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REGULARITY OF POWERS OF BINOMIAL EDGE IDEALS OF COMPLETE MULTIPARTITE GRAPHS

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Abstract. Let $G = K_{n_1,n_2,\dots,n_r}$ be a complete multipartite graph on $[n]$ with $n > r > 1$ and J_G being its binomial edge ideal. It is proved that the Castelnuovo-Mumford regularity $\operatorname{reg}(J_G^t)$ is $2t+1$ for any positive integer t.

Keywords: Castelnuovo-Mumford regularity; binomial edge ideal; multipartite graph

MSC 2020: 13D02, 05E40

1. Introduction

The notion of binomial edge ideals of graphs was first introduced in [7] and [11] independently. Since then, many researches on binomial edge ideals of graphs have been done, cf. [8]. One of challenging problems in this area is the computation of the Castelnuovo-Mumford regularity of powers of binomial edge ideals. Because of the difficulties, only a few results have been obtained. When the graph is a cycle, a star or complete, such regularities were successfully computed in [10]. The regularities of powers of binomial edge ideals of a closed graph were obtained in [5]. The aim of this paper is to compute the regularities of powers of binomial edge ideals of complete multipartite graphs.

Let $G = (V(G), E(G))$ be a finite simple graph with $V(G) = [n] := \{1, 2, \dots, n\}$ and $S = K[x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n]$ be a polynomial ring in $2n$ variables over a field K. Then the *binomial edge ideal* of G, denoted by J_G, is the ideal generated by all binomials

$$f_{ij} := x_i y_j - x_j y_i, \quad 1 \leqslant i < j \leqslant n, \quad \{i, j\} \in E(G).$$

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On the other hand, for a finitely generated graded S-module M, the (Castelnuovo-Mumford) regularity of M, denoted by $\operatorname{reg}(M)$, is an important algebraic invariant of M. For a long time, mathematicians have made great efforts to find homogeneous ideals I whose powers possess linear regularities, i.e., $\operatorname{reg}(I^t) = at + b$. When G is a complete multipartite graph, we show that its binomial edge ideal J_G is such a homogeneous ideal. In fact, we prove that $\operatorname{reg}(J_G^t) = 2t + 1$.

After preparing some auxiliary results in Section 2, we reduce the problem on the power J_G^t to certain suitable powers by using symbolic powers in Section 3. Section 4 is the main part of this paper. We simplify some colon ideals, which makes it possible to estimate the regularity of sums of powers of binomial edge ideals of complete graphs and another ideal. Then we prove the main result $\operatorname{reg}(J_G^t) = 2t+1$ in Section 5.

2. Preliminaries

Let $S = K[x_1, x_2, \ldots, x_n]$ be a polynomial ring over a field K and M a finitely generated graded S-module. Let

$$\ldots \to F_j \xrightarrow{\varphi_j} \ldots \to F_1 \xrightarrow{\varphi_1} F_0 \to M \to 0$$

be a graded minimal free resolution of M, where $F_j = \bigoplus\limits_i S(-a_{ji})$. $\operatorname{Im}(\varphi_j)$ is called the jth syzygy module of M. One says that M is m-regular if $a_{ji} - j \leqslant m$ for all i, j and defines the *Castelnuovo-Mumford regularity* (or *regularity*) of M by

$$\operatorname{reg}(M) = \min\{m\colon\ M \text{ is } m\text{-regular}\}.$$

Let I be a nonzero homogeneous ideal of S. Notice that the ith syzygy of I is just the $(i+1)$st syzygy of S/I. It follows that $\operatorname{reg}(I) = \operatorname{reg}(S/I) + 1$. For the properties of regularity, we refer to [4].

The following Regularity Lemma will be used frequently.

Lemma 2.1 (Regularity Lemma, cf. [4], Corollary 20.19)**.** *Let $0 \to M \to N \to L \to 0$ be a short exact sequence of graded S-modules. Then*
(1) $\operatorname{reg}(N) \leqslant \max\{\operatorname{reg}(M), \operatorname{reg}(L)\}$;
(2) $\operatorname{reg}(M) \leqslant \max\{\operatorname{reg}(N), \operatorname{reg}(L) + 1\}$.

We will apply the following two lemmas in the sequel.

Lemma 2.2 (Cf. the proof of Lemma 2.10 of [3])**.** *Let I be a monomial ideal and x a variable of S. Then*

$$\operatorname{reg}((I, x)) \leqslant \operatorname{reg}(I).$$

Lemma 2.3 ([9], Lemma 3.2). *Let $S_1 = K[x_1, x_2, \dots, x_n]$, $S_2 = K[y_1, y_2, \dots, y_N]$ and $S_3 = K[x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_N]$. Then, for nonzero homogeneous ideals $I \subseteq S_1$ and $J \subseteq S_2$, we have*

$$\operatorname{reg}(S_3/(I+J)) = \operatorname{reg}(S_1/I) + \operatorname{reg}(S_2/J).$$

Then, in the above lemma, $\operatorname{reg}((I, y_i)) = \operatorname{reg}(I)$.

Let $<$ be a monomial order on S. For any $f \neq 0 \in S$, the initial monomial of f is denoted by $\operatorname{in}_<(f)$. Let I be a nonzero ideal of S. Then the *initial ideal* of I, denoted by $\operatorname{in}_<(I)$, is the ideal generated by all monomials $\operatorname{in}_<(f)$, $f \neq 0 \in I$. In some cases, one can reduce the problem on I to the monomial ideal $\operatorname{in}_<(I)$.

Lemma 2.4 ([6], Theorem 3.3.4). *Let I be a nonzero homogeneous proper ideal of S. Then, for any monomial order $<$ on S,*

$$\operatorname{reg}(I) \leqslant \operatorname{reg}(\operatorname{in}_<(I)).$$

Lemma 2.5 ([4], Lemma 15.5). *Let $I \subseteq J$ be nonzero proper ideals of S. If there is a monomial order $<$ on S such that $\operatorname{in}_<(I) = \operatorname{in}_<(J)$, then $I = J$.*

In our cases, we are interested in colon ideals. The following lemma is useful.

Lemma 2.6. *Let $I_1, I_2, \dots, I_r$ be nonzero proper ideals of S and $f \neq 0 \in S$. If there is a monomial order $<$ on S such that*

$$\operatorname{in}_<(I_1 + I_2 + \ldots + I_r) = \operatorname{in}_<(I_1) + \operatorname{in}_<(I_2) + \ldots + \operatorname{in}_<(I_r) \quad \textit{and}$$
$$\operatorname{in}_<(I_i : f) = \operatorname{in}_<(I_i) : \operatorname{in}_<(f), \quad i = 1, 2, \dots, r,$$

then

$$(I_1 + I_2 + \ldots + I_r) : f = (I_1 : f) + (I_2 : f) + \ldots + (I_r : f).$$

P r o o f. It is clear that $(I_1 : f) + (I_2 : f) + \ldots + (I_r : f) \subseteq (I_1 + I_2 + \ldots + I_r) : f$. Then, by Lemma 2.5, we only need to show that

$$\operatorname{in}_<((I_1 + I_2 + \ldots + I_r) : f) \subseteq \operatorname{in}_<((I_1 : f) + (I_2 : f) + \ldots + (I_r : f)).$$

Let $g \in (I_1 + I_2 + \ldots + I_r) : f$. Then $gf \in I_1 + I_2 + \ldots + I_r$ and

$$\begin{aligned}
\operatorname{in}_<(g) &\in \operatorname{in}_<(I_1 + I_2 + \ldots + I_r) : \operatorname{in}_<(f) = (\operatorname{in}_<(I_1) + \operatorname{in}_<(I_2) + \ldots + \operatorname{in}_<(I_r)) : \operatorname{in}_<(f) \\
&= (\operatorname{in}_<(I_1) : \operatorname{in}_<(f)) + (\operatorname{in}_<(I_2) : \operatorname{in}_<(f)) + \ldots + (\operatorname{in}_<(I_r) : \operatorname{in}_<(f)) \\
&= \operatorname{in}_<(I_1 : f) + \operatorname{in}_<(I_2 : f) + \ldots + \operatorname{in}_<(I_r : f) \\
&\subseteq \operatorname{in}_<((I_1 : f) + (I_2 : f) + \ldots + (I_r : f)).
\end{aligned}$$

The result follows. $\square$

Let $<$ be a monomial order on S and $I = (g_1, g_2, \ldots, g_s)$ be a nonzero ideal of S. If $\mathrm{in}_<(I) = (\mathrm{in}_<(g_1), \mathrm{in}_<(g_2), \ldots, \mathrm{in}_<(g_s))$, then $\{g_1, g_2, \ldots, g_s\}$ is called a *Gröbner basis* of I with respect to $<$.

