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Jian Liu

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TRIANGULATED CATEGORIES OF PERIODIC COMPLEXES AND ORBIT CATEGORIES

Jian Liu, Shanghai

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Abstract. We investigate the triangulated hull of orbit categories of the perfect derived category and the bounded derived category of a ring concerning the power of the suspension functor. It turns out that the triangulated hull corresponds to the full subcategory of compact objects of certain triangulated categories of periodic complexes. This specializes to Stai and Zhao's result on the finite dimensional algebra of finite global dimension. As the first application, if A, B are flat algebras over a commutative ring and they are derived equivalent, then the corresponding derived categories of n-periodic complexes are triangle equivalent. As the second application, we get the periodic version of the Koszul duality.

Keywords: periodic complex; orbit category; triangulated hull; derived category; derived equivalence; dg category; Koszul duality

MSC 2020: 18G80, 16E45, 18E20, 18G35

1. Introduction

Given an additive category $\mathcal{A}$ and an integer $n \geqslant 1$, a complex (X, ∂_X) over $\mathcal{A}$ is called *n-periodic* if $X^i = X^{i+n}$ and $\partial_X^i = \partial_X^{i+n}$ for all i. A chain map f between n-periodic complexes is an n-periodic morphism if $f^i = f^{i+n}$ for all i. A 1-periodic complex is just a differential object which first appeared in Cartan and Eilenberg's book, see [11]. It was systematically studied by Avramov, Buchweitz, and Iyengar, see [1]. Two morphisms $f, g\colon X \to Y$ of n-periodic complexes are called *homotopic* if there is a homotopy map $\{\sigma^i\colon X^i \to Y^{i-1}\}_{i\in\mathbb{Z}}$ from f to g such that $\sigma^i = \sigma^{i+n}$ for all i. Then one can form the homotopy category $\mathsf{K}_n(\mathcal{A})$ of n-periodic complexes and the derived category $\mathsf{D}_n(\mathcal{A})$ of n-periodic complexes when $\mathcal{A}$ is abelian. They are both triangulated categories, see [30] or Section 3.

Let R be a left noetherian ring. The homotopy category $\mathsf{K}(R\text{-Inj})$ of complexes of injective R-modules and the derived category $\mathsf{D}(R\text{-Mod})$ of complexes of

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R-modules are compactly generated, see [20] and [24], respectively. Inspired by this, we prove that the homotopy category $\mathsf{K}_n(R\text{-Inj})$ of n-periodic complexes of injecitve R-modules and the derived category $\mathsf{D}_n(R\text{-Mod})$ of n-periodic complexes of R-modules are compactly generated, see Theorem 3.1. Moreover, the canonical functor $\mathsf{K}_n(R\text{-Inj}) \to \mathsf{D}_n(R\text{-Mod})$ induces a recollement, see Theorem 3.1.

Let $T\colon\ \mathcal{A} \to \mathcal{A}$ be an autoequivalence. Following [22], the *orbit category* $\mathcal{A}/T$ is defined as follows: it has the same objects as $\mathcal{A}$ and the morphism spaces

$$\mathrm{Hom}_{\mathcal{A}/T}(X, Y) := \coprod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X, T^i Y).$$

The composition in $\mathcal{A}/T$ is defined in a natural way. As the name suggests, the objects in the same T-orbit are isomorphic.

The following question has been considered in the literature: given a triangulated category $\mathcal{T}$ with suspension functor Σ, is there a triangulated structure of $\mathcal{T}/\Sigma^n$ such that the projection functor $\mathcal{T} \to \mathcal{T}/\Sigma^n$ is exact? Neeman found the answer to this question is negative; see discussions in [22]. Let R be a finite dimensional hereditary algebra over a field. Peng and Xiao in [30] observed the orbit category $\mathsf{D}^b(R\text{-mod})/\Sigma^2$ of the bounded derived category of finitely generated R-modules, introduced by Happel (see [16]) under the name "root category", is triangulated. Indeed, they proved that it is equivalent to the homotopy category of 2-periodic complexes of finitely generated projective R-modules. This established a link between the orbit category and the triangulated category of periodic complexes. By making use of this triangulated structure, they constructed the so-called Ringel-Hall Lie algebra determined by $\mathsf{D}^b(R\text{-mod})/\Sigma^2$ and gave a realization of all symmetrizable Kac-Moody Lie algebras, see [31].

If R is a finite dimensional algebra over a field with finite global dimension, it was independently proved by Stai (see [35]) and Zhao (see [37]) that $\mathsf{D}^b(R\text{-mod})/\Sigma^n$ embeds into its triangulated hull $\mathsf{D}_n(R\text{-mod})$, where R-mod is the category of finitely generated R-modules. We are motivated by the natural question: what is the triangulated hull of $\mathsf{D}^b(R\text{-mod})/\Sigma^n$ for a general ring R?

It is proved in Section 4 that the triangulated hull of $\mathsf{D}^b(R\text{-mod})/\Sigma^n$ coincides with the full subcategory of compact objects of $\mathsf{K}_n(R\text{-Inj})$, see Corollary 4.1. Its proof relies on some techniques from [24], [35].

Two rings A, B are called *derived equivalent* if there exists a triangle equivalence $\mathsf{D}(A\text{-Mod}) \simeq \mathsf{D}(B\text{-Mod})$. In general, whether two rings are derived equivalent is difficult to grasp. Therefore, it is important to investigate the invariant under the derived equivalence. By introducing the tilting complex, Rickard in [32] established the derived Morita theory of rings. After that, Keller in [20] generalized Rickard's derived Morita theory through the language of differential graded categories.

In Section 5, we compare the triangle equivalences $\mathsf{D}(A\text{-Mod}) \simeq \mathsf{D}(B\text{-Mod})$ and $\mathsf{D}_n(A\text{-Mod}) \simeq \mathsf{D}_n(B\text{-Mod})$ for two rings A, B. It turns out that these two equivalences are closely related, see Theorem 5.1.

Over the past forty years, the Koszul duality phenomenon has played an important role in representation theory. For instance, the DG version of the Koszul duality was used by Benson, Iyengar, and Krause (see [6]) to classify subcategories of stable module category of finite groups. In Section 6 we study the periodic version of the Koszul duality.We prove that there exists a triangle equivalence between the derived category of n-periodic complexes of graded modules over symmetric algebra and the homotopy category of n-periodic complexes of graded-injective modules over exterior algebra, see Theorem 6.1. Its proof relies on the classical Koszul duality and the studies in previous sections.

2. Notations and preliminaries

Throughout the article, R is a left noetherian ring, R-Mod (or R-mod) is the category of left (or finitely generated left, respectively) R-modules. The full subcategory of R-Mod consisting of all projective (or injective) R-modules is denoted by R-Proj (or R-Inj, respectively). For an additive category $\mathcal{A}$, $\mathsf{C}(\mathcal{A})$ is the category of complexes over $\mathcal{A}$ with the suspension functor Σ^l ($\Sigma^l(X)^i := X^{i+l}$, $\partial^i_{\Sigma^l(X)} := (-1)^l\partial^{i+l}_X$). Denote by $\mathsf{K}(\mathcal{A})$ the homotopy category of complexes over $\mathcal{A}$. When $\mathcal{A}$ is abelian, let $\mathsf{D}(\mathcal{A})$ denote the derived category of complexes over $\mathcal{A}$.

A complex of R-modules is *perfect* provided that it is quasi-isomorphic to a bounded complex of finitely generated projective R-modules. The symbol $\mathrm{per}(R)$ stands for the full subcategory of $\mathsf{D}(R\text{-Mod})$ consisting of all perfect complexes.

2.1. Thick subcategories and localizing subcategories. Let $\mathcal{T}$ be a triangulated category and $\mathcal{C}$ be a triangulated subcategory of $\mathcal{T}$. We say $\mathcal{C}$ is *thick* (or *localizing*) if it is closed under direct summands (or coproducts, respectively). For a set S of objects in $\mathcal{T}$, we let $\mathrm{thick}_{\mathcal{T}}(S)$ denote the smallest thick subcategories of $\mathcal{T}$ containing S. This can be realized as the intersection of all thick subcategories of $\mathcal{T}$ containing S; it has an inductive construction, see [2], Subsection 2.2.4.

If $\mathcal{T}$ has coproducts, then a technique of Eilenberg's swindle implies that any localizing subcategory is thick.

It is well-known that $\mathrm{per}(R) = \mathrm{thick}_{\mathsf{D}(R\text{-Mod})}(R)$ is thick, see [10], Lemma 1.2.1.

2.2. Let $F\colon \mathcal{T} \to \mathcal{T}'$ be an exact functor between triangulated categories. Then the *kernel* of F defined by

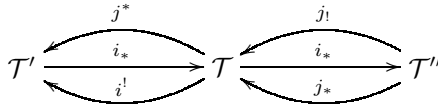
$$\operatorname{Ker} F := \{X \in \mathcal{T}\colon\ F(X) \cong 0\}$$

is a thick subcategory of $\mathcal{T}$. When the functor F is full, the *essential image* of F defined by

$$\operatorname{Im} F := \{Y \in \mathcal{T}'\colon\ Y \cong F(X) \text{ for some } X \in \mathcal{T}\}$$

is a triangulated subcategory of $\mathcal{T}'$.

2.3. Recollement. Following Beilinson, Bernstein and Deligne (see [4]), we call the diagram



of triangulated categories and exact functors the *recollement* if the following conditions are satisfied.

(1) (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$ and (j^*, j_*) are adjoint pairs.
(2) i_*, $j_!$ and j_* are fully faithful.
(3) $\operatorname{Im} i_* = \operatorname{Ker} j^*$, that is, $j^*(X) = 0$ if and only if $X \cong i_*(Y)$ for some $Y \in \mathcal{T}'$.

Next, we record a useful result for its proof, see [24], Section 3.

2.4. Let (F, G) be a sequence $\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}''$ of exact functors between triangulated categories. We say (F, G) is a *localization sequence* if the following conditions hold.

(1) F is fully faithful and F has a right adjoint.
(2) G has a right adjoint G_ϱ and G_ϱ is fully faithful.
(3) For an object X of $\mathcal{T}$, $G(X) = 0$ if and only if $X \cong F(X')$ for some $X' \in \mathcal{T}'$.

The sequence (F, G) is called a *colocalization sequence* if the sequence $(F^{\mathrm{op}}, G^{\mathrm{op}})$ of opposite functors is a localization sequence.

Let x be a thick subcategory of $\mathcal{T}$. Assume the sequence $x \xrightarrow{\mathrm{inc}} \mathcal{T} \xrightarrow{Q} \mathcal{T}/x$ is a localization sequence. Denote by π (or ι) the right adjoint of the functor inc (or Q, respectively). Then the sequence $\mathcal{T}/x \xrightarrow{\iota} \mathcal{T} \xrightarrow{\pi} x$ is a colocalization sequence, see [22], Lemma 3.2. In particular, π induces a triangle equivalence

$$\mathcal{T}/\operatorname{Im} \iota \xrightarrow{\sim} x.$$

Note that a sequence $\mathcal{T}' \to \mathcal{T} \to \mathcal{T}''$ induces a recollement (see 2.3) if and only if the sequence is both a localization sequence and a colocalization sequence.

2.5. Compactly generated triangulated categories. Let $\mathcal{T}$ be a triangulated category with coproducts. An object $X \in \mathcal{T}$ is called *compact* provided that the Hom functor $\mathrm{Hom}_{\mathcal{T}}(X, -)$ commutes with coproducts. That is, for any class of objects $Y_i (i \in I)$ in $\mathcal{T}$, the canonical map

$$\mathrm{can}\colon \coprod_{i\in I} \mathrm{Hom}_{\mathcal{T}}(X, Y_i) \to \mathrm{Hom}_{\mathcal{T}}\left(X, \coprod_{i\in I} Y_i\right)$$

is an isomorphism. We let $\mathcal{T}^{\mathrm{c}}$ denote the full subcategory of $\mathcal{T}$ formed by compact objects in $\mathcal{T}$. It is not hard to show that $\mathcal{T}^{\mathrm{c}}$ is a thick subcategory of $\mathcal{T}$.

