda\neq 0$) anti-quasitriangular AC bialgebra over a field K ($\mathrm{char}(K) = 0$) is trivial, which coincides with Part (2) in Proposition 2.8.

Let us recall that a *Rota-Baxter system* is a triple (A,P,Q), where A is an associative algebra, $P,Q\colon A\to A$ are two linear maps such that for all $a,b\in A$,

$$P(a)P(b) = P(P(a)b + aQ(b)), \tag{26}$$

$$Q(a)Q(b) = Q(P(a)b + aQ(b)). \tag{27}$$

Definition 2.17. A *special Rota-Baxter system* is a Rota-Baxter system satisfying the following condition:

$$aP(b) + Q(b)a = P(b)a + aQ(b) \quad \forall\, a,b\in A. \tag{28}$$

Proposition 2.18. *Let A be an associative algebra, $r,s\in A\otimes A$ such that (r,s) is an anti-associative Yang-Baxter pair. For all $a\in A$, define*

$$P(a) := r^2ar^1, \quad Q(a) := s^2as^1.$$

Then (A,P,Q) is a special Rota-Baxter system.

P r o o f. Similarly to [5], Proposition 3.4, we can check that (26) and (27) hold and (28) can be proved by (21). □

Remark 2.19.

(1) If $P = Q$, then (28) holds automatically. So in this case, a special Rota-Baxter system and a Rota-Baxter system coincide, both turn to be a Rota-Baxter algebra $(A, P = Q)$ of weight 0.

(2) A special Rota-Baxter system over a commutative associative algebra is just an usual Rota-Baxter system.

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