Let $f, g \neq 0 \in S$ and, c_f and c_g be coefficients of $\mathrm{in}_<(f)$ and $\mathrm{in}_<(g)$, respectively. The *S-polynomial* of f and g is defined as

$$S(f,g) = \frac{\mathrm{lcm}(\mathrm{in}_<(f), \mathrm{in}_<(g))}{c_f \cdot \mathrm{in}_<(f)} f - \frac{\mathrm{lcm}(\mathrm{in}_<(f), \mathrm{in}_<(g))}{c_g \cdot \mathrm{in}_<(g)} g.$$

We say that f has a *standard expression* with respect to $g_1, g_2, \ldots, g_s$ with remainder 0 if

$$f = f_1 g_1 + f_2 g_2 + \ldots + f_s g_s, \quad \mathrm{in}_<(f_i g_i) \leqslant \mathrm{in}_<(f), \quad i = 1, 2, \ldots, s.$$

Then, by *Buchberger's Criterion*, $\{g_1, g_2, \ldots, g_s\}$ is a Gröbner basis of I if and only if for all $i \neq j$, $S(g_i, g_j)$ has a standard expression with respect to $g_1, g_2, \ldots, g_s$ with remainder 0.

Remark 2.7. Notice that in Lemma 2.6, $\mathrm{in}_<(I_1 + I_2 + \ldots + I_r) = \mathrm{in}_<(I_1) + \mathrm{in}_<(I_2) + \ldots + \mathrm{in}_<(I_r)$ holds if and only if there is a Gröbner basis G_i of I_i with respect to $<$, $i = 1, 2, \ldots, r$, such that $G_1 \cup G_2 \cup \ldots \cup G_r$ is a Gröbner basis of $I_1 + I_2 + \ldots + I_r$ with respect to $<$.

Set $[m, n] = \{m, m+1, \ldots, n\}$ for any $m \leqslant n$. Write $[m]$ for $[1, m]$. Let $G = ([n], E(G))$ be a finite simple graph on $[n]$ and $S = K[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]$ be a polynomial ring in $2n$ variables over a field K. Then the *binomial edge ideal* of G, denoted by J_G, is the ideal generated by all binomials

$$f_{ij} := x_i y_j - x_j y_i, \quad 1 \leqslant i < j \leqslant n, \quad \{i, j\} \in E(G).$$

Note that, in fact, f_{ij} is just a 2-minor of the generic matrix

$$\begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ y_1 & y_2 & \ldots & y_n \end{pmatrix}.$$

Then, by the properties of minors of a determinant, we have:

Lemma 2.8.
(1) *For any $a < b < c$, $x_a f_{bc} - x_b f_{ac} + x_c f_{ab} = 0$ and $y_a f_{bc} - y_b f_{ac} + y_c f_{ab} = 0$.*
(2) *For any $a < b < c < d$, $f_{ad} f_{bc} - f_{ac} f_{bd} + f_{ab} f_{cd} = 0$ (Plücker relation).*

Let $T \subseteq [n]$. Denote the complete graph on T by K_T. Then $J_{K_n} = \{f_{ij}\colon 1 \leqslant i < j \leqslant n\}$. Let G be the complete r-partite graph $K_{n_1,n_2,\ldots,n_r}$, i.e., $V(G)$ is the disjoint union of $V_i := \Bigl[1 + \sum\limits_{t=1}^{i-1} n_t, \sum\limits_{t=1}^{i} n_t\Bigr]$, $i = 1, 2, \ldots, r$, and $E(G) = \{\{a, b\}\colon a$ and b is not in the same $V_i\}$. If $r = 1$, then $J_G = 0$. In the following, we always assume that $r > 1$. When $n_1 = n_2 = \ldots = n_r = 1$, $G = K_{1,1,\ldots,1}$ is a complete graph.

For any $T \subseteq [n]$, set $A_T = (x_i, y_i\colon i \in T)$ and $A_m = A_{[m]}$ for any $1 \leqslant m \leqslant n$. Set $A_0 = 0$ by convention. Note that, for any $1 \leqslant m \leqslant n$ and $s \geqslant 1$, $\operatorname{reg}(A_m^s) = s$ as A_m^s has a linear resolution. On the other hand, let $<$ be the lexicographic order with $x_1 > x_2 > \ldots > x_n > y_1 > y_2 > \ldots > y_n$, it is well-known that f_{ij}, $1 \leqslant i < j \leqslant n$, form a Gröbner basis of J_{K_n} with respect to $<$, which implies that $\mathrm{in}_<(J_{K_n}) = (x_iy_j, 1 \leqslant i < j \leqslant n)$. It is also known that $\operatorname{reg}(\mathrm{in}_<(J_{K_n})) = 2$ by [13], Theorem 2.2. Furthermore, we have:

Lemma 2.9. *Let $0 \leqslant m \leqslant n$. Then, for any $s \geqslant 1$, $\operatorname{reg}(\mathrm{in}_<(J_{K_n}) + A_m^s) \leqslant s + 1$.*

Proof. Let us use induction on m and s together to prove that

$$\operatorname{reg}(\mathrm{in}_<(J_{K_n}) + A_m^s) \leqslant s + 1.$$

If $m = 0$, the result is clear. If $s = 1$, then by Lemma 2.2, $\operatorname{reg}(\mathrm{in}_<(J_{K_n}) + A_m) \leqslant \operatorname{reg}(\mathrm{in}_<(J_{K_n})) = 2$.

Now assume that $m > 0$ and $s > 1$, and the result is true for $(s, m-1)$ and $(s-1, m)$. Consider the following short exact sequences:

$$\begin{aligned}
0 &\to \frac{S}{(\mathrm{in}_<(J_{K_n}) + A_m^s) : x_1}(-1) \xrightarrow{\cdot x_1} \frac{S}{\mathrm{in}_<(J_{K_n}) + A_m^s} \\
&\to \frac{S}{\mathrm{in}_<(J_{K_n}) + A_m^s + (x_1)} \to 0, \\
0 &\to \frac{S}{(\mathrm{in}_<(J_{K_n}) + A_m^s + (x_1)) : y_1}(-1) \xrightarrow{\cdot y_1} \frac{S}{\mathrm{in}_<(J_{K_n}) + A_m^s + (x_1)} \\
&\to \frac{S}{\mathrm{in}_<(J_{K_n}) + A_m^s + (x_1, y_1)} \to 0.
\end{aligned}$$

Note that $\mathrm{in}_<(J_{K_n}) + A_m^s + (x_1, y_1) = \mathrm{in}_<(J_{K_{[2,n]}}) + A_{[2,m]}^s + (x_1, y_1)$ and $(\mathrm{in}_<(J_{K_n}) + A_m^s + (x_1)) : y_1 = \mathrm{in}_<(J_{K_n}) + A_m^{s-1} + (x_1)$. Then $\operatorname{reg}(\mathrm{in}_<(J_{K_n}) + A_m^s + (x_1, y_1)) = \operatorname{reg}(\mathrm{in}_<(J_{K_{[2,n]}}) + A_{[2,m]}^s) \leqslant s + 1$ by the induction hypothesis on m and

$$\operatorname{reg}((\mathrm{in}_<(J_{K_n}) + A_m^s + (x_1)) : y_1) \leqslant \operatorname{reg}(\mathrm{in}_<(J_{K_n}) + A_m^{s-1}) \leqslant s$$

by the induction hypothesis on s. Then applying Regularity Lemma 2.1 to the above second short exact sequence, we obtain that

$$\operatorname{reg}(\mathrm{in}_<(J_{K_n}) + A_m^s + (x_1)) \leqslant s + 1.$$

It is easy to see that $(\mathrm{in}_<(J_{K_n}) + A_m^s) : x_1 = (\mathrm{in}_<(J_{K_n}) : x_1) + (A_m^s : x_1) = (y_2, \ldots, y_n) + A_m^{s-1}$, hence $\mathrm{reg}((\mathrm{in}_<(J_{K_n}) + A_m^s) : x_1) \leqslant \mathrm{reg}(A_m^{s-1}) = s-1$. Then $\mathrm{reg}(\mathrm{in}_<(J_{K_n}) + A_m^s) \leqslant s+1$ follows by applying Regularity Lemma 2.1 to the above first short exact sequence. □

The following simple result will be used in Section 4.