The category $\mathcal{T}$ is said to be *compactly generated* if there exists a set S of compact objects such that any object Y satisfying $\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^i(Y)) = 0$ for all $X \in S$ and $i \in \mathbb{Z}$ is a zero object; the condition is equivalent to the fact that $\mathcal{T}$ is equal to the smallest localizing subcategory containing S, see [27], Lemma 3.2. In this case, $\mathcal{T}^{\mathrm{c}} = \mathrm{thick}_{\mathcal{T}}(S)$, see [26], Lemma 2.2. For instance, $\mathsf{D}(R\text{-Mod})$ is compactly generated by the compact object R.

A set S of objects in $\mathcal{T}$ is called a *compact generating set* provided that $S \subseteq \mathcal{T}^{\mathrm{c}}$ and $\mathcal{T}$ is compactly generated by S. The following result is well-known. For its proof, we refer the reader to [6], Lemma 4.5; compare [3], Lemma 1 and [20], Lemma 4.2.

Lemma 2.1. *Let $F\colon \mathcal{T} \to \mathcal{T}'$ be an exact functor between compactly generated triangulated categories. Assume F preserves coproducts and $S \subseteq \mathcal{T}^{\mathrm{c}}$ is a compact generating set. Then F is fully faithful if and only if the induced maps*

$$\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^i(Y)) \to \mathrm{Hom}_{\mathcal{T}'}(FX, F\Sigma^i(Y))$$

are isomorphic for all $X, Y \in S$ and $i \in \mathbb{Z}$. In this case, F is dense if and only if $\mathrm{Im}\, F$ *contains a compact generating set of $\mathcal{T}'$.*

2.6. dg **categories and** dg **functors.** An additive category $\mathcal{A}$ is called a dg *category* provided that for each $X, Y \in \mathcal{A}$, the morphism space $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is a complex and the composition

$$\mathrm{Hom}_{\mathcal{A}}(Y, Z) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$$

is a chain map. An additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is called a dg *functor* provided that F commutes with the differential.

Let $\mathcal{A}$ be an additive category. Denote by $\mathsf{C}_{\mathrm{dg}}(\mathcal{A})$ the dg category of complexes over $\mathcal{A}$ whose morphism spaces are Hom complex defined by

$$\mathrm{Hom}_{\mathcal{A}}(X, Y)^i = \prod_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X^p, Y^{p+i})$$

with differential $\partial(f) = \partial_Y \circ f + (-1)^{|f|} f \circ \partial_X$.

The *homotopy category* $\mathrm{H}^0(\mathcal{A})$ of $\mathcal{A}$ is defined to be the category with same objects as $\mathcal{A}$ whose morphism spaces are the zeroth cohomology of the corresponding Hom complexes in $\mathcal{A}$. Observe that

$$\mathrm{H}^0(\mathsf{C}_{\mathrm{dg}}(\mathcal{A})) = \mathsf{K}(\mathcal{A}).$$

2.7. Derived categories of dg **categories.** We briefly discuss the derived category of a dg category, see [20] for more details.

Let $\mathcal{A}$ be a small dg category. A dg *module* over $\mathcal{A}$ is a dg functor

$$M\colon\ \mathcal{A}^{\mathrm{op}} \longrightarrow \mathsf{C}_{\mathrm{dg}}(\mathbb{Z}\text{-Mod}).$$

Then the category of dg $\mathcal{A}$-modules, denoted by $\mathrm{Mod}_{\mathrm{dg}}(\mathcal{A})$, is still a dg category, see [20], Section 1.2. Its homotopy category $\mathrm{H}^0(\mathrm{Mod}_{\mathrm{dg}}(\mathcal{A}))$ is a triangulated category, see [20], Lemma 2.2. A dg $\mathcal{A}$-module is called *acyclic* if $M(X)$ is acyclic for each object $X \in \mathcal{A}$. The *derived category* of $\mathcal{A}$ is defined to be the Verdier quotient of $\mathrm{H}^0(\mathrm{Mod}_{\mathrm{dg}}(\mathcal{A}))$ by its full subcategory of acyclic dg $\mathcal{A}$-modules. We have the Yoneda embedding

$$\mathrm{Y}\colon\ \mathrm{H}^0(\mathcal{A}) \to \mathsf{D}(\mathcal{A}), \quad X \mapsto \mathrm{Hom}_{\mathcal{A}}(-, X).$$

It is well-known that $\mathsf{D}(\mathcal{A})$ is compactly generated by the image of Y, see [20], Subsection 4.2.

2.8. Pretriangulated category. Keep the notation as above. The dg category $\mathcal{A}$ is called *pretriangulated* if $\mathrm{Im}\, Y$ is a triangulated category. In this case, $\mathrm{H}^0(\mathcal{A})$ inherits a natural triangulated structure and there is (up to direct summands) a triangle equivalence

$$\mathrm{H}^0(\mathcal{A}) \xrightarrow{\sim} \mathsf{D}(\mathcal{A})^{\mathrm{c}}.$$

2.9. dg **enhancement.** Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}$ be a dg category. The dg category $\mathcal{A}$ is said to be a *dg enhancement* of $\mathcal{T}$ provided that $\mathcal{A}$ is pretriangulated and $\mathcal{T}$ is triangle equivalent to $\mathrm{H}^0(\mathcal{A})$ endowed with the natural triangulated structure, see 2.8. In this case, any triangulated subcategory x of $\mathcal{T}$ has a dg enhancement. Indeed, denote by $\mathcal{A}'$ the full dg subcategory of $\mathcal{A}$ consisting of objects in the essential image of x. Then $\mathcal{A}'$ is a dg enhancement of x, see [19], Section 2.2.

Let $\mathcal{A}$ be an additive category. Then $\mathsf{C}_{\mathrm{dg}}(\mathcal{A})$ is pretriangulated and is a dg enhancement of $\mathsf{K}(\mathcal{A})$.

Example 2.1. By above, $\mathsf{C}_{\mathrm{dg}}(R\text{-Mod})$ (or $\mathsf{C}_{\mathrm{dg}}(R\text{-Inj})$) is a dg enhancement of $\mathsf{K}(R\text{-Mod})$ (or $\mathsf{K}(R\text{-Inj})$, respectively). Denote by $\mathrm{per}_{\mathrm{dg}}(R)$ the full dg subcategory of $\mathsf{C}_{\mathrm{dg}}(R\text{-Mod})$ consisting of all perfect complexes. Then $\mathrm{per}_{\mathrm{dg}}(R)$ is a dg enhancement of $\mathrm{per}(R)$.

Next we give an example that is used in Section 4. We write $\mathsf{C}_{\mathrm{dg}}^{+,f}(R\text{-Inj})$ to be the full subcategory of $\mathsf{C}_{\mathrm{dg}}(R\text{-Inj})$ formed by bounded below complexes whose total cohomology are finitely generated R-modules. Induced by taking injective resolution, there exists a triangle equivalence

$$\mathsf{D}^b(R\text{-mod}) \xrightarrow{\sim} \mathrm{H}^0(\mathsf{C}_{\mathrm{dg}}^{+,f}(R\text{-Inj})).$$

3. Triangulated categories of periodic complexes

Throughout the article, $n \geqslant 1$ is an integer. In this section we investigate periodic complexes. Remarkably, there exists an adjoint pair between the classical triangulated category and the corresponding triangulated category of periodic complexes. It is proved that many properties of the latter can be determined by the former. The main result in this section is Theorem 3.1.

Let $\mathcal{A}$ be an additive category, denote by $\mathsf{C}_n(\mathcal{A})$ the category of n-periodic complexes over $\mathcal{A}$ whose morphism spaces are n-periodic morphisms, see the introduction. For each $l \in \mathbb{Z}$, there is a canonical suspension functor Σ^l on $\mathsf{C}_n(\mathcal{A})$ which maps X to $\Sigma^l(X)$ ($\Sigma^l(X)^i := X^{i+l}$, $\partial^i_{\Sigma^l(X)} := (-1)^l\partial_X^{i+l}$) and acts trivially on morphisms.

3.1. Homotopy category of n-periodic complexes. Let $\mathcal{A}$ be an additive category and $X, Y \in \mathsf{C}_n(\mathcal{A})$. Two morphisms $f, g\colon X \to Y$ are called *homotopic* if there exists a sequence $\{\sigma^i\colon X^i \to Y^{i-1}\}_{i\in\mathbb{Z}}$ of morphisms over $\mathcal{A}$ such that $f^i - g^i = \sigma^{i+1} \circ \partial_X^i + \partial_Y^{i-1} \circ \sigma^i$ and $\sigma^i = \sigma^{i+n}$ for all $i \in \mathbb{Z}$.

The *homotopy category* of n-periodic complexes over $\mathcal{A}$, denoted by $\mathsf{K}_n(\mathcal{A})$, is defined by identifying homotopy in $\mathsf{C}_n(\mathcal{A})$. It is a triangulated category with suspension functor Σ, see [30], Section 7.

Let $f\colon X \to Y$ be a morphism in $\mathsf{C}_n(\mathcal{A})$. The *mapping cone* $C(f)$ of f is

$$C(f)^i := X^{i+1} \coprod Y^i, \quad \partial^i_{C(f)} := \begin{pmatrix} -\partial_X^{i+1} & 0 \\ f^{i+1} & \partial_Y^i \end{pmatrix}.$$

In $\mathsf{K}_n(\mathcal{A})$, f can be embedded in a canonical exact triangle

$$X \xrightarrow{f} Y \xrightarrow{\binom{0}{1}} C(f) \xrightarrow{(1\ 0)} \Sigma(X).$$

As Peng and Xiao [30], Subsection 7.1 mentioned, $\mathsf{C}_n(\mathcal{A})$ is a subcategory of $\mathsf{C}(\mathcal{A})$ (usually not full) and $\mathsf{K}_n(\mathcal{A})$ is usually not a subcategory of $\mathsf{K}(\mathcal{A})$.

3.2. Derived category of n-periodic complexes. Let $\mathcal{A}$ be an abelian category. An n-periodic complex X is called *acyclic* if it is acyclic as complex, i.e., $\mathrm{H}^i(X) := \mathrm{Ker}(\partial_X^i)/\mathrm{Im}(\partial_X^{i-1}) = 0$ for all $i \in \mathbb{Z}$. The *derived category* of n-periodic complexes over $\mathcal{A}$, denoted by $\mathsf{D}_n(\mathcal{A})$, is the Verdier quotient category of $\mathsf{K}_n(\mathcal{A})$ by its full subcategory of acyclic n-periodic complexes.

Following the definition of the compression for the case $n = 1$ in [1], Subsection 1.3, we define the compression for arbitrary $n \geqslant 1$, see also [35].

3.3. Compression. Let $\mathcal{A}$ be an additive category with coproducts. For a complex $X \in \mathsf{C}(\mathcal{A})$

$$\cdots \longrightarrow X^{i-1} \xrightarrow{\partial_X^{i-1}} X^i \xrightarrow{\partial_X^i} X^{i+1} \longrightarrow \cdots$$

The *compression* $\Delta(X)$ of X is defined by

$$\ldots \to \coprod_{j\equiv i-1 \pmod n} X^j \to \coprod_{j\equiv i \pmod n} X^j \to \coprod_{j\equiv i+1 \pmod n} X^j \to \ldots$$

with the natural differential induced by the differential of X, where the ith component of $\Delta(X)$ is $\coprod\limits_{j\equiv i \pmod n} X^j$. This gives an additive functor $\Delta\colon \mathsf{C}(\mathcal{A}) \to \mathsf{C}_n(\mathcal{A})$.