Lemma 2.10. *Let I be an ideal of a ring R and $f \in R$. Then, in the polynomial ring $R[x]$,*

$$(I, x) : f = (I : f) + (x).$$

3. Powers of binomial edge ideals of complete multipartite graphs

Let I be an ideal of a ring R. The tth symbolic power of I is, by definition,

$$I^{(t)} = \bigcap_{\mathfrak{p} \in \min(I)} (I^t R_{\mathfrak{p}} \cap R),$$

where $\min(I)$ is the set of minimal prime ideals of I.

Let $I = Q_1 \cap Q_2 \cap \ldots \cap Q_m$ be an irredundant primary decomposition of I with $\min(I) = \{P_1, P_2, \ldots, P_s\}$ and $P_i = \sqrt{Q_i}$, $i = 1, 2, \ldots, s$. Then it is well-known that

$$I^{(t)} = Q_1^{(t)} \cap Q_2^{(t)} \cap \ldots \cap Q_s^{(t)}.$$

Now we consider the case that I is the binomial edge ideal of a complete multipartite graph.

Let $G = K_{n_1, n_2, \ldots, n_r}$ be a complete multipartite graph on $[n]$. Then, by [12], Theorem 4.3, $J_G^t = J_G^{(t)}$ for any $t \geqslant 1$. On the other hand, assume that $n_1 \leqslant n_2 \leqslant \ldots \leqslant n_r$, $r > 1$, and $n_r > 1$, i.e., G is not a complete graph. Furthermore, assume that $n_1 = n_2 = \ldots = n_s = 1$ and $n_{s+1} > 1$. Then, by [12], Lemma 2.2, all the associated prime ideals of J_G are minimal, which are exactly J_{K_n}, P_i, $i = s+1, \ldots, r$, where $P_i = \Big(x_j, y_j \colon j \notin \Big[1 + \sum\limits_{k=1}^{i-1} n_k, \sum\limits_{k=1}^{i} n_k\Big]\Big)$. It is clear that $P_i^{(t)} = P_i^t$. On the other hand, $J_{K_n}^{(t)} = J_{K_n}^t$ is well-known, see [1], Theorem 10.4. Thus, it follows that

$$J_G^t = J_{K_n}^t \cap P_{s+1}^t \cap P_{s+2}^t \cap \ldots \cap P_r^t.$$

We will use this equality to compute the regularity $\mathrm{reg}(J_G^t)$.

Remark 3.1. Our strategy for computing $\operatorname{reg}(J_G^t)$ is as follows. We firstly estimate $\operatorname{reg}(J_{K_n}^t + P_i^t)$, which is the crucial result of this paper, then estimate $\operatorname{reg}(J_{K_n}^t \cap P_i^t)$, and finally get $\operatorname{reg}(J_{K_n}^t \cap P_{s+1}^t \cap P_{s+2}^t \cap \ldots \cap P_r^t)$, i.e., $\operatorname{reg}(J_G^t)$.

4. Regularity related to powers of binomial edge ideals of complete graphs

Let K be a field and $S = K[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]$. Let T_1 and T_2 be two subsets of S. Set $T_1T_2 = \{t_1t_2\colon\ t_i \in T_i,\ i = 1, 2\}$. For any ideal I of a ring R, $I^i = R$ for $i \leqslant 0$ by convention. Let $<$ be the lexicographic order with $x_1 > x_2 > \ldots > x_n > y_1 > y_2 > \ldots > y_n$.

Remark 4.1. It is known that $\operatorname{reg}(J_{K_n}^t) = \operatorname{reg}(\operatorname{in}_<(J_{K_n})^t) = 2t$ for any $t \geqslant 1$ as observed by [10], Observation 3.2 (2). In this section, we will estimate the regularity of the sum of $J_{K_n}^t$ and some power of an ideal as A_T.

Set $\mathcal{G}(J_{K_n}) = \{f_{ij}\colon\ 1 \leqslant i < j \leqslant n\}$.

Lemma 4.2 ([2], Theorem 2.1). *For any $t \geqslant 1$, $(\mathcal{G}(J_{K_n}))^t$ is a Gröbner basis of $J_{K_n}^t$ with respect to $<$ and $\operatorname{in}_<(J_{K_n}^t) = (\operatorname{in}_<(J_{K_n}))^t$.*

Similarly, for any $1 \leqslant a < b \leqslant n$, we can define $\mathcal{G}(J_{K_{[a,b]}})$ and prove that $(\mathcal{G}(J_{K_{[a,b]}}))^t$ is a Gröbner basis of $J_{K_{[a,b]}}^t$ with respect to $<$.

Lemma 4.3. *For any $1 \leqslant a < b \leqslant n$ and $t \geqslant 1$, $\mathcal{G}(J_{K_{[a,b]}})(\mathcal{G}(J_{K_n}))^t$ is a Gröbner basis of $J_{K_{[a,b]}} J_{K_n}^t$ with respect to $<$ and $\operatorname{in}_<(J_{K_{[a,b]}} J_{K_n}^t) = \operatorname{in}_<(J_{K_{[a,b]}})(\operatorname{in}_<(J_{K_n}))^t$.*

P r o o f. Note that $\mathcal{G}(J_{K_{[a,b]}})(\mathcal{G}(J_{K_n}))^t$ generates $J_{K_{[a,b]}} J_{K_n}^t$ and the set of initial terms of $\mathcal{G}(J_{K_{[a,b]}})(\mathcal{G}(J_{K_n}))^t$ generates $\operatorname{in}_<(J_{K_{[a,b]}})(\operatorname{in}_<(J_{K_n}))^t$. It is sufficient to show that

$$\operatorname{in}_<(J_{K_{[a,b]}} J_{K_n}^t) \subseteq \operatorname{in}_<(J_{K_{[a,b]}})(\operatorname{in}_<(J_{K_n}))^t.$$

On the contrary, suppose that there exists a monomial $u \in \operatorname{in}_<(J_{K_{[a,b]}} J_{K_n}^t) \setminus \operatorname{in}_<(J_{K_{[a,b]}})(\operatorname{in}_<(J_{K_n}))^t$. As $\operatorname{in}_<(J_{K_{[a,b]}} J_{K_n}^t) \subseteq \operatorname{in}_<(J_{K_n}^{t+1}) = (\operatorname{in}_<(J_{K_n}))^{t+1}$, we can write $u = w \prod\limits_{k=1}^{t+1} x_{i_k} y_{j_k}$ with no $x_{i_k} y_{j_k} \in \operatorname{in}_<(J_{K_{[a,b]}})$. On the other hand, as $\operatorname{in}_<(J_{K_{[a,b]}} J_{K_n}^t) \subseteq \operatorname{in}_<(J_{K_{[a,b]}})$, there exists some $x_p y_q \in \operatorname{in}_<(J_{K_{[a,b]}})$ such that $x_p y_q | u$, which admits four possibilities:

(1) $x_py_q|w$. Then $u \in \mathrm{in}_<(J_{K_{[a,b]}})(\mathrm{in}_<(J_{K_n}))^{t+1} \subseteq \mathrm{in}_<(J_{K_{[a,b]}})(\mathrm{in}_<(J_{K_n}))^t$, a contradiction.

(2) $x_p|w$ and $y_q|\prod\limits_{k=1}^{t+1} x_{i_k}y_{j_k}$. Let $y_q = y_{j_l}$. Then

$$u = x_py_q\frac{w}{x_p}x_{i_l}\prod_{j\neq l} x_{i_k}y_{j_k} \in \mathrm{in}_<(J_{K_{[a,b]}})(\mathrm{in}_<(J_{K_n}))^t,$$

a contradiction.

(3) $x_p|\prod\limits_{k=1}^{t+1} x_{i_k}y_{j_k}$ and $y_q|w$. The same as (2).

(4) $x_py_q|\prod\limits_{k=1}^{t+1} x_{i_k}y_{j_k}$. Let $p = i_s$ and $q = j_l$. Then $s \neq l$, $i_l < a < b < j_s$ and

$$u = w(x_py_q)(x_{i_l}y_{j_s})\prod_{k\neq s,l} x_{i_k}y_{j_k} \in \mathrm{in}_<(J_{K_{[a,b]}})(\mathrm{in}_<(J_{K_n}))^t,$$

which is also a contradiction. Then the result follows. □

For a monomial ideal I of S, the set of minimal generators of I is denoted by $G(I)$.

Now, we verify the conditions in Lemma 2.6 in order to apply this lemma to get following Proposition 4.9.

Proposition 4.4. *For any* $1 \leqslant a < b \leqslant n$, $T \subseteq [n]$ *and* $s, t \geqslant 1$,

$$\mathrm{in}_<(J_{K_n}^t + J_{K_{[a,b]}}J_{K_n}^{t-2} + A_T^s) = (\mathrm{in}_<(J_{K_n}))^t + \mathrm{in}_<(J_{K_{[a,b]}})(\mathrm{in}_<(J_{K_n}))^{t-2} + A_T^s.$$

P r o o f. It is sufficient to show that $(\mathcal{G}(J_{K_n}))^t \cup \mathcal{G}(J_{K_{[a,b]}})(\mathcal{G}(J_{K_n}))^{t-2} \cup G(A_T^s)$ is a Gröbner basis of $J_{K_n}^t + J_{K_{[a,b]}}J_{K_n}^{t-2} + A_T^s$ with respect to $<$. For any h_1, h_2 from two different sets of the above three sets, let us show that the S-polynomial $S(h_1, h_2)$ has a standard expression with respect to the union of two sets with remainder 0.

(I) $h_1 \in (\mathcal{G}(J_{K_n}))^t$ or $\mathcal{G}(J_{K_{[a,b]}})(\mathcal{G}(J_{K_n}))^{t-2}$ and $h_2 \in G(A_T^s)$. Then $h_1 \in (\mathcal{G}(J_{K_n}))^{t'}$ where $t' = t$ or $t-1$. Write

$$h_1 = \prod_{k=1}^{t'} f_{i_kj_k} = u_1 + c_2u_2 + \ldots + c_ru_r,$$

where $u_1 > u_2 > \ldots > u_r$ are monomials of the same degrees and $c_i \in K^*$, $i = 2, 3, \ldots, r$. Then

$$S(h_1, h_2) = \sum_{i=2}^{r} \frac{\mathrm{lcm}(u_1, h_2)}{u_1}c_iu_i = \sum_{i=2}^{r} \frac{c_ih_2}{\gcd(u_1, h_2)}u_i.$$

Note that $u_1 = x_{i_1}x_{i_2}\ldots x_{i_{t'}}y_{j_1}y_{j_2}\ldots y_{j_{t'}}$ and if $u_1 \in A_T^d$ for some $d \geqslant 0$, then $u_i \in A_T^d$ for all $i = 2, 3, \ldots, r$. If $d \geqslant s$, then it is clear that $c_ih_2u_i/\gcd(u_1, h_2) \in A_T^s$,

$i = 2, 3, \ldots, r$. If $d < s$, then $h_2/\gcd(u_1, h_2) \in A_T^{s-d}$. Thus again $c_i h_2/\gcd(u_1, h_2) u_i \in A_T^s$, $i = 2, 3, \ldots, r$. Hence, $S(h_1, h_2)$ has a standard expression with respect to $G(A_T^s) \cup \{h_1\}$ with remainder 0.

(II) $h_1 \in (\mathcal{G}(J_{K_n}))^t$ and $h_2 \in \mathcal{G}(J_{K_{[a,b]}})$. Let us show that $S(h_1, h_2)$ has a standard expression with respect to $(\mathcal{G}(J_{K_n}))^t \cup \mathcal{G}(J_{K_{[a,b]}})$ with remainder 0 . We may assume that $h_1 = \prod_{k=1}^{t} f_{i_k j_k}$ with no $f_{i_k j_k} \in J_{K_{[a,b]}}$, $h_2 = f_{pq}$ and $\gcd(\mathrm{in}_<(h_1), \mathrm{in}_<(h_2)) \neq 1$. Then $\gcd(\mathrm{in}_<(h_1), \mathrm{in}_<(h_2)) = x_p$ or y_q or $x_p y_q$.

(i) $\gcd(\mathrm{in}_<(h_1), \mathrm{in}_<(h_2)) = x_p$. Let $p = i_1$. Then $p < q \leqslant b < j_1$. One has

$$\begin{aligned}
S(h_1, h_2) &= y_q h_1 - \left(y_{j_1} \prod_{k=2}^{t} x_{i_k} y_{j_k} \right) h_2 \\
&= y_q f_{p j_1} \prod_{k=2}^{t} f_{i_k j_k} - (y_{j_1} \prod_{k=2}^{t} x_{i_k} y_{j_k}) f_{pq} \\
&= (y_{j_1} f_{pq} + y_p f_{q j_1}) \prod_{k=2}^{t} f_{i_k j_k} - \left(y_{j_1} \prod_{k=2}^{t} x_{i_k} y_{j_k} \right) f_{pq} \\
&= y_p f_{q j_1} \prod_{k=2}^{t} f_{i_k j_k} + y_{j_1} \left(\prod_{k=2}^{t} f_{i_k j_k} - \prod_{k=2}^{t} x_{i_k} y_{j_k} \right) f_{pq},
\end{aligned}$$

which is a standard expression with respect to $f_{q j_1} \prod_{k=2}^{t} f_{i_k j_k}$ and f_{pq} with remainder 0 because $\mathrm{in}_<(y_p f_{q j_1}) < \mathrm{in}_<(y_q f_{i_1 j_1})$ and $\mathrm{in}_<\left(\prod_{k=2}^{t} f_{i_k j_k} - \prod_{k=2}^{t} x_{i_k} y_{j_k} \right) < \mathrm{in}_<\left(\prod_{k=2}^{t} f_{i_k j_k} \right)$.

(ii) $\gcd(\mathrm{in}_<(h_1), \mathrm{in}_<(h_2)) = y_q$. The same as (i).

(iii) $\gcd(\mathrm{in}_<(h_1), \mathrm{in}_<(h_2)) = x_p y_q$. We may assume that $p = i_1$ and $q = j_2$. Then $i_2 < a \leqslant p < q \leqslant b < j_1$. We have

$$\begin{aligned}
S(h_1, h_2) &= \prod_{k=1}^{t} f_{i_k j_k} - x_{i_2} y_{j_1} \left(\prod_{k=3}^{t} x_{i_k} y_{j_k} \right) f_{pq} \\
&= f_{p j_1} f_{i_2 q} \prod_{k=3}^{t} f_{i_k j_k} - x_{i_2} y_{j_1} \left(\prod_{k=3}^{t} x_{i_k} y_{j_k} \right) f_{pq} \\
&= (f_{pq} f_{i_2 j_1} + f_{i_2 p} f_{q j_1}) \prod_{k=3}^{t} f_{i_k j_k} - x_{i_2} y_{j_1} \left(\prod_{k=3}^{t} x_{i_k} y_{j_k} \right) f_{pq} \\
&= f_{i_2 p} f_{q j_1} \prod_{k=3}^{t} f_{i_k j_k} + (f_{i_2 j_1} - x_{i_2} y_{j_1}) \left(\prod_{k=3}^{t} x_{i_k} y_{j_k} \right) f_{pq},
\end{aligned}$$

which is also a standard expression with respect to $f_{i_2 p} f_{q j_1} \prod_{k=3}^{t} f_{i_k j_k}$ and f_{pg} with remainder 0 because $\mathrm{in}_<(f_{i_2 p} f_{q j_1}) < \mathrm{in}_<(f_{p j_1} f_{i_2 q})$ and $\mathrm{in}_<(f_{i_2 j_1} - x_{i_2} y_{j_1}) < \mathrm{in}_<(f_{i_2 j_1})$.