Clearly, there is a natural exact functor $\nabla\colon \mathsf{C}_n(\mathcal{A}) \to \mathsf{C}(\mathcal{A})$ which maps a periodic complex to itself. We observe that (Δ, ∇) is an adjoint pair. For each X in $\mathsf{C}(\mathcal{A})$, it is not hard to see there is an isomorphism $\nabla\Delta(X) \cong \coprod\limits_{i\in\mathbb{Z}} \Sigma^{ni}(X)$. Moreover, the unit $\eta_X\colon X \to \nabla\Delta(X)$ corresponding to the adjoint pair is the composition

$$X \xrightarrow{\mathrm{can}} \coprod_{i\in\mathbb{Z}} \Sigma^{ni}(X) \cong \nabla\Delta(X).$$

3.4. Keep the notation as in Subsection 3.3. One can check directly that Δ and ∇ preserve homotopy, suspensions and mapping cones. Hence, they induce an adjoint pair of exact functors between the homotopy categories

$$\mathsf{K}(\mathcal{A}) \underset{\nabla}{\overset{\Delta}{\rightleftarrows}} \mathsf{K}_n(\mathcal{A}).$$

If $\mathcal{A}$ is also abelian, we observe ∇ induces an exact functor $\nabla\colon \mathsf{D}_n(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$. If further $\mathcal{A}$ is an AB4 category (i.e., an abelian category with coproducts and the coproduct is an exact functor), then Δ preserves acyclic objects. Thus, Δ naturally induces an exact functor $\Delta\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}_n(\mathcal{A})$. Moreover, (Δ, ∇) is an adjoint between the derived categories, see [29], Lemma 1.1.

3.5. If $\mathcal{A}$ is an additive category with coproducts, then it can be checked directly that both $\mathsf{K}(\mathcal{A})$ and $\mathsf{K}_n(\mathcal{A})$ have coproducts. If $\mathcal{A}$ is an AB4 category, then both $\mathsf{D}(\mathcal{A})$ and $\mathsf{D}_n(\mathcal{A})$ have coproducts, see [25], Proposition 3.5.1.

In addition, in these cases, the degree-wise coproduct of objects in $\mathsf{K}(A)$ (or $\mathsf{K}_n(\mathcal{A})$, $\mathsf{D}(A)$, $\mathsf{D}_n(\mathcal{A})$) is the categorical coproduct.

Results similar to those of Subsection 3.5 hold when we replace the coproduct by the product and replace an AB4 category by an AB4* category (i.e., an abelian category with products and the product is an exact functor).

Lemma 3.1.

(1) *Let $\mathcal{A}$ be an additive category with coproducts and X be an object in $\mathsf{K}(\mathcal{A})$. Then X is compact in $\mathsf{K}(\mathcal{A})$ if and only if $\Delta(X)$ is compact in $\mathsf{K}_n(\mathcal{A})$.*

(2) *Let $\mathcal{A}$ be an AB4 category and X be an object in $\mathsf{D}(\mathcal{A})$. Then X is compact in $\mathsf{D}(\mathcal{A})$ if and only if $\Delta(X)$ is compact in $\mathsf{D}_n(\mathcal{A})$.*

P r o o f. We prove (1). The proof of (2) is similar. First, assume X is compact in $\mathsf{K}(\mathcal{A})$. Since (Δ, ∇) is an adjoint pair and ∇ preserves coproducts (cf. 3.5), Δ preserves compact objects, see [27], Theorem 5.1. Thus, $\Delta(X)$ is compact in $\mathsf{K}_n(\mathcal{A})$.

For the converse, assume $\Delta(X)$ is compact in $\mathsf{K}_n(\mathcal{A})$. For a class of objects Y_i $(i \in I)$ in $\mathsf{K}(\mathcal{A})$, consider the commutative diagram

$$\begin{array}{ccc}
\coprod_{i\in I} \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, Y_i) & \xrightarrow{\ \mathrm{can}\ } & \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}\Big(X, \coprod_{i\in I} Y_i\Big) \\
\downarrow{\scriptstyle \coprod_{i\in I}(\eta_{Y_i})_*} & & \downarrow{\scriptstyle (\eta_{\coprod_{i\in I} Y_i})_*} \\
\coprod_{i\in I} \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, \nabla\Delta(Y_i)) & & \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}\Big(X, \nabla\Delta\Big(\coprod_{i\in I} Y_i\Big)\Big) \\
\uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \cong} \\
\coprod_{i\in I} \mathrm{Hom}_{\mathsf{K}_n(\mathcal{A})}(\Delta(X), \Delta(Y_i)) & \xrightarrow{\ \cong\ } & \mathrm{Hom}_{\mathsf{K}_n(\mathcal{A})}\Big(\Delta(X), \Delta\Big(\coprod_{i\in I} Y_i\Big)\Big)
\end{array}$$

where the vertical isomorphisms are induced by the adjoint pair (Δ, ∇) and the horizontal one is based on the assumption. Since the unit $\eta_M \colon M \to \nabla\Delta(M)$ is split injection for each $M \in \mathsf{K}(\mathcal{A})$ (see Subsection 3.3), we conclude that can is an isomorphism. □

Example 3.1. Let $n = 1$. Following [1], a differential R-module (P, δ_P) admits a *finite projective flag* if $P = P_0 \coprod P_1 \coprod \ldots \coprod P_l$ and δ_P is of the form

$$\begin{pmatrix} 0 & \partial_{1,0} & \partial_{2,0} & \ldots & \partial_{l-1,0} & \partial_{l,0} \\ 0 & 0 & \partial_{2,1} & \ldots & \partial_{l-1,1} & \partial_{l,1} \\ 0 & 0 & 0 & \ldots & \partial_{l-1,2} & \partial_{l,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & \partial_{l,l-1} \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix},$$

where each P_i is a finitely generated projective R-module. Set $F^i = (P_0 \coprod \ldots \coprod P_i, \delta_P)$ $(0 \leqslant i \leqslant l)$. These are differential submodules of (P, δ_P). It follows that (P, δ_P) has a filtration

$$F^0 \subseteq F^1 \subseteq \ldots \subseteq F^l = (P, \delta_P)$$

such that $F^i/F^{i-1} \cong (P_i, 0)$ for each i. Since $\Delta(R) = (R, 0) \in \mathsf{D}_1(R\text{-Mod})^{\mathrm{c}}$ (see Lemma 3.1), the differential modules that admit finite projective flags are compact objects in $\mathsf{D}_1(R\text{-Mod})$.

If $\mathcal{A}$ is an abelian category, then an object X in $\mathsf{D}_n(\mathcal{A})$ is zero if and only if $\nabla(X)$ is zero in $\mathsf{D}(\mathcal{A})$. Similar result holds in the homotopy category, see the next lemma.

Lemma 3.2. *Let $\mathcal{A}$ be an additive category and X be an object in $\mathsf{K}_n(\mathcal{A})$. Then X is zero in $\mathsf{K}_n(\mathcal{A})$ if and only if $\nabla(X)$ is zero in $\mathsf{K}(\mathcal{A})$.*

P r o o f. The forward direction is trivial. For the converse, assume $\nabla(X)$ is zero in $\mathsf{K}(\mathcal{A})$. Then there exists $s^i \in \mathrm{Hom}_{\mathcal{A}}(X^i, X^{i-1})$ for all $i \in \mathbb{Z}$ such that

$$\mathrm{id}_{X^i} = s^{i+1} \circ \partial_X^i + \partial_X^{i-1} \circ s^i. \tag{3.1}$$

We define $\sigma^i \colon X^i \to X^{i-1}$ as

$$\sigma^i = \begin{cases} s^n \circ \partial_X^{-1} \circ s^0 & \text{if } i \equiv 0 \pmod{n}, \\ s^j & \text{if } i \equiv j \pmod{n} \text{ and } 1 \leqslant j \leqslant n-1. \end{cases}$$

Our aim is to show that this gives the homotopy map from id_X to 0 in $\mathsf{K}_n(\mathcal{A})$. Due to the choice of σ^i, it remains to check that $\mathrm{id}_{X^0} = \sigma^1 \circ \partial_X^0 + \partial_X^{-1} \circ \sigma^0$ and $\mathrm{id}_{X^{n-1}} = \sigma^0 \circ \partial_X^{n-1} + \partial_X^{n-2} \circ \sigma^{n-1}$. Indeed, these are direct consequences of (3.1). Thus, X is zero in $\mathsf{K}_n(\mathcal{A})$. □

Lemma 3.3.

(1) *Let $\mathcal{A}$ be an additive category with coproducts. If $\mathsf{K}(\mathcal{A})$ is compactly generated, then so is $\mathsf{K}_n(\mathcal{A})$ and it is compactly generated by the image of $\mathsf{K}(\mathcal{A})^{\mathrm{c}}$ under the compression functor.*

(2) *Let $\mathcal{A}$ be an AB4 category. If $\mathsf{D}(\mathcal{A})$ is compactly generated, then so is $\mathsf{D}_n(\mathcal{A})$ and it is compactly generated by the image of $\mathsf{D}(\mathcal{A})^{\mathrm{c}}$ under the compression functor.*

P r o o f. We prove (1). The proof of (2) is similar. Suppose $\mathsf{K}(\mathcal{A})$ is compactly generated. Lemma 3.1 yields $\Delta(\mathsf{K}(\mathcal{A})^{\mathrm{c}}) \subseteq \mathsf{K}_n(\mathcal{A})^{\mathrm{c}}$. Let $X \in \mathsf{K}_n(\mathcal{A})$ and

$$\mathrm{Hom}_{\mathsf{K}_n(\mathcal{A})}(\Delta(\mathsf{K}(\mathcal{A})^{\mathrm{c}}), X) = 0.$$

In order to show $\mathsf{K}_n(\mathcal{A})$ is compactly generated by $\Delta(\mathsf{K}(\mathcal{A})^{\mathrm{c}})$, we need to prove $X = 0$ in $\mathsf{K}_n(\mathcal{A})$. By the adjoint pair (Δ, ∇) we have

$$\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(\mathsf{K}(\mathcal{A})^{\mathrm{c}}, \nabla(X)) = 0.$$

Then the assumption implies $\nabla(X) = 0$. It follows immediately from Lemma 3.2 that $X = 0$ in $\mathsf{K}_n(\mathcal{A})$. The proof has been completed. □

The following result is due to Neeman, see [27], Theorem 4.1 and [28], Theorem 8.6.1.

3.6. Let x be a compactly generated triangulated category and $F\colon x \to \mathcal{T}$ be an exact functor between triangulated categories. Then

(1) F has a right adjoint if and only if F preserves coproducts.

(2) F has a left adjoint if and only if F preserves products.

As we assume R is a left noetherian ring, the direct sum of injective R-modules is still injective, see [15], Theorem 3.1.17. Hence, R-Inj is an additive category with coproducts.

3.7. Krause proved that $\mathsf{K}(R\text{-Inj})$ and the full subcategory of $\mathsf{K}(R\text{-Inj})$ formed by acyclic complexes (denoted by $\mathsf{K}^{\mathrm{ac}}(R\text{-Inj})$) are compactly generated triangulated categories, see [24], Proposition 2.3, Corollary 5.4. Moreover, Krause in Corollary 4.3 of [24] observed that the canonical sequence

$$\mathsf{K}^{\mathrm{ac}}(R\text{-Inj}) \xrightarrow{\mathrm{inc}} \mathsf{K}(R\text{-Inj}) \xrightarrow{Q} \mathsf{D}(R\text{-Mod})$$

induces a recollement; the definition of recollement is recalled in Subsection 2.3.

Let $\mathsf{K}_n^{\mathrm{ac}}(R\text{-Inj})$ denote the full subcategory of $\mathsf{K}_n(R\text{-Inj})$ formed by acyclic complexes. This is a localizing subcategory of $\mathsf{K}(R\text{-Inj})$. Next, we give the periodic version of Krause's result in Theorem 3.1.

Theorem 3.1. *Let R be a left noetherian ring. Then*

(1) $\mathsf{K}_n^{\mathrm{ac}}(R\text{-Inj}), \mathsf{K}_n(R\text{-Inj})$ *and* $\mathsf{D}_n(R\text{-Mod})$ *are compactly generated triangulated categories.*

(2) *The sequence*

$$\mathsf{K}_n^{\mathrm{ac}}(R\text{-Inj}) \xrightarrow{\mathrm{inc}} \mathsf{K}_n(R\text{-Inj}) \xrightarrow{Q} \mathsf{D}_n(R\text{-Mod})$$

induces a recollement

$$\mathsf{K}_n^{\mathrm{ac}}(R\text{-Inj}) \xrightarrow{\mathrm{inc}} \mathsf{K}_n(R\text{-Inj}) \xrightarrow{Q} \mathsf{D}_n(R\text{-Mod}) \ .$$

P r o o f. (1) Combining with Subsection 3.7, it follows immediately from Lemma 3.3 that $\mathsf{K}_n(R\text{-Inj})$ and $\mathsf{D}_n(R\text{-Mod})$ are compactly generated. Also, with the same proof of Lemma 3.3, $\mathsf{K}_n^{\mathrm{ac}}(R\text{-Inj})$ is compactly generated.

(2) Next, we borrow Krause's idea in the proof of Corollary 4.3 of [24]. Since Q preserves coproducts and products (cf. Subsection 3.5), Q has both a left adjoint and a right adjoint by (1) and Subsection 3.6. Combining with Subsection 2.4, it remains to show Q induces a triangle equivalence $\mathsf{K}_n(R\text{-Inj})/\mathsf{K}_n^{\mathrm{ac}}(R\text{-Inj}) \cong \mathsf{D}_n(R\text{-Mod})$. Again Subsection 2.4 yields this is equivalent to show the right adjoint of Q is fully faithful.

Denote by Q_ϱ the right adjoint of Q. It is clear that the inclusion functor $J\colon \mathsf{K}_n(R\text{-Inj}) \to \mathsf{K}_n(R\text{-Mod})$ preserves products. Then Lemmata 3.3 and 3.6 imply that J has a left adjoint J_λ. Hence, there are adjoint pairs

$$\mathsf{K}_n(R\text{-Mod}) \underset{J}{\overset{J_\lambda}{\rightleftarrows}} \mathsf{K}_n(R\text{-Inj}) \underset{Q_\varrho}{\overset{Q}{\rightleftarrows}} \mathsf{D}_n(R\text{-Mod}).$$

Since J is a fully faithful right adjoint of J_λ, $\mathrm{Hom}_{\mathsf{K}_n(R\text{-Mod})}(\mathrm{Ker}\, J_\lambda, \mathsf{K}_n(R\text{-Inj})) = 0$. This implies $\mathrm{Ker}\, J_\lambda \subseteq \mathsf{K}_n^{\mathrm{ac}}(R\text{-Mod})$. Thus, for each $M \in \mathsf{K}_n(R\text{-Mod})$, the unit $\eta_M\colon M \to J_\lambda(M)$ is a quasi-isomorphism. It follows that $Q(M) \cong (Q \circ J_\lambda)(M)$. That is, $Q \circ J_\lambda$ is isomorphic to the localization functor $\mathsf{K}_n(R\text{-Mod}) \to \mathsf{D}_n(R\text{-Mod})$. By Subsection 2.4, $J \circ Q_\varrho$ is fully faithful. As J is also fully faithful, we infer that Q_ϱ is too. This completes the proof. □

3.8. Let $\mathcal{A}$ be an abelian category. An n-periodic complex $X \in \mathsf{K}_n(\mathcal{A})$ is called *homotopy injective* (or *homotopy projective*) if

$$\mathrm{Hom}_{\mathsf{K}_n(\mathcal{A})}(Y, X) = 0 \quad (\text{or } \mathrm{Hom}_{\mathsf{K}_n(\mathcal{A})}(X, Y) = 0, \text{ respectively})$$

for each acyclic complex $Y \in \mathsf{K}_n(\mathcal{A})$. Denote by $\mathsf{K}_n^{\mathsf{i}}(\mathcal{A})$ (or $\mathsf{K}_n^{\mathsf{p}}(\mathcal{A})$) the full subcategory of $\mathsf{K}_n(\mathcal{A})$ consisting of all homotopy injective (or homotopy projective, respectively) complexes. They naturally inherit the structure of triangulated categories.

Let Q_ϱ denote the right adjoint of $Q\colon \mathsf{K}_n(R\text{-Inj}) \to \mathsf{D}_n(R\text{-Mod})$. Using the adjointness, it is easy to check $Q_\varrho(X)$ is homotopy injective for each n-periodic complex X and the unit $X \to Q_\varrho(X)$ is a quasi-isomorphism. Thus, we get:

Corollary 3.1. *Q_ϱ induces a triangle equivalence*

$$\mathsf{D}_n(R\text{-Mod}) \xrightarrow{\sim} \mathsf{K}^{\mathsf{i}}_n(R\text{-Mod}).$$

Remark 3.1.

(1) Tang and Huang in Theorem 5.11 of [36] proved an analog of the above result for higher differential objects. The two results coincide when $n = 1$.

(2) Stai in Section 3 of [35] obtained the dual version of the above result. That is, the localization functor $Q\colon \mathsf{K}_n(R\text{-Mod}) \to \mathsf{D}_n(R\text{-Mod})$ has a left adjoint and the left adjoint induces a triangle equivalence

$$\mathsf{D}_n(R\text{-Mod}) \xrightarrow{\sim} \mathsf{K}^{\mathsf{p}}_n(R\text{-Mod}).$$

4. The triangulated hull of the orbit categories

In this section, the main result is Theorem 4.1.

4.1. Let $\mathcal{A}$ be an additive category and $T\colon \mathcal{A} \to \mathcal{A}$ be an autoequivalence. As mentioned in the introduction, the objects in the same T-orbit are isomorphic in the orbit category $\mathcal{A}/T$. We remind the reader that, in general, F is not isomorphic to the identity functor in the orbit category $\mathcal{A}/T$, see [23] and [35], Proposition 5.6. However, there is a natural isomorphism $\pi \cong \pi \circ T$, where $\pi\colon \mathcal{A} \to \mathcal{A}/T$ is the projection functor. Moreover, this gives rise to the universal property of the orbit category:

If the functor $F\colon \mathcal{A} \to \mathcal{B}$ satisfies $F \circ T \cong F$, then there exists a natural functor $\overline{F}\colon \mathcal{A}/T \to \mathcal{B}$ such that $\overline{F} \circ \pi = F$.

4.2. Let $\mathcal{A}$ be an additive category. Recall the degree shift functor (n) on $\mathsf{C}(\mathcal{A})$: for a complex X, $X(n)^i := X^{i+n}$, $\partial^i_{X(n)} := \partial^{i+n}_X$; (n) acts trivially on morphisms. There is a natural isomorphism $\Sigma^n(X) \xrightarrow{\cong} X(n)$ which maps $x \in X^i$ to $(-1)^{ni}x$.

If further $\mathcal{A}$ is an additive category with coproducts (or AB4 category), then $\Delta \circ \Sigma^n \cong \Delta \circ (n) = \Delta$. By the universal property of the orbit category, Δ induces

$$\overline{\Delta}\colon \mathsf{K}(\mathcal{A})^{\mathrm{c}}/\Sigma^n \longrightarrow \mathsf{K}_n(\mathcal{A})^{\mathrm{c}} \text{ (or } \overline{\Delta}\colon \mathsf{D}(\mathcal{A})^{\mathrm{c}}/\Sigma^n \longrightarrow \mathsf{D}_n(\mathcal{A})^{\mathrm{c}}, \text{ respectively}).$$

We first strengthen Lemma 3.3 to the following result, compare [35], Lemma 3.13.

Proposition 4.1.

(1) *Let $\mathcal{A}$ be an additive category with coproducts. If $\mathsf{K}(\mathcal{A})$ is compactly generated, then there is a fully faithful embedding*

$$\overline{\Delta}\colon\ \mathsf{K}(\mathcal{A})^{\mathrm{c}}/\Sigma^n \longrightarrow \mathsf{K}_n(\mathcal{A})^{\mathrm{c}}$$

and $\mathsf{K}_n(\mathcal{A})$ is compactly generated by its image.

(2) *Let $\mathcal{A}$ be an AB4 category. If $\mathsf{D}(\mathcal{A})$ is compactly generated, then there is a fully faithful embedding*

$$\overline{\Delta}\colon\ \mathsf{D}(\mathcal{A})^{\mathrm{c}}/\Sigma^n \longrightarrow \mathsf{D}_n(\mathcal{A})^{\mathrm{c}}$$

and $\mathsf{D}_n(\mathcal{A})$ is compactly generated by its image.

P r o o f. We prove (1). The proof of (2) is similar. By Lemma 3.3, it remains to show $\overline{\Delta}$ is fully faithful. For $X, Y \in \mathsf{K}(\mathcal{A})^{\mathrm{c}}$, we have

$$\mathrm{Hom}_{\mathsf{K}_n(\mathcal{A})}(\Delta(X), \Delta(Y)) \cong \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, \nabla\Delta(Y)) \cong \coprod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, \Sigma^{ni}(Y)),$$

where the second isomorphism holds because X is compact and $\nabla\Delta(Y) \cong \coprod\limits_{i\in\mathbb{Z}} \Sigma^{ni}(Y)$, see Subsection 3.3. It follows immediately from the isomorphism above that the induced functor $\overline{\Delta}\colon\ \mathsf{K}(\mathcal{A})^{\mathrm{c}}/\Sigma^n \to \mathsf{K}_n(\mathcal{A})^{\mathrm{c}}$ is fully faithful. □

4.3. A homogeneous morphism f in a dg category is called *closed* if f is of degree 0 and $\partial(f) = 0$. We call a natural transformation η between dg functors *closed* if η_X is closed for all $X \in \mathcal{A}$.

The following is the universal property of the orbit category of the dg category.

Lemma 4.1. *Let $T\colon\ \mathcal{A} \to \mathcal{A}$ be a dg autoequivalence of a dg category $\mathcal{A}$ and $F\colon\ \mathcal{A} \to \mathcal{B}$ be a dg functor between dg categories such that there exists a closed natural isomorphism $\eta\colon\ F \circ T \to F$. Then F induces a dg functor $\overline{F}\colon\ \mathcal{A}/T \to \mathcal{B}$ such that $\overline{F} \circ \pi = F$, where $\pi\colon\ \mathcal{A} \to \mathcal{A}/T$ is the projection functor.*

P r o o f. By assumption, η induces closed isomorphisms $F \circ T^i \xrightarrow{\sim} F$ (denoted by η^i). We define $\overline{F}\colon\ \mathcal{A}/T \to \mathcal{B}$ as $\overline{F}(M) = F(M)$ for all $M \in \mathcal{A}$; for each homogeneous morphism $\alpha\colon\ M \to T^j(N)$ in $\mathrm{Hom}_{\mathcal{A}/T}(M, N)$, $\overline{F}(\alpha)$ is defined by the composition

$$F(M) \xrightarrow{F(\alpha)} F(T^j(N)) \xrightarrow[\cong]{\eta^j_N} F(N).$$

By assumption, F is a dg functor and η is closed, we have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{A}}(X, T^j(Y)) & \xrightarrow{F} & \mathrm{Hom}_{\mathcal{B}}(F(X), F(T^j(Y))) & \xrightarrow{(\eta_Y^j)_*} & \mathrm{Hom}_{\mathcal{B}}(F(X), F(Y)) \\
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\
\mathrm{Hom}_{\mathcal{A}}(X, T^j(Y)) & \xrightarrow{F} & \mathrm{Hom}_{\mathcal{B}}(F(X), F(T^j(Y))) & \xrightarrow{(\eta_Y^j)_*} & \mathrm{Hom}_{\mathcal{B}}(F(X), F(Y)).
\end{array}$$

This means $\overline{F}$ is a dg functor. Clearly $\overline{F} \circ \pi = F$. □

Example 4.1. Let $\mathcal{C}$ be an additive category. Set $\mathcal{A} = \mathsf{C}_{\mathrm{dg}}(\mathcal{C})$ and $\mathcal{B} = \mathsf{C}_{\mathrm{dg}}(\mathbb{Z})$. The suspension functor $\Sigma^n \colon \mathsf{C}_{\mathrm{dg}}(\mathcal{C}) \to \mathsf{C}_{\mathrm{dg}}(\mathcal{C})$ is a dg autoequivalence. If X is an n-periodic complex in $\mathcal{A}$, then

$$\mathrm{Hom}_{\mathcal{A}}(\Sigma^n(Y), X) \cong \mathrm{Hom}_{\mathcal{A}}(Y(n), X) \cong \mathrm{Hom}_{\mathcal{A}}(Y, X),$$

where the first isomorphism is induced by $\Sigma^n(Y) \cong Y(n)$ and the second one maps $\alpha \colon Y(n)^i \to X^j$ to $\alpha \colon Y^{n+i} \to X^{n+j}$. Set $F = \mathrm{Hom}_{\mathcal{A}}(-, X)$ and $T = \Sigma^n$. We conclude that there is a closed natural isomorphism $F \circ T \cong F$. This is an example that satisfies the assumption of Lemma 4.1.