(III) $t > 2$ and $(\mathcal{G}(J_{K_n}))^t \cup \mathcal{G}(J_{K_{[a,b]}})(\mathcal{G}(J_{K_n}))^{t-2}$ forms a Gröbner basis with respect to $<$. As in the proof of Lemma 4.3, it is sufficient to show that

$$\mathrm{in}_<(J_{K_n}^t + J_{K_{[a,b]}} J_{K_n}^{t-2}) \subseteq (\mathrm{in}_<(J_{K_n}))^t + \mathrm{in}_<(J_{K_{[a,b]}})(\mathrm{in}_<(J_{K_n}))^{t-2}.$$

Suppose on the contrary that there is one monomial $u \in \mathrm{in}_<(J_{K_n}^t + J_{K_{[a,b]}} J_{K_n}^{t-2}) \setminus (\mathrm{in}_<(J_{K_n}))^t + \mathrm{in}_<(J_{K_{[a,b]}})(\mathrm{in}_<(J_{K_n}))^{t-2}$. Then $u \in (\mathrm{in}_<(J_{K_n}))^{t-1}$, $u \notin (\mathrm{in}_<(J_{K_n}))^t$ and $u \notin \mathrm{in}_<(J_{K_{[a,b]}})(\mathrm{in}_<(J_{K_n}))^{t-2}$. On the other hand, $\mathrm{in}_<(J_{K_n}^t + J_{K_{[a,b]}} J_{K_n}^{t-2}) \subseteq \mathrm{in}_<(J_{K_n}^t + J_{K_{[a,b]}}) = (\mathrm{in}_<(J_{K_n}))^t + \mathrm{in}_<(J_{K_{[a,b]}})$ by (II), which implies that $u \in \mathrm{in}_<(J_{K_{[a,b]}})$. Thus, $u = w \prod\limits_{k=1}^{t-1} x_{i_k} y_{j_k}$ with no $x_{i_k} y_{j_k} \in \mathrm{in}_<(J_{K_{[a,b]}})$ and $x_p y_q | u$ for some $a \leqslant p < q \leqslant b$. This situation leads to a contradiction by the same argument as in the proof of Lemma 4.3. This completes the proof. $\square$

Lemma 4.5. *For any $T \subseteq [n]$, $1 \leqslant k < l \leqslant n$ and $s \geqslant 1$, $A_T^s : f_{kl} = A_T^{s-|\{k,l\}\cap T|}$ and $\mathrm{in}_<(A_T^s : f_{kl}) = A_T^s : \mathrm{in}_<(f_{kl})$.*

Proof. $|\{k,l\} \cap T|$ has three possible values 0, 1, 2. If $|\{k,l\} \cap T| = 0$, then no variable in f_{kl} appears in A_T, hence $A_T^s : f_{kl} = A_T^s$. If $|\{k,l\} \cap T| = 1$, then $f_{kl} \in A_T$ and $A_T^{s-1} \subseteq A_T^s : f_{kl}$. On the other hand, one has

$$\mathrm{in}_<(A_T^s : f_{kl}) \subseteq \mathrm{in}_<(A_T^s) : \mathrm{in}_<(f_{kl}) = A_T^s : x_k y_l \subseteq A_T^{s-1}.$$

Then $A_T^s : f_{kl} = A_T^{s-1}$ and $\mathrm{in}_<(A_T^s : f_{kl}) = A_T^s : \mathrm{in}_<(f_{kl})$. Finally, suppose that $|\{k,l\} \cap T| = 2$. Then $f_{kl} \in A_T^2$ and $A_T^{s-2} \subseteq A_T^s : f_{kl}$. From $\mathrm{in}_<(A_T^s : f_{kl}) \subseteq A_T^s : x_k y_l \subseteq A_T^{s-2}$, one has $A_T^s : f_{kl} = A_T^{s-2}$ and $\mathrm{in}_<(A_T^s : f_{kl}) = A_T^s : \mathrm{in}_<(f_{kl})$. $\square$

It is easy to show the following lemma, as $\mathrm{in}_<(J_{K_n})$ is a monomial ideal.

Lemma 4.6. *For any $t \geqslant 1$, $(\mathrm{in}_<(J_{K_n}))^t : x_1 y_2 = (\mathrm{in}_<(J_{K_n}))^{t-1}$.*

Lemma 4.7. *For any $t \geqslant 1$, $J_{K_n}^t : f_{12} = J_{K_n}^{t-1}$ and $\mathrm{in}_<(J_{K_n}^t : f_{12}) = \mathrm{in}_<(J_{K_n}^t) : \mathrm{in}_<(f_{12})$.*

Proof. It is clear that $J_{K_n}^{t-1} \subseteq J_{K_n}^t : f_{12}$. On the other hand, since

$$\mathrm{in}_<(J_{K_n}^t : f_{12}) \subseteq \mathrm{in}_<(J_{K_n}^t) : \mathrm{in}_<(f_{12}) = (\mathrm{in}_<(J_{K_n}))^t : \mathrm{in}_<(f_{12}) = \mathrm{in}_<(J_{K_n}^{t-1})$$

by Lemma 4.6, it follows that $J_{K_n}^t : f_{12} = J_{K_n}^{t-1}$ and $\mathrm{in}_<(J_{K_n}^t : f_{12}) = \mathrm{in}_<(J_{K_n}^t) : \mathrm{in}_<(f_{12})$. $\square$

Lemma 4.8. *For any $1 < c \leqslant n$ and $t \geqslant 0$, $J_{K_{[2,c]}} J_{K_n}^t : f_{12} = J_{K_{[2,c]}} J_{K_n}^{t-1}$ and $\mathrm{in}_<(J_{K_{[2,c]}} J_{K_n}^t : f_{12}) = \mathrm{in}_<(J_{K_{[2,c]}} J_{K_n}^t) : \mathrm{in}_<(f_{12})$.*

Proof. Clearly $J_{K_{[2,c]}} J_{K_n}^{t-1} \subseteq J_{K_{[2,c]}} J_{K_n}^{t} : f_{12}$. Note that, by Lemma 4.3,

$$\mathrm{in}_<(J_{K_{[2,c]}} J_{K_n}^t : f_{12}) \subseteq \mathrm{in}_<(J_{K_{[2,c]}} J_{K_n}^t) : x_1y_2 = \mathrm{in}_<(J_{K_{[2,c]}})(\mathrm{in}_<(J_{K_n}))^t : x_1y_2.$$

However, $\mathrm{in}_<(J_{K_{[2,c]}})(\mathrm{in}_<(J_{K_n}))^t$ is a product of monomial ideals and x_1y_2 is coprime with all the generators of $\mathrm{in}_<(J_{K_{[2,c]}})$. This implies that

$$\mathrm{in}_<(J_{K_{[2,c]}})(\mathrm{in}_<(J_{K_n}))^t : x_1y_2 = \mathrm{in}_<(J_{K_{[2,c]}})((\mathrm{in}_<(J_{K_n}))^t : x_1y_2).$$

Hence, by Lemma 4.6,

$$\mathrm{in}_<(J_{K_{[2,c]}} J_{K_n}^t : f_{12}) \subseteq \mathrm{in}_<(J_{K_{[2,c]}})(\mathrm{in}_<(J_{K_n}))^{t-1} = \mathrm{in}_<(J_{K_{[2,c]}} J_{K_n}^{t-1}).$$

Then $J_{K_{[2,c]}} J_{K_n}^t : f_{12} = J_{K_{[2,c]}} J_{K_n}^{t-1}$ and $\mathrm{in}_<(J_{K_{[2,c]}} J_{K_n}^t : f_{12}) = \mathrm{in}_<(J_{K_{[2,c]}} J_{K_n}^t) : \mathrm{in}_<(f_{12})$. $\square$