4.4. Let R be a left noetherian ring. Krause in Proposition 2.3 of [24] proved that $\mathsf{K}(R\text{-Inj})$ is compactly generated. Moreover, he observed that the localization functor $\mathsf{K}(R\text{-Mod}) \to \mathsf{D}(R\text{-Mod})$ induces the triangle equivalence

$$\mathsf{K}(R\text{-Inj})^{\mathrm{c}} \xrightarrow{\sim} \mathsf{D}^b(R\text{-mod}).$$

The inverse is induced by taking injective resolution. In particular, $\mathsf{K}(R\text{-Inj})^{\mathrm{c}}$ is the full subcategory of $\mathsf{K}(R\text{-Inj})$ consisting of complexes with finitely generated total cohomology.

Recall that $\mathrm{per}_{\mathrm{dg}}(R)$ is the dg category of perfect complexes over R and $\mathsf{C}_{\mathrm{dg}}^{+,f}(R\text{-Inj})$ is the dg category of bounded below complexes of injective R-modules with finitely generated total cohomology. They are dg enhancements of $\mathrm{per}(R)$ and $\mathsf{D}^b(R\text{-mod})$, respectively, see Subsection 4.4 and Example 2.1.

Next, we realize examples of triangulated categories in Theorem 3.1 as derived categories of dg categories, compare [12], Theorem 2.2 and [24], Appendix A.

Theorem 4.1. *Let R be a left noetherian ring. There are triangle equivalences*

$$\mathsf{K}_n(R\text{-Inj}) \xrightarrow{\sim} \mathsf{D}(\mathsf{C}_{\mathrm{dg}}^{+,f}(R\text{-Inj})/\Sigma^n) \quad \textit{and} \quad \mathsf{D}_n(R\text{-Mod}) \xrightarrow{\sim} \mathsf{D}(\mathrm{per}_{\mathrm{dg}}(R)/\Sigma^n).$$

P r o o f. We prove the first equivalence. The proof of the second one is similar. For each complex X of R-modules, set $X^\wedge = \mathrm{Hom}_R(-, X)$. By Lemma 4.1 and Example 4.1, the map $I \mapsto \overline{I^\wedge}\,|_{\mathsf{C}^{+,f}_{\mathrm{dg}}(R\text{-Inj})/\Sigma^n}$ induces an exact functor

$$\Phi\colon\ \mathsf{K}_n(R\text{-Inj}) \to \mathsf{D}(\mathsf{C}^{+,f}_{\mathrm{dg}}(R\text{-Inj})/\Sigma^n).$$

The functor Φ preserves coproducts. Indeed, for each object $J \in \mathsf{C}^{+,f}_{\mathrm{dg}}(R\text{-Inj})$ and a family $I_i \in \mathsf{K}_n(R\text{-Inj})$ $(i \in S)$, we have isomorphisms

$$\begin{aligned}\mathrm{H}^l\Big(\Phi\Big(\coprod_{i\in S} I_i\Big)(J)\Big) &\cong \mathrm{Hom}_{\mathsf{K}(R\text{-Inj})}\Big(\Sigma^{-l}(J), \coprod_{i\in S} I_i\Big) \cong \coprod_{i\in S} \mathrm{Hom}_{\mathsf{K}(R\text{-Inj})}(\Sigma^{-l}(J), I_i)\\ &\cong \coprod_{i\in S} \mathrm{H}^l(\Phi(I_i)(J)) \cong \mathrm{H}^l\Big(\coprod_{i\in S} \Phi(I_i)(J)\Big)\end{aligned}$$

for each $l \in \mathbb{Z}$, where the second isomorphism holds because J is a compact object in $\mathsf{K}(R\text{-Inj})$, see Subsection 4.4. Hence, in $\mathsf{D}(\mathsf{C}^{+,f}_{\mathrm{dg}}(R\text{-Inj})/\Sigma^n)$,

$$\Phi\Big(\coprod_{i\in S} I_i\Big) \cong \coprod_{i\in S} \Phi(I_i).$$

We observe that there exists a commutative diagram

$$\begin{array}{ccc}\mathrm{H}^0(\mathsf{C}^{+,f}_{\mathrm{dg}}(R\text{-Inj}/\Sigma^n) & \xrightarrow{\overline{\Delta}} & \mathsf{K}_n(R\text{-Inj})^{\mathrm{c}}\\ \Big\downarrow{\scriptstyle \mathrm{Y}} & & \Big\downarrow{\scriptstyle \mathrm{inc}}\\ \mathsf{D}(\mathsf{C}^{+,f}_{\mathrm{dg}}(R\text{-Inj})/\Sigma^n) & \xleftarrow{\Phi} & \mathsf{K}_n(R\text{-Inj}),\end{array} \tag{4.1}$$

where Y is the Yoneda embedding. From Proposition 4.1,

$$\overline{\Delta}\colon\ \mathrm{H}^0(\mathsf{C}^{+,f}_{\mathrm{dg}}(R\text{-Inj})/\Sigma^n) \longrightarrow \mathsf{K}_n(R\text{-Inj})^{\mathrm{c}}$$

is fully faithful and $\mathsf{K}_n(R\text{-Inj})$ is compactly generated by the image of $\overline{\Delta}$. As $\mathsf{D}(\mathsf{C}^{+,f}_{\mathrm{dg}}(R\text{-Inj})/\Sigma^n)$ is compactly generated by the image of Y, we conclude that Φ is an equivalence by Lemma 2.1. □

Remark 4.1.

(1) Let k be a field. When R is a finite dimensional k-algebra with finite global dimension, the triangle equivalence $\mathsf{D}_n(R\text{-Mod}) \xrightarrow{\sim} \mathsf{D}(\mathrm{per}_{\mathrm{dg}}(R)/\Sigma^n)$ was proved by Stai with a different method, see [35], Section 4.

(2) Let $\mathcal{B}$ (or $\mathcal{A}$) denote $\mathrm{per}_{\mathrm{dg}}(R)/\Sigma^n$ (or $\mathsf{C}^{+,f}_{\mathrm{dg}}(R\text{-Inj})/\Sigma^n$, respectively). We can regard $\mathcal{B}$ as a full dg subcategory of $\mathcal{A}$, see Subsection 4.4. Then we can form a dg quotient category $\mathcal{A}/\mathcal{B}$, see Keller's construction in [21], Section 4. The restriction functor $\mathsf{D}(\mathcal{A}/\mathcal{B}) \to \mathsf{D}(\mathcal{A})$ is fully faithful and its essential image is equal to the kernel of the restriction functor $\mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$, see [21], Section 4 and [13], Proposition 4.6. Combining this with Theorems 3.1 and 4.1, we conclude that there is a triangle equivalence

$$\mathsf{K}^{\mathrm{ac}}_n(R\text{-Inj}) \simeq \mathsf{D}(\mathcal{A}/\mathcal{B}).$$

4.5. Let $\mathcal{A}$ be a dg enhancement of a triangulated category $\mathcal{T}$. Assume the functor $F\colon\ \mathcal{T} \to \mathcal{T}$ is an autoequivalence and it lifts to a dg equivalence $\mathcal{A} \to \mathcal{A}$ (still denoted by F). Then we can form an orbit category $\mathcal{A}/F$ which naturally inherits a structure of the dg category and gives the desired enhancement of $\mathcal{T}/F$. Hence,

$$\mathcal{T}/F \xrightarrow{\sim} \mathrm{H}^0(\mathcal{A}/F) \xrightarrow{\mathrm{Y}} \mathsf{D}(\mathcal{A}/F),$$

where Y is the Yoneda embedding. The *triangulated hull* of $\mathcal{T}/F$ is chosen to be the triangulated subcategory of $\mathsf{D}(\mathcal{A}/F)$ generated by the image of Y. It is up to direct summands equivalent to $\mathsf{D}(\mathcal{A}/F)^{\mathrm{c}}$. Thus, we use $\mathsf{D}(\mathcal{A}/F)^{\mathrm{c}}$ to represent the triangulated hull of $\mathcal{T}/F$ in the article, see Keller's definition in [22], Section 5 for a broader definition of the triangulated hull.

The inverse of the equivalence $\mathsf{K}(R\text{-Inj})^{\mathrm{c}} \xrightarrow{\sim} \mathsf{D}^b(R\text{-mod})$ (see Subsection 4.4) is induced by taking injective resolution. We denote it by i.

Corollary 4.1. *Let R be a left noetherian ring. Compression of complexes induces functors*

$$\overline{\Delta \circ \mathsf{i}}\colon\ \mathsf{D}^b(R\text{-mod})/\Sigma^n \longrightarrow \mathsf{K}_n(R\text{-Inj})^{\mathrm{c}} \quad \textit{and} \quad \overline{\Delta}\colon\ \mathrm{per}(R)/\Sigma^n \longrightarrow \mathsf{D}_n(R\text{-Mod})^{\mathrm{c}}$$

and these yield embeddings of the orbit categories into their triangulated hull.

P r o o f. For the first one: $\mathsf{C}^{+,f}_{\mathrm{dg}}(R\text{-Inj})$ is the dg enhancement of $\mathsf{D}^b(R\text{-mod})$. Then $\mathsf{C}^{+,f}_{\mathrm{dg}}(R\text{-Inj})/\Sigma^n$ is the desired dg enhancement of $\mathsf{D}^b(R\text{-mod})/\Sigma^n$. Given this and Subsection 4.5, the desired result follows from Theorem 4.1.

For the second one: $\mathrm{per}_{\mathrm{dg}}(R)$ is the dg enhancement of $\mathrm{per}(R)$. Then the remaining proof is parallel to the first one. □

Remark 4.2. Fix a *locally noetherian* Grothendieck category $\mathcal{A}$. That is, $\mathcal{A}$ is an AB4 category with exact direct colimit, and $\mathcal{A}$ has a set $\mathcal{A}_0$ of noetherian objects such that every object in $\mathcal{A}$ is a quotient of a coproduct of objects in $\mathcal{A}_0$. Denote by $\mathcal{A}$-noeth the full subcategory of $\mathcal{A}$ formed by noetherian objects, and by $\mathcal{A}$-Inj the full subcategory of $\mathcal{A}$ formed by injective objects. With the same argument of Theorem 4.1, we have $\mathsf{K}_n(\mathcal{A}\text{-Inj}) \xrightarrow{\sim} \mathsf{D}(\mathsf{C}_{\mathrm{dg}}^{+,f}(\mathcal{A}\text{-Inj})/\Sigma^n)$, where $\mathsf{C}_{\mathrm{dg}}^{+,f}(\mathcal{A}\text{-Inj})$ is the dg category of bounded below complexes of injective objects with noetherian total cohomology. Then the same proof of Corollary 4.1 yields that the compression of complexes

$$\overline{\Delta \circ \mathsf{i}}\colon\ \mathsf{D}^b(\mathcal{A}\text{-noeth})/\Sigma^n \longrightarrow \mathsf{K}_n(\mathcal{A}\text{-Inj})^{\mathrm{c}}$$

induces an embedding of the orbit category into its triangulated hull.

4.6. It was proved by Stai [35], Lemma 3.5 that when R has finite global dimension, then every object in $\mathsf{C}_1(R\text{-mod})$ is quasi-isomorphic to one admitting a finite projective flag (see the definition in Example 3.1). Thus, $\mathsf{D}_1(R\text{-mod})$ is equal to the thick subcategory generated by $\Delta(R)$. As he also mentioned, this extends to any $n \geqslant 1$. Combining with Proposition 4.1, there is a natural triangle equivalence

$$\mathsf{D}_n(R\text{-mod}) \xrightarrow{\sim} \mathsf{D}_n(R\text{-Mod})^{\mathrm{c}}.$$

With the same method of Stai, one can show: if $\mathcal{A}$ is an abelian category with enough projective objects and every object in $\mathcal{A}$ has finite projective dimension, then

$$\mathsf{D}_n(\mathcal{A}) = \mathrm{thick}_{\mathsf{D}_n(\mathcal{A})}(\{\Delta(P)\colon\ P \text{ is projective in } \mathcal{A}\}).$$

The following result was proved independently by Stai (see [35], Theorem 4.3) and Zhao (see [37], Theorem 2.10) when R is a finite dimensional algebra with finite global dimension over a field.

Corollary 4.2. *Let R be a left noetherian ring with finite global dimension, then the compression of complexes*

$$\overline{\Delta}\colon\ \mathsf{D}^b(R\text{-mod})/\Sigma^n \longrightarrow \mathsf{D}_n(R\text{-mod})$$

is an embedding of the orbit category into its triangulated hull.

P r o o f. When R has finite global dimension, $\mathsf{D}^b(R\text{-mod}) = \mathrm{per}(R)$. Combining with Subsection 4.6, the desired result follows from Corollary 4.1. □

4.7. When R is hereditary, $\mathsf{D}^b(R\text{-mod})/\Sigma^n$ is triangulated and hence, it is (up to direct summands) equivalent to its triangulated hull, see [22], Theorem 1 and [35], Proposition 5.3. When $n = 1$ and R is a path algebra of finite connected acyclic quiver, Ringel and Zhang in Theorem 1 of [34] proved that $\mathsf{D}^b(R\text{-mod})/\Sigma$ is equivalent to a stable category of certain Frobenius category.

5. Derived equivalence as derived tensor product

For two rings A and B, the purpose of this section is to compare the triangle equivalences $\mathsf{D}(A\text{-Mod}) \simeq \mathsf{D}(B\text{-Mod})$ and $\mathsf{D}_n(A\text{-Mod}) \simeq \mathsf{D}_n(B\text{-Mod})$. It turns out that these two equivalences are closely related, see Theorem 5.1.

5.1. Tensor products. Let X be a complex of B-A bimodules. For an n-periodic complex Y in $\mathsf{C}_n(A\text{-Mod})$, the tensor product $X \otimes_A Y$ is an n-periodic complex in $\mathsf{C}_n(B\text{-Mod})$. Thus, $X \otimes_A -$ gives a functor

$$X \boxtimes_A -\colon \mathsf{C}_n(A\text{-Mod}) \to \mathsf{C}_n(B\text{-Mod}).$$

The notation $X \boxtimes_A -$ is to distinguish it from $X \otimes_A -\colon \mathsf{C}(A\text{-Mod}) \to \mathsf{C}(B\text{-Mod})$. Moreover, the diagram

$$\begin{array}{ccc} \mathsf{C}(A\text{-Mod}) & \xrightarrow{X\otimes_A -} & \mathsf{C}(B\text{-Mod}) \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta} \\ \mathsf{C}_n(A\text{-Mod}) & \xrightarrow{X\boxtimes_A -} & \mathsf{C}_n(B\text{-Mod}) \end{array} \tag{5.1}$$

is commutative; see [1], equation (1.9.4) for the case $n = 1$.

5.2. Keep the same assumption as Subsection 5.1. Since $X \boxtimes_A -$ preserves homotopy, suspensions and mapping cones, it induces an exact functor $X \boxtimes_A -\colon \mathsf{K}_n(A\text{-Mod}) \to \mathsf{K}_n(B\text{-Mod})$. We define the *derived tensor product* $X \boxtimes^{\mathsf{L}}_A -$ by the composition

$$\mathsf{D}_n(A\text{-Mod}) \xrightarrow{\mathsf{p}} \mathsf{K}_n(A\text{-Mod}) \xrightarrow{X\boxtimes_A -} \mathsf{K}_n(B\text{-Mod}) \xrightarrow{Q} \mathsf{D}_n(B\text{-Mod}) ,$$

where p is the left adjoint of the canonical functor $\mathsf{K}_n(A\text{-Mod}) \to \mathsf{D}_n(A\text{-Mod})$, see Remark 3.1 for its existence. The compression functor $\Delta\colon \mathsf{K}(A\text{-Mod}) \to \mathsf{K}_n(A\text{-Mod})$ preserves homotopy projective objects because its right adjoint preserves acyclic

complexes. Combining this with (5.1), we observe that there exists a commutative diagram

$$\begin{array}{ccc} \mathsf{D}(A\text{-Mod}) & \xrightarrow{X\otimes^{\mathbf{L}}_{A}-} & \mathsf{D}(B\text{-Mod}) \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta} \\ \mathsf{D}_n(A\text{-Mod}) & \xrightarrow{X\boxtimes^{\mathbf{L}}_{A}-} & \mathsf{D}_n(B\text{-Mod}). \end{array}$$

For a triangulated category $\mathcal{T}$, we write $\Sigma_{\mathcal{T}}$ to be the suspension functor of $\mathcal{T}$.

Lemma 5.1. *Let $F\colon \mathcal{T} \to \mathcal{T}'$ be an exact functor between triangulated categories. Then F is fully faithful if and only if the induced functor $\overline{F}\colon \mathcal{T}/\Sigma^n_{\mathcal{T}} \to \mathcal{T}'/\Sigma^n_{\mathcal{T}'}$ is fully faithful. Moreover, F is an equivalence if and only if $\overline{F}$ is an equivalence.*

P r o o f. Since $\overline{F}(X) = F(X)$ for each object $X \in \mathcal{T}$, the second statement follows from the first one. Fix objects $X, Y \in \mathcal{T}$, we observe that the map

$$\overline{F}\colon \coprod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(X, \Sigma^{ni}_{\mathcal{T}}Y) \longrightarrow \coprod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}'}(F(X), \Sigma^{ni}_{\mathcal{T}'}F(Y))$$

is the direct sum of the composition maps

$$\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^{ni}_{\mathcal{T}}Y) \xrightarrow{F} \mathrm{Hom}_{\mathcal{T}'}(F(X), F(\Sigma^{ni}_{\mathcal{T}}Y)) \cong \mathrm{Hom}_{\mathcal{T}'}(F(X), \Sigma^{ni}_{\mathcal{T}'}F(Y)),$$

where the isomorphism is induced by the canonical isomorphism $F\Sigma_{\mathcal{T}} \cong \Sigma_{\mathcal{T}'}F$. The desired result follows. □

Lemma 5.2. *Let F, G, Φ_1, Φ_2 be exact functors between compactly generated triangulated categories such that the diagram*

$$\begin{array}{ccc} x_1 & \xrightarrow{F} & x_2 \\ {\scriptstyle\Phi_1}\downarrow & & \downarrow{\scriptstyle\Phi_2} \\ \mathcal{T}_1 & \xrightarrow{G} & \mathcal{T}_2 \end{array}$$

commutes. Assume F, G preserve coproducts and Φ_i preserves compact objects for $i = 1, 2$. Moreover, we assume Φ_i induces a fully faithful functor

$$\overline{\Phi_i}\colon x_i^{\mathrm{c}}/\Sigma^n_{x_i} \to \mathcal{T}_i^{\mathrm{c}}$$

such that $\mathcal{T}_i$ is compactly generated by its image for $i = 1, 2$. Then we have the implications:

(1) *F is an equivalence $\Rightarrow$ G is an equivalence.*

(2) *F preserves compact objects and G is an equivalence $\Rightarrow$ F is fully faithful.*

P r o o f. Combining with the assumption, the condition of (1) or (2) implies the diagram

$$\begin{array}{ccc} x_1^{\mathrm{c}}/\Sigma_{x_1}^n & \xrightarrow{\overline{F}} & x_2^{\mathrm{c}}/\Sigma_{x_2}^n \\ {\scriptstyle\overline{\Phi_1}}\downarrow & & \downarrow{\scriptstyle\overline{\Phi_2}} \\ \mathcal{T}_1^{\mathrm{c}} & \xrightarrow{G} & \mathcal{T}_2^{\mathrm{c}} \end{array}$$

commutes. Indeed, this is trivial for (2). For (1), it remains to show that G preserves compact objects. The assumption and the condition of (1) yield $G(\operatorname{Im}\overline{\Phi_1}) \subseteq \mathcal{T}_2^{\mathrm{c}}$. Since $\mathcal{T}_1$ is compactly generated by $\operatorname{Im}\overline{\Phi_1}$, we have $\operatorname{thick}_{\mathcal{T}_1}(\operatorname{Im}\overline{\Phi_1}) = \mathcal{T}_1^{\mathrm{c}}$, see Subsection 2.5. On the other hand, the full subcategory $\{X \in \mathcal{T}_1 \colon G(X) \in \mathcal{T}_2^{\mathrm{c}}\}$ of $\mathcal{T}_1$ is thick. Thus, G preserves compact objects.

(1) Assume F is equivalence. Then $\overline{F}$ is an equivalence. For $i = 1, 2$, $\operatorname{Im}\overline{\Phi_i}$ is a compact generating set of $\mathcal{T}_i$. Clearly $\operatorname{Im}\overline{\Phi_i}$ is closed under suspensions. Then we apply Lemma 2.1 to conclude that $G \colon \mathcal{T}_1 \to \mathcal{T}_2$ is an equivalence.

(2) By assumption, G induces an equivalence $G \colon \mathcal{T}_1^{\mathrm{c}} \xrightarrow{\sim} \mathcal{T}_2^{\mathrm{c}}$. Since $\overline{\Phi_i}$ is fully faithful for $i = 1, 2$, $\overline{F}$ is fully faithful. Then Lemma 5.1 yields the functor $F \colon x_1^{\mathrm{c}} \to x_2^{\mathrm{c}}$ is fully faithful. According to Lemma 2.1, F is fully faithful. □

Example 5.1. Let R be a commutative noetherian ring with a dualizing complex ω. Iyengar and Krause in Theorem I of [17] proved that

$$\omega \otimes_R - \colon \mathsf{K}(R\text{-Proj}) \longrightarrow \mathsf{K}(R\text{-Inj})$$

is a triangle equivalence. Combining this result with Proposition 4.1 and Lemma 5.2, we immediately get that there is a triangle equivalence

$$\omega \boxtimes_R - \colon \mathsf{K}_n(R\text{-Proj}) \xrightarrow{\sim} \mathsf{K}_n(R\text{-Inj}).$$

Let $\operatorname{Thick}\mathcal{T}$ be the *lattice of thick subcategories* of a triangulated category $\mathcal{T}$.