Proposition 4.9. *Let $1 \leqslant a < b \leqslant n$ and $T \subseteq [n]$. Then, for any $t \geqslant 2$,*

$$(J_{K_n}^t + A_T^s + J_{K_{b-1}} J_{K_n}^{t-2}) : f_{ab} = J_{K_n}^{t-1} + A_T^{s-|\{a,b\}\cap T|} + J_{K_{b-1}} J_{K_n}^{t-3}.$$

Proof. Note that $(J_{K_n}^t + A_T^s + J_{K_{b-1}} J_{K_n}^{t-2}) : f_{ab} = (J_{K_n}^t + A_T^s + J_{K_{b-1}} J_{K_n}^{t-2}) : (-f_{ab})$ and $-f_{ab} = x_by_a - x_ay_b$. Reverse the order of $1, 2, \ldots, n$, it is equivalent to showing that, for $1 \leqslant a < b \leqslant n$,

$$(J_{K_n}^t + A_T^s + J_{K_{[a+1,n]}} J_{K_n}^{t-2}) : f_{ab} = J_{K_n}^{t-1} + A_T^{s-|\{a,b\}\cap T|} + J_{K_{[a+1,n]}} J_{K_n}^{t-3}.$$

Then, reordering $1, 2, \ldots, n$, we may assume that $a = 1$, $b = 2$ and need to prove that, for any $2 \leqslant c \leqslant n$,

$$(J_{K_n}^t + A_T^s + J_{K_{[2,c]}} J_{K_n}^{t-2}) : f_{12} = J_{K_n}^{t-1} + A_T^{s-|\{1,2\}\cap T|} + J_{K_{[2,c]}} J_{K_n}^{t-3}.$$

Then, in virtue of Lemma 2.6 and by Proposition 4.4, Lemmas 4.5, 4.7 and 4.8, we get the result. $\square$

The following result plays an important role in the proof of our main theorem.

Proposition 4.10. *Let $t \geqslant 1$, $r = \binom{n}{2}^{t-1}$ and $(\mathcal{G}(J_{K_n}))^{t-1} = \{F_1, F_2, \ldots, F_r\}$ ordered by a lexicographic order $<_e$ induced by*

$$f_{12} >_e f_{13} >_e \ldots >_e f_{1n} >_e f_{23} >_e f_{24} >_e \ldots >_e f_{2n} >_e f_{34} >_e \ldots >_e f_{n-1,n}.$$

Then, for any $s \geqslant t \geqslant 1$, $1 \leqslant m \leqslant n$ and $l = 1, 2, \ldots, r$, if $F_l \notin J_{K_n}^t + A_m^s + (F_1, F_2, \ldots, F_{l-1})$, write $F_l = \prod\limits_{k=1}^{u} f_{a_k,b_k}^{c_k}$ with $f_{a_1,b_1} >_e f_{a_2,b_2} >_e \ldots >_e f_{a_u,b_u}$, $c_k > 0$, $k = 1, 2, \ldots, u$, and $\sum\limits_{k=1}^{u} c_k = t - 1$, and obtain

$$(J_{K_n}^t + A_m^s + (F_1, F_2, \ldots, F_{l-1})) : F_l = J_{K_{[a_u,n]}} + A_{[a_u,m]}^{s'} + A_{a_u-1},$$

where $s' = s - \sum\limits_{k=1}^{u} c_k |\{a_k, b_k\} \cap [m]|$.

Proof. Firstly note that from $f_{a_1,b_1} >_e f_{a_2,b_2} >_e \ldots >_e f_{a_u,b_u}$ we have $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_u$. Further, it also holds that $b_1 \leqslant b_2 \leqslant \ldots \leqslant b_u$. Assume, on the contrary, that $b_i > b_j$ for some $i < j$, then $a_i < a_j < b_j < b_i$. By Lemma 2.8, $f_{a_i,b_i} f_{a_j,b_j} = f_{a_i,b_j} f_{a_j,b_i} - f_{a_i,a_j} f_{b_j,b_i}$. But $f_{a_i,b_i} f_{a_j,b_j} <_e f_{a_i,b_j} f_{a_j,b_i}, f_{a_i,a_j} f_{b_j,b_i}$ and it follows that

$$\begin{aligned}F_l &= f_{a_i,b_i} f_{a_j,b_j} \Bigl(\prod_{k \neq i,j} f_{a_k,b_k}^{c_k}\Bigr)(f_{a_i,b_i}^{c_i-1} f_{a_j,b_j}^{c_j-1}) \\ &= (f_{a_i,b_j} f_{a_j,b_i} - f_{a_i,a_j} f_{b_j,b_i})\Bigl(\prod_{k \neq i,j} f_{a_k,b_k}^{c_k}\Bigr)(f_{a_i,b_i}^{c_i-1} f_{a_j,b_j}^{c_j-1}) \in (F_1, \ldots, F_{l-1}),\end{aligned}$$

a contradiction. Note also that, for $i = 1, 2, \ldots, a_u - 1$,

$$(*) \qquad (x_i, y_i) F_l \subseteq (F_1, F_2, \ldots, F_{l-1}),$$

because $x_i f_{a_u,b_u} = x_{a_u} f_{i,b_u} - x_{b_u} f_{i,a_u}$ by Lemma 2.8 and $f_{a_u,b_u} <_e f_{i,b_u}, f_{i,a_u}$, then it follows that

$$\begin{aligned}x_i F_l &= x_i f_{a_u,b_u} f_{a_u,b_u}^{c_u-1} \prod_{k=1}^{u-1} f_{a_k,b_k}^{c_k} \\ &= (x_{a_u} f_{i,b_u} - x_{b_u} f_{i,a_u}) f_{a_u,b_u}^{c_u-1} \prod_{k=1}^{u-1} f_{a_k,b_k}^{c_k} \in (F_1, F_2, \ldots, F_{l-1})\end{aligned}$$

and similarly $y_i F_l \in (F_1, F_2, \ldots, F_{l-1})$. Then $A_{a_u-1} \subseteq (F_1, F_2, \ldots, F_{l-1}) : F_l$. By Lemma 4.5, $A_{[a_u,m]}^{s'} + A_{a_u-1} = (A_m^s : F_l) + A_{a_u-1}$. Hence

$$J_{K_{[a_u,n]}} + A_{[a_u,m]}^{s'} + A_{a_u-1} \subseteq (J_{K_n}^t + A_m^s + (F_1, F_2, \ldots, F_{l-1})) : F_l.$$

Let us show the converse containment.

Set $J = (F \in (\mathcal{G}(J_{K_n}))^{t-1-c_1} \colon f_{a_1,b_1}^{t-1-c_1} \geqslant_e F >_e f_{a_2,b_2}^{c_2} \cdots f_{a_u,b_u}^{c_u})$. Then

$$(**) \qquad (F_1, F_2, \ldots, F_{l-1}) \subseteq J_{K_{b_1-1}} J_{K_n}^{t-2} + f_{a_1,b_1}^{c_1} J.$$

Let us use the induction on u to show that

$$(J_{K_n}^t + A_m^s + J_{K_{b_1-1}}J_{K_n}^{t-2} + f_{a_1,b_1}^{c_1}J) : F_l \subseteq J_{K_{[a_u,n]}} + A_{[a_u,m]}^{s'} + A_{a_u-1},$$

then the above converse containment follows.

If $u = 1$, we need to show that

$$(J_{K_n}^t + A_m^s + J_{K_{b_1-1}}J_{K_n}^{t-2}) : f_{a_1,b_1}^{t-1} \subseteq J_{K_{[a_1,n]}} + A_{[a_1,m]}^{s-(t-1)|\{a_1,b_1\}\cap[m]|} + A_{a_1-1},$$

which is true by applying Proposition 4.9 $(t-1)$ times as

$$J_{K_{[a_1,n]}} + A_{[a_1,m]}^{s-(t-1)|\{a_1,b_1\}\cap[m]|} + A_{a_1-1} \supseteq J_{K_n} + A_m^{s-(t-1)|\{a_1,b_1\}\cap[m]|} + J_{K_{b_1-1}}.$$

Now assume that $u > 1$ and the result is true for $u-1$. Then

$$\begin{aligned}&(J_{K_n}^t + A_m^s + J_{K_{b_1-1}}J_{K_n}^{t-2} + f_{a_1,b_1}^{c_1}J) : F_l \\ &\qquad = ((J_{K_n}^t + A_m^s + J_{K_{b_1-1}}J_{K_n}^{t-2} + f_{a_1,b_1}^{c_1}J) : f_{a_1,b_1}^{c_1}) : f_{a_2,b_2}^{c_2} \cdots f_{a_u,b_u}^{c_u}.\end{aligned}$$

But $(J_{K_n}^t + A_m^s + J_{K_{b_1-1}}J_{K_n}^{t-2} + f_{a_1,b_1}^{c_1}J) : f_{a_1,b_1}^{c_1} = (J_{K_n}^t + A_m^s + J_{K_{b_1-1}}J_{K_n}^{t-2}) : f_{a_1,b_1}^{c_1} + J$. Then, by Proposition 4.9,

$$\begin{aligned}&(J_{K_n}^t + A_m^s + J_{K_{b_1-1}}J_{K_n}^{t-2} + f_{a_1,b_1}^{c_1}J) : f_{a_1,b_1}^{c_1} \\ &\qquad = J_{K_n}^{t-c_1} + A_m^{s-c_1|\{a_1,b_1\}\cap[m]|} + J_{K_{b_1-1}}J_{K_n}^{t-2-c_1} + J.\end{aligned}$$

Note that $A_{a_2-1} \subseteq (F_1, F_2, \ldots, F_{l-1}) : F_l$ by $(*)$. Using $(**)$, we have

$$\begin{aligned}&(J_{K_n}^t + A_m^s + J_{K_{b_1-1}}J_{K_n}^{t-2} + f_{a_1,b_1}^{c_1}J) : F_l \\ &\quad = (J_{K_n}^{t-c_1} + A_m^{s-c_1|\{a_1,b_1\}\cap[m]|} + J_{K_{b_1-1}}J_{K_n}^{t-2-c_1} + J) : f_{a_2,b_2}^{c_2} \cdots f_{a_u,b_u}^{c_u} + A_{a_2-1}.\end{aligned}$$

Then

$$\begin{aligned}&(J_{K_n}^t + A_m^s + J_{K_{b_1-1}}J_{K_n}^{t-2} + f_{a_1,b_1}^{c_1}J) : F_l \\ &\quad \subseteq (J_{K_n}^{t-c_1} + A_m^{s-c_1|\{a_1,b_1\}\cap[m]|} + J_{K_{b_1-1}}J_{K_n}^{t-2-c_1} + J + A_{a_2-1}) : f_{a_2,b_2}^{c_2} \cdots f_{a_u,b_u}^{c_u} \\ &\quad = (J_{K_{[a_2,n]}}^{t-c_1} + A_{[a_2,m]}^{s-c_1|\{a_1,b_1\}\cap[m]|} + J_{K_{[a_2,b_1-1]}}J_{K_{[a_2,n]}}^{t-2-c_1} + J' + A_{a_2-1}) : f_{a_2,b_2}^{c_2} \cdots f_{a_u,b_u}^{c_u},\end{aligned}$$

where $J' = (F \in (\mathcal{G}(J_{K_{[a_2,n]}}))^{t-1-c_1} : F >_e f_{a_2,b_2}^{c_2} \cdots f_{a_u,b_u}^{c_u})$. Again, by the same argument as for $(F_1, F_2, \ldots, F_{l-1}) \subseteq J_{K_{b_1-1}}J_{K_n}^{t-2} + f_{a_1,b_1}^{c_1}J$, we have $J' \subseteq$

$J_{K_{[a_2,b_2-1]}} J^{t-2-c_1}_{K_{[a_2,n]}} + f^{c_2}_{a_2,b_2} J''$ where $J'' = (F \in (\mathcal{G}(J_{K_{[a_2,n]}}))^{t-1-c_1-c_2} : f^{t-1-c_1-c_2}_{a_2,b_2} \geqslant_e F >_e f^{c_3}_{a_3,b_3} \dots f^{c_u}_{a_u,b_u})$. It turns out that

$$\begin{aligned}
&J^{t-c_1}_{K_{[a_2,n]}} + A^{s-c_1|\{a_1,b_1\}\cap[m]|}_{[a_2,m]} + J_{K_{[a_2,b_1-1]}} J^{t-2-c_1}_{K_{[a_2,n]}} + J' + A_{a_2-1}\\
&\quad \subseteq J^{t-c_1}_{K_{[a_2,n]}} + A^{s-c_1|\{a_1,b_1\}\cap[m]|}_{[a_2,m]} + J_{K_{[a_2,b_1-1]}} J^{t-2-c_1}_{K_{[a_2,n]}}\\
&\quad\quad + J_{K_{[a_2,b_2-1]}} J^{t-2-c_1}_{K_{[a_2,n]}} + f^{c_2}_{a_2,b_2} J'' + A_{a_2-1}\\
&\quad = J^{t-c_1}_{K_{[a_2,n]}} + A^{s-c_1|\{a_1,b_1\}\cap[m]|}_{[a_2,m]} + J_{K_{[a_2,b_2-1]}} J^{t-2-c_1}_{K_{[a_2,n]}} + f^{c_2}_{a_2,b_2} J'' + A_{a_2-1}
\end{aligned}$$

as $b_1 \leqslant b_2$. Therefore

$$\begin{aligned}
&(J^t_{K_n} + A^s_m + J_{K_{b_1-1}} J^{t-2}_{K_n} + f^{c_1}_{a_1,b_1} J) : F_l\\
&\quad \subseteq (J^{t-c_1}_{K_{[a_2,n]}} + A^{s-c_1|\{a_1,b_1\}\cap[m]|}_{[a_2,m]} + J_{K_{[a_2,b_2-1]}} J^{t-2-c_1}_{K_{[a_2,n]}}\\
&\quad\quad + f^{c_2}_{a_2,b_2} J'' + A_{a_2-1}) : f^{c_2}_{a_2,b_2} \dots f^{c_u}_{a_u,b_u}\\
&\quad = (J^{t-c_1}_{K_{[a_2,n]}} + A^{s-c_1|\{a_1,b_1\}\cap[m]|}_{[a_2,m]} + J_{K_{[a_2,b_2-1]}} J^{t-2-c_1}_{K_{[a_2,n]}}\\
&\quad\quad + f^{c_2}_{a_2,b_2} J'') : f^{c_2}_{a_2,b_2} \dots f^{c_u}_{a_u,b_u} + A_{a_2-1}
\end{aligned}$$

where the last equality follows by Lemma 2.10.

Note that from $F_l \notin J^t_{K_n} + A^s_m + (F_1, F_2, \dots, F_{l-1})$ we get that $f^{c_2}_{a_2,b_2} \dots f^{c_u}_{a_u,b_u} \notin J^{t-c_1}_{K_{[a_2,n]}} + A^{s-c_1|\{a_1,b_1\}\cap[m]|}_{[a_2,m]} + J'$. Then, applying the induction hypothesis, we have

$$\begin{aligned}
&(J^{t-c_1}_{K_{[a_2,n]}} + A^{s-c_1|\{a_1,b_1\}\cap[m]|}_{[a_2,m]} + J_{K_{[a_2,b_2-1]}} J^{t-2-c_1}_{K_{[a_2,n]}} + f^{c_2}_{a_2,b_2} J'') : f^{c_2}_{a_2,b_2} \dots f^{c_u}_{a_u,b_u}\\
&\quad \subseteq J_{K_{[a_u,n]}} + A^{s^*}_{[a_u,m]} + A_{[a_2,a_u-1]},
\end{aligned}$$

where $s^* = s - c_1|\{a_1,b_1\}\cap[m]| - \sum\limits_{k=2}^{u} c_k|\{a_k,b_k\}\cap[a_2,m]|$. As noted at the beginning of the proof, $a_2 \leqslant a_3 \leqslant \ldots \leqslant a_u$ and $a_2 \leqslant b_2 \leqslant b_3 \leqslant \ldots \leqslant b_u$, so we see that $\{a_k,b_k\}\cap[a_2,m] = \{a_k,b_k\}\cap[m]$ for $k \geqslant 2$. It follows that $s^* = s'$. Then

$$\begin{aligned}
(J^t_{K_n} + A^s_m + J_{K_{b_1-1}} J^{t-2}_{K_n} + f^{c_1}_{a_1,b_1} J) : F_l &\subseteq J_{K_{[a_u,n]}} + A^{s'}_{[a_u,m]} + A_{[a_2,a_u-1]} + A_{a_2-1}\\
&= J_{K_{[a_u,n]}} + A^{s'}_{[a_u,m]} + A_{a_u-1},
\end{aligned}$$

as required. □

Theorem 4.11. *For any $1 \leqslant m \leqslant n$ and $s \geqslant t \geqslant 1$, $\operatorname{reg}(J^t_{K_n} + A^s_m) \leqslant 2s$.*