5.3. Suppose $\mathcal{A}$ is an additive category with coproducts (or AB4 category). We write $\mathcal{T}$ to be $\mathsf{K}(\mathcal{A})$ (or $\mathsf{D}(\mathcal{A})$) and $\mathcal{T}'$ to be $\mathsf{K}_n(\mathcal{A})$ (or $\mathsf{D}_n(\mathcal{A})$, respectively). For a thick subcategory x of $\mathcal{T}^{\mathrm{c}}$, we let $F(x)$ be the smallest thick subcategory of $\mathcal{T}'^{\,\mathrm{c}}$ containing all objects $\Delta(X)$ such that $X \in x$. For a thick subcategory x' of $\mathcal{T}'^{\,\mathrm{c}}$, we let $G(x')$ be the smallest thick subcategory of $\mathcal{T}^{\mathrm{c}}$ containing all objects X in $\mathcal{T}^{\mathrm{c}}$ such that $\Delta(X) \in x'$. Thus, we have maps of lattices

$$\operatorname{Thick}\mathcal{T}^{\mathrm{c}} \underset{G}{\overset{F}{\rightleftarrows}} \operatorname{Thick}T'^{\,\mathrm{c}}.$$

The next result is inspired by a recent result of Iyengar, Letz, Pollitz, and the author, see [18], Corollary 5.9. It is important in the proof of Theorem 5.1.

Lemma 5.3. *Keep the assumptions as in Subsection* 5.3. *Then* $G \circ F = \mathrm{id}$. *In particular, the map of lattices* $F\colon \operatorname{Thick}\mathcal{T}^{\mathrm{c}} \to \operatorname{Thick}\mathcal{T}'^{\mathrm{c}}$ *is injective.*

P r o o f. Fix a thick subcategory x of $\mathcal{T}^c$. In order to show $GF(x) = x$, it suffices to show for $X, Y \in \mathcal{T}^c$,

$$X \in \operatorname{thick}_{\mathcal{T}}(Y) \Leftrightarrow \Delta(X) \in \operatorname{thick}_{\mathcal{T}'}(\Delta(Y)).$$

The forward direction is trivial, see [2], Lemma 2.4. For the converse, assume $\Delta(X)$ is an object in $\operatorname{thick}_{\mathcal{T}'}(\Delta(Y))$. Then we have $\nabla\Delta(X) \in \operatorname{thick}_{\mathcal{T}}(\nabla\Delta(Y))$. Since $\nabla\Delta(M) \cong \coprod_{i\in\mathbb{Z}} \Sigma^{ni}(M)$ for each $M \in \mathcal{T}$ (see Subsection 3.3), X is in the localizing subcategory of $\mathcal{T}$ generated by Y. As X, Y are compact objects in $\mathcal{T}$, we conclude by Subsection 2.5 that X is in $\operatorname{thick}_{\mathcal{T}}(Y)$. The proof has been completed. □

Theorem 5.1. *Let* A, B *be two rings and* X *be a complex of* B-A-*bimodules. Then the functor* $X \otimes^{\mathsf{L}}_A -\colon \mathsf{D}(A\text{-Mod}) \to \mathsf{D}(B\text{-Mod})$ *is a triangle equivalence if and only if the functor* $X \boxtimes^{\mathsf{L}}_A -\colon \mathsf{D}_n(A\text{-Mod}) \to \mathsf{D}_n(B\text{-Mod})$ *is a triangle equivalence.*

P r o o f. First, assume $X\otimes^{\mathsf{L}}_A -$ is a triangle equivalence. It follows immediately from Proposition 4.1, Subsection 5.2 and Lemma 5.2 that $X\boxtimes^{\mathsf{L}}_R -$ is a triangle equivalence.

Now, assume $X\boxtimes^{\mathsf{L}}_A -$ is a triangle equivalence. It restricts to an equivalence between the full categories of compact objects. Combining with the commutative diagram in Subsection 5.2, we conclude by Lemma 3.1 that $X \otimes^{\mathsf{L}}_A -\colon \mathsf{D}(A\text{-Mod}) \to \mathsf{D}(B\text{-Mod})$ preserves compact objects. It follows from Proposition 4.1 and Lemma 5.2 that $X\otimes^{\mathsf{L}}_A -$ is fully faithful.

To show $X \otimes^{\mathsf{L}}_A -$ is an equivalence, by Lemma 2.1 it remains to show the essential image of $X \otimes^{\mathsf{L}}_A -\colon \mathsf{D}(A\text{-Mod})^{\mathrm{c}} \to \mathsf{D}(B\text{-Mod})^{\mathrm{c}}$, denoted by x, is a compact generating set of $\mathsf{D}(B\text{-Mod})$. Consider the commutative diagram

$$\begin{array}{ccc} \mathsf{D}(A\text{-Mod})^{\mathrm{c}} & \xrightarrow{X\otimes^{\mathsf{L}}_A -} & \mathsf{D}(B\text{-Mod})^{\mathrm{c}} \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta} \\ \mathsf{D}_n(A\text{-Mod})^{\mathrm{c}} & \xrightarrow[\sim]{X\boxtimes^{\mathsf{L}}_A -} & \mathsf{D}_n(B\text{-Mod})^{\mathrm{c}}. \end{array}$$

We apply Proposition 4.1 to get $\operatorname{thick}_{\mathsf{D}_n(B\text{-Mod})}(\Delta(x)) = \mathsf{D}_n(B\text{-Mod})^{\mathrm{c}}$. Then Lemma 5.3 yields the smallest thick subcategory of $\mathsf{D}(B\text{-Mod})^{\mathrm{c}}$ containing x is the whole of $\mathsf{D}(B\text{-Mod})^{\mathrm{c}}$. Hence, x is a compact generating set of $\mathsf{D}(B\text{-Mod})$. The proof has been completed. □

Two rings are *derived equivalent* provided that $\mathsf{D}(A\text{-Mod})$ and $\mathsf{D}(B\text{-Mod})$ are equivalent as triangulated categories.

5.4. It is an open question whether any triangle equivalence

$$\mathsf{D}(A\text{-Mod}) \xrightarrow{\sim} \mathsf{D}(B\text{-Mod})$$

is isomorphic to a derived tensor functor $X \otimes^{\mathsf{L}}_{A} -$, where X is a complex of B-A bimodules. Such derived equivalence is called a *standard equivalence*.

However, if A, B are two algebras over a commutative ring k such that they are flat as k-modules, then any triangle equivalence $\mathsf{D}(A\text{-Mod}) \xrightarrow{\sim} \mathsf{D}(B\text{-Mod})$ is standard, see [20], Corollary 9.2 and [33], Section 3.

Combining with Theorem 5.1, the statement in Subsection 5.4 implies the following result.

Corollary 5.1. *Let k be a commutative ring and A, B be flat k-algebras. If A and B are derived equivalent, then $\mathsf{D}_n(A\text{-Mod})$ and $\mathsf{D}_n(B\text{-Mod})$ are equivalent as triangulated categories.*

If A is a left noetherian ring with finite global dimension, then $\mathsf{D}_n(A\text{-Mod})^{\mathrm{c}} = \mathsf{D}_n(A\text{-mod})$; see Subsection 4.6. As a consequence of Corollary 5.1, we have:

Corollary 5.2. *Let k be a commutative ring and A, B be flat k-algebras. If A, B are noetherian with finite global dimensions and A, B are derived equivalent, then $\mathsf{D}_n(A\text{-mod})$ and $\mathsf{D}_n(B\text{-mod})$ are equivalent as triangulated categories.*

Remark 5.1. The above corollary extends a result of Zhao, see [37], Theorem. In her paper, she proved the above result holds for finite dimensional algebras with finite global dimensions over a field.

6. Koszul duality for periodic complexes

Throughout this section, k is a field and S is the graded polynomial algebra $k[x_1, \dots, x_c]$ with $\deg(x_i) = 1$. We let Λ denote the Koszul dual of S. More precisely, Λ is the graded exterior algebra over k on variables $\xi_1, \dots, \xi_c$ of degree -1.

For a graded algebra A, denote by A-Gr (or A-gr) the category of left (or finitely generated left, respectively) graded A-modules. A graded A-module is called *graded-injective* provided that it is an injective object in A-Gr. It is well-known that A-Gr has enough projective objects and enough injective objects, see [9], Section 1.5 and Theorem 3.6.2.

Let A-GrInj denote the category of graded-injective A-module. As Λ is noetherian, one can show that the direct sum of graded-injective Λ-module is graded-injective; the proof is parallel to the non-graded version, see [15], Theorem 3.1.17.

The main purpose of this section is to give the following periodic version of the Koszul duality.

Theorem 6.1. *There exists a triangle equivalence*

$$\mathsf{K}_n(\Lambda\text{-GrInj}) \xrightarrow{\sim} \mathsf{D}_n(S\text{-Gr}).$$

We give the proof of the above result at the end of this section. As a consequence, we have:

Corollary 6.1. *There is an embedding*

$$\mathsf{D}^b(\Lambda\text{-gr})/\Sigma^n \longrightarrow \mathsf{D}_n(S\text{-gr})$$

of the orbit category into its triangulated hull.

Before giving the proof of the corollary, we recall a result.

6.1. Due to Krause (see [24], Proposition 2.3), $\mathsf{K}(\Lambda\text{-GrInj})$ is compactly generated. Moreover, the localization functor $\mathsf{K}(\Lambda\text{-Gr}) \to \mathsf{D}(\Lambda\text{-Gr})$ induces a triangle equivalence

$$\mathsf{K}(\Lambda\text{-GrInj})^{\mathrm{c}} \xrightarrow{\sim} \mathsf{D}^b(\Lambda\text{-gr}).$$

Its inverse is induced by taking grade-injective resolution, denoted by i.

P r o o f of Corollary 6.1. Keep the notation as in Subsection 6.1, Remark 4.2 implies that the compression

$$\overline{\Delta \circ \mathsf{i}}\colon\ \mathsf{D}^b(\Lambda\text{-gr})/\Sigma^n \longrightarrow \mathsf{K}_n(\Lambda\text{-GrInj})^{\mathrm{c}}$$

induces an embedding of $\mathsf{D}^b(\Lambda\text{-gr})/\Sigma^n$ into its triangulated hull. It follows from Theorem 6.1 that $\mathsf{K}_n(\Lambda\text{-GrInj})^{\mathrm{c}}$ is triangle equivalent to $\mathsf{D}_n(S\text{-Gr})^{\mathrm{c}}$. Choose $\mathcal{A} = S\text{-gr}$ in Subsection 4.6, we conclude that $\mathsf{D}_n(S\text{-gr})$ is the smallest thick subcategory containing $\Delta(S(i))$ for all $i \in \mathbb{Z}$. It is precisely $\mathsf{D}_n(S\text{-Gr})^{\mathrm{c}}$, see Subsection 2.5 and Proposition 4.1. This completes the proof. □

6.2. Recall the functor $\Phi\colon \mathsf{C}(S\text{-Gr}) \to \mathsf{C}(\Lambda\text{-Gr})$, see [5], [7] or [14] for more details. Set $(-)^* := \mathrm{Hom}_k(-, k)$. For a graded S-module $M = \coprod_{i\in\mathbb{Z}} M_i$, $\Phi(M)$ is defined by the complex

$$\ldots \xrightarrow{\partial} \Lambda^* \otimes_k M_{i-1} \xrightarrow{\partial} \Lambda^* \otimes_k M_i \xrightarrow{\partial} \Lambda^* \otimes_k M_{i+1} \xrightarrow{\partial} \ldots,$$

where $\partial(f \otimes m) := (-1)^{l+i} \sum_{j=1}^{c} \xi_j f \otimes x_j m$ for $f \in (\Lambda^*)_l$ and $m \in M_i$, the sign makes sure that ∂ is Λ-linear. For a complex $M\colon \ \dots \xrightarrow{d} M^{j-1} \xrightarrow{d} M^j \xrightarrow{d} M^{j+1} \xrightarrow{d} \dots$ in $\mathsf{C}(S\text{-Gr})$, $\Phi(M)$ is defined by the total complex of the double complex

(6.1)
$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle 1\otimes d} & & \downarrow{\scriptstyle 1\otimes d} & & \\
\cdots & \xrightarrow{\partial} & \Lambda^* \otimes_k M_i^j & \xrightarrow{\partial} & \Lambda^* \otimes_k M_{i+1}^j & \xrightarrow{\partial} & \cdots \\
 & & \downarrow{\scriptstyle 1\otimes d} & & \downarrow{\scriptstyle 1\otimes d} & & \\
\cdots & \xrightarrow{\partial} & \Lambda^* \otimes_k M_i^{j+1} & \xrightarrow{\partial} & \Lambda^* \otimes_k M_{i+1}^{j+1} & \xrightarrow{\partial} & \cdots \\
 & & \downarrow{\scriptstyle 1\otimes d} & & \downarrow{\scriptstyle 1\otimes d} & & \\
 & & \vdots & & \vdots & &
\end{array}$$

where the lth component of $\Phi(M)$ is $\coprod_{i+j=l} \Lambda^* \otimes_k M_i^j$.

6.3. Keep the notation as above. Since Φ preserves homotopy, suspensions and mapping cones, it induces an exact functor $\Phi\colon \mathsf{K}(S\text{-Gr}) \to \mathsf{K}(\Lambda\text{-Gr})$. The image of this functor lies in $\mathsf{K}(\Lambda\text{-GrInj})$ because Λ^* is graded-injective. Bernstein, Gel'fand, and Gel'fand (see [7], Theorem 3) proved that Φ naturally induces a triangle equivalence

$$\Phi\colon \mathsf{D}^b(S\text{-gr}) \xrightarrow{\sim} \mathsf{D}^b(\Lambda\text{-gr}),$$

see also [5], Theorem 2.12.1. This is known as the BGG correspondence. Moreover, it fits into the commutative diagram

(6.2)
$$\begin{array}{ccc}
\mathsf{D}^b(S\text{-gr}) & \xrightarrow{\sim} & \mathsf{D}^b(\Lambda\text{-gr}) \\
{\scriptstyle \mathrm{inc}}\downarrow & & \downarrow{\scriptstyle \mathrm{i}} \\
\mathsf{D}(S\text{-Gr}) & \longrightarrow & \mathsf{K}(\Lambda\text{-GrInj}),
\end{array}$$

where the bottom map is the composition

$$\mathsf{D}(S\text{-Gr}) \xrightarrow{\mathsf{p}} \mathsf{K}(S\text{-Gr}) \xrightarrow{\Phi} \mathsf{K}(\Lambda\text{-GrInj}),$$

where p is the left adjoint of the localization functor $\mathsf{K}(S\text{-Gr}) \to \mathsf{D}(S\text{-Gr})$, see [8], Proposition 2.12 for its exisence.

The essential images of the vertical functors in (6.2) are precisely the full subcategories of compact objects in the bottom categories. This is clear for the left one as the global dimension of S is finite. See Subsection 6.1 for the right one. Combining with the fact that $\Phi \circ \mathsf{p}$ preserves coproducts, Lemma 2.1 yields $\Phi \circ \mathsf{p}$ is an equivalence, see [24], Example 5.7.

Now we define the exact functor $\mathsf{D}_n(S\text{-Gr}) \to \mathsf{K}_n(\Lambda\text{-GrInj})$.

6.4. For an n-periodic complex $M \in \mathsf{C}_n(S\text{-Gr})$, the total complex $\Phi(M)$ (see (6.1)) is an n-periodic complex in $\mathsf{C}_n(\Lambda\text{-Gr})$. Therefore this gives a functor

$$\Phi' \colon \mathsf{C}_n(S\text{-Gr}) \longrightarrow \mathsf{C}_n(\Lambda\text{-Gr})$$

which maps M to $\Phi(M)$. Also, Φ' induces an exact functor $\Phi' \colon \mathsf{K}_n(S\text{-Gr}) \to \mathsf{K}_n(\Lambda\text{-Gr})$ between the homotopy categories and its image lies in $\mathsf{K}_n(\Lambda\text{-GrInj})$. Consider the composition

$$\mathsf{D}_n(S\text{-Gr}) \xrightarrow{\mathsf{p}'} \mathsf{K}_n(S\text{-Gr}) \xrightarrow{\Phi'} \mathsf{K}_n(\Lambda\text{-GrInj}),$$

where p' is the left adjoint of the localization functor $\mathsf{K}_n(S\text{-Gr}) \to \mathsf{D}_n(S\text{-Gr})$; its existence can refer the non-graded version of Remark 3.1.

P r o o f of Theorem 6.1. It follows from Proposition 4.1 and Subsection 6.1 that $\mathsf{D}_n(S\text{-Gr})$ and $\mathsf{K}_n(\Lambda\text{-GrInj})$ are compactly generated triangulated categories. Combining with Subsection 6.3, we observe that there exists a commutative diagram

$$\begin{array}{ccc} \mathsf{D}(S\text{-Gr}) & \xrightarrow[\sim]{\Phi\circ\mathsf{p}} & \mathsf{K}(\Lambda\text{-GrInj}) \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta} \\ \mathsf{D}_n(S\text{-Gr}) & \xrightarrow{\Phi'\circ\mathsf{p}'} & \mathsf{K}_n(\Lambda\text{-GrInj}). \end{array}$$

Since $\Phi' \circ \mathsf{p}'$ preserves coproducts, Proposition 4.1 and Lemma 5.2 imply $\Phi' \circ \mathsf{p}'$ is a triangle equivalence. □

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References

[1] *L. L. Avramov, R.-O. Buchweitz, S. B. Iyengar*: Class and rank of differential modules. Invent. Math. *169* (2007), 1–35. zbl MR doi

[2] *L. L. Avramov, R.-O. Buchweitz, S. B. Iyengar, C. Miller*: Homology of perfect complexes. Adv. Math. *223* (2010), 1731–1781. zbl MR doi

[3] *A. A. Beilinson*: Coherent sheaves on P^n and problems of linear algebra. Funkts. Anal. Prilozh. *12* (1978), 68–69. (In Russian.) zbl MR

[4] *A. A. Beilinson, J. Bernstein, P. Deligne*: Faisceaux pervers. Analysis and Topology on Singular Spaces. I. Astérisque 100. Société mathématique de France, Paris, 1982, pp. 5–171. (In French.) zbl MR

[5] *A. A. Beilinson, V. Ginzburg, W. Soergel*: Koszul duality patterns in representation theory. J. Am. Math. Soc. *9* (1996), 473–527. zbl MR doi

[6] *D. J. Benson, S. B. Iyengar, H. Krause*: Stratifying modular representations of finite groups. Ann. Math. (2) *174* (2011), 1643–1684. zbl MR doi

[7] *I. N. Bernstein, I. M. Gel'fand, S. I. Gel'fand*: Algebraic vector bundles on P^n and problems of linear algebra. Funkts. Anal. Prilozh. *12* (1978), 66–67. (In Russian.) zbl MR

[8] *M. Bökstedt, A. Neeman*: Homotopy limits in triangulated categories. Compos. Math. *86* (1993), 209–234. zbl MR

[9] *W. Bruns, J. Herzog*: Cohen-Macaulay Rings. Cambridge Studies in Advanced Mathematics 39. Cambridge University Press, Cambridge, 1998. zbl MR doi

[10] *R.-O. Buchweitz*: Maximal Cohen-Macaulay Modules and Tate Cohomology. Mathematical Surveys and Monographs 262. AMS, Providence, 2021. zbl MR doi

[11] *H. Cartan, S. Eilenberg*: Homological Algebra. Princeton Mathematical Series 19. Princeton University Press, Princeton, 1956. zbl MR

[12] *X.-W. Chen, J. Liu, R. Wang*: Singular equivalences induced by bimodules and quadratic monomial algebras. To appear in Algebr. Represent. Theory. doi

[13] *V. Drinfeld*: DG quotients of DG categories. J. Algebra *272* (2004), 643–691. zbl MR doi

[14] *D. Eisenbud, G. Fløystad, F.-O. Schreyer*: Sheaf cohomology and free resolutions over exterior algebras. Trans. Am. Math. Soc. *355* (2003), 4397–4426. zbl MR doi

[15] *E. E. Enochs, O. M. G. Jenda*: Relative Homological Algebra. De Gruyter Expositions in Mathematics 30. Walter De Gruyter, Berlin, 2000. zbl MR doi

[16] *D. Happel*: On the derived category of a finite-dimensional algebra. Comment. Math. Helv. *62* (1987), 339–389. zbl MR doi

[17] *S. B. Iyengar, H. Krause*: Acyclicity versus total acyclicity for complexes over Noetherian rings. Doc. Math. *11* (2006), 207–240. zbl MR

[18] *S. B. Iyengar, J. C. Letz, J. Liu, J. Pollitz*: Exceptional complete intersection maps of local rings. Pac. J. Math. *318* (2022), 275–293. zbl MR doi

[19] *M. Kalck, D. Yang*: Derived categories of graded gentle one-cycle algebras. J. Pure Appl. Algebra *222* (2018), 3005–3035. zbl MR doi

[20] *B. Keller*: Deriving DG categories. Ann. Sci. Éc. Norm. Supér. (4) *27* (1994), 63–102. zbl MR doi

[21] *B. Keller*: On the cyclic homology of exact categories. J. Pure Appl. Algebra *136* (1999), 1–56. zbl MR doi

[22] *B. Keller*: On triangulated orbit categories. Doc. Math. *10* (2005), 551–581. zbl MR

[23] *B. Keller*: Corrections to 'On triangulated orbit categories'. Available at `https://webusers.imj-prg.fr/~bernhard.keller/publ/corrTriaOrbit.pdf` (2009), 5 pages.

[24] *H. Krause*: The stable derived category of a Noetherian scheme. Compos. Math. *141* (2005), 1128–1162. zbl MR doi

[25] *H. Krause*: Localization theory for triangulated categories. Triangulated Categories. London Mathematical Society Lecture Note Series 375. Cambridge University Press, Cambridge, 2010, pp. 161–235. zbl MR doi
[26] *A. Neeman*: The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel. Ann. Sci. Éc. Norm. Supér. (4) *25* (1992), 547–566. zbl MR doi
[27] *A. Neeman*: The Grothendieck duality theorem via Bousfield's techniques and Brown representability. J. Am. Math. Soc. *9* (1996), 205–236. zbl MR doi
[28] *A. Neeman*: Triangulated Categories. Annals of Mathematics Studies 148. Princeton University Press, Princeton, 2001. zbl MR doi
[29] *D. Orlov*: Derived categories of coherent sheaves and triangulated categories of singularities. Algebra, Arithmetic, and Geometry. Volume II. Progress in Mathematics 270. Birkhäuser, Boston, 2009, pp. 503–531. zbl MR doi
[30] *L. Peng, J. Xiao*: Root categories and simple Lie algebras. J. Algebra *198* (1997), 19–56. zbl MR doi
[31] *L. Peng, J. Xiao*: Triangulated categories and Kac-Moody algebras. Invent. Math. *140* (2000), 563–603. zbl MR doi
[32] *J. Rickard*: Morita theory for derived categories. J. Lond. Math. Soc., II. Ser. *39* (1989), 436–456. zbl MR doi
[33] *J. Rickard*: Derived equivalences as derived functors. J. Lond. Math. Soc., II. Ser. *43* (1991), 37–48. zbl MR doi
[34] *C. M. Ringel, P. Zhang*: Representations of quivers over the algebra of dual numbers. J. Algebra *475* (2017), 327–360. zbl MR doi
[35] *T. Stai*: The triangulated hull of periodic complexes. Math. Res. Lett. *25* (2018), 199–236. zbl MR doi
[36] *X. Tang, Z. Huang*: Higher differential objects in additive categories. J. Algebra *549* (2020), 128–164. zbl MR doi
[37] *X. Zhao*: A note on the equivalence of m-periodic derived categories. Sci. China, Math. *57* (2014), 2329–2334. zbl MR doi

Author's address: Jian Liu, School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, P. R. China, e-mail: liuj231@sjtu.edu.cn.