Proof. We prove the result by using induction on t. If $t = 1$, then, as $\operatorname{in}_<(J_{K_n} + A^s_m) = \operatorname{in}_<(J_{K_n}) + A^s_m$ by Proposition 4.4 and $\operatorname{reg}(\operatorname{in}_<(J_{K_n}) + A^s_m) \leqslant s+1$ by Lemma 2.9, we have

$$\operatorname{reg}(J_{K_n} + A^s_m) \leqslant s + 1 \leqslant 2s.$$

Now assume that $t > 1$ and the result is true for $t-1$. Set $(\mathcal{G}(J_{K_n}))^{t-1} = \{F_1, F_2, \ldots, F_r\}$ as in above Proposition 4.10. We have the exact sequences

$$
\begin{aligned}
&0 \to \frac{S}{(J_{K_n}^t + A_m^s) : F_1}(-(2t-2)) \xrightarrow{\cdot F_1} \frac{S}{J_{K_n}^t + A_m^s} \to \frac{S}{J_{K_n}^t + A_m^s + (F_1)} \to 0,\\
&0 \to \frac{S}{(J_{K_n}^t + A_m^s + (F_1)) : F_2}(-(2t-2)) \xrightarrow{\cdot F_2} \frac{S}{J_{K_n}^t + A_m^s + (F_1)}\\
&\quad \to \frac{S}{J_{K_n}^t + A_m^s + (F_1, F_2)} \to 0,\\
&\quad \vdots\\
&0 \to \frac{S}{(J_{K_n}^t + A_m^s + (F_1, \ldots, F_{r-2})) : F_{r-1}}(-(2t-2))\\
&\quad \xrightarrow{\cdot F_{r-1}} \frac{S}{J_{K_n}^t + A_m^s + (F_1, \ldots, F_{r-2})} \to \frac{S}{J_{K_n}^t + A_m^s + (F_1, \ldots, F_{r-1})} \to 0,\\
&0 \to \frac{S}{(J_{K_n}^t + A_m^s + (F_1, \ldots, F_{r-1})) : F_r}(-(2t-2))\\
&\quad \xrightarrow{\cdot F_r} \frac{S}{J_{K_n}^t + A_m^s + (F_1, \ldots, F_{r-1})} \to \frac{S}{J_{K_n}^{t-1} + A_m^s} \to 0.
\end{aligned}
$$

For $l = 1, 2, \ldots, r$, if $F_l \notin J_{K_n}^t + A_m^s + (F_1, \ldots, F_{l-1})$, write $F_l = \prod_{k=1}^{u} f_{a_k,b_k}^{c_k}$ with $f_{a_1,b_1} >_e f_{a_2,b_2} >_e \ldots >_e f_{a_u,b_u}$, $c_k > 0, k = 1, 2, \ldots, u$, and $\sum_{k=1}^{u} c_k = t-1$, then, by Proposition 4.10

$$(J_{K_n}^t + A_m^s + (F_1, \ldots, F_{l-1})) : F_l = J_{K_{[a_u,n]}} + A_{[a_u,m]}^{s'} + A_{a_u-1},$$

where $s' = s - \sum_{k=1}^{u} c_k |\{a_k, b_k\} \cap [m]|$. For the regularity, by Lemma 2.3, $\operatorname{reg}(J_{K_{[a_u,n]}} + A_{[a_u,m]}^{s'} + A_{a_u-1}) = \operatorname{reg}(J_{K_{[a_u,n]}} + A_{[a_u,m]}^{s'}) + \operatorname{reg}(A_{a_u-1}) = \operatorname{reg}(J_{K_{[a_u,n]}} + A_{[a_u,m]}^{s'})$. Note that if $a_u > m$ then $A_{[a_u,m]} = \emptyset$ and if $a_u \leqslant m$ then $\{a_k, b_k\} \cap [m] \neq \emptyset$ for all $k = 1, \ldots, u$, hence $s' = s - \sum_{k=1}^{u} c_k |\{a_k, b_k\} \cap [m]| \leqslant s - \sum_{k=1}^{u} c_k = s - t + 1$. Then either $\operatorname{reg}(J_{K_{[a_u,n]}} + A_{[a_u,m]}^{s'} + A_{a_u-1}) = \operatorname{reg}(J_{K_{[a_u,n]}}) = 2$ or $\operatorname{reg}(J_{K_{[a_u,n]}} + A_{[a_u,m]}^{s'} + A_{a_u-1}) = \operatorname{reg}(J_{K_{[a_u,n]}} + A_{[a_u,m]}^{s'}) \leqslant s' + 1 \leqslant s - t + 2$ by Lemmas 2.4 and 2.9. It follows that $\operatorname{reg}(J_{K_{[a_u,n]}} + A_{[a_u,m]}^{s'} + A_{a_u-1}) \leqslant 2s - 2t + 2$ in any cases.

By the induction hypothesis, $\operatorname{reg}(S/(J_{K_n}^{t-1} + A_m^s)) \leqslant 2s-1$. Then by applying Regularity Lemma 2.1 to the above last exact sequence, we get that $\operatorname{reg}(S/(J_{K_n}^t + A_m^s + (F_1, \ldots, F_{r-1}))) \leqslant 2s-1$. Similarly, from the above last but one exact sequence, one has $\operatorname{reg}(S/(J_{K_n}^t + A_m^s + (F_1, \ldots, F_{r-2}))) \leqslant 2s-1$, and so on. Finally, from the first exact sequence , we obtain that $\operatorname{reg}(S/(J_{K_n}^t + A_m^s)) \leqslant 2s-1$, as required. □

Corollary 4.12. *For any* $1 \leqslant m \leqslant n$ *and* $s \geqslant t \geqslant 1$, $\operatorname{reg}(J_{K_n}^t \cap A_m^s) \leqslant 2s+1$.

Proof. Since $\operatorname{reg}(S/J_{K_n}^t) = 2t-1$, $\operatorname{reg}(S/A_m^s) = s-1$ and $\operatorname{reg}(S/(J_{K_n}^t + A_m^s)) \leqslant 2s-1$ by Theorem 4.11, the result follows by applying Regularity Lemma 2.1 to the short exact sequence

$$0 \to \frac{S}{J_{K_n}^t \cap A_m^s} \to \frac{S}{J_{K_n}^t} \oplus \frac{S}{A_m^s} \to \frac{S}{J_{K_n}^t + A_m^s} \to 0.$$

□

5. The main result

Now we come to the computation of regularity of powers of binomial edge ideals of complete multipartite graphs. First we have

Proposition 5.1. *Let* $J_{K_{n_1,n_2,\ldots,n_r}}$ *be a complete* r*-partite graph on* $[n]$ *with* $n > r > 1$. *Then* $\operatorname{reg}(J_{K_{n_1,n_2,\ldots,n_r}}) = 3$.

Proof. Assume that $n_1 \leqslant n_2 \leqslant \ldots \leqslant n_r$. Let us prove the result by induction on r.

If $r = 2$, it is known by [15], Theorem 1.1. Now assume that $r > 2$ and the result is true for $r-2$. Let G be the graph on $[n_1]$ with the empty edge set. Then $K_{n_1,n_2,\ldots,n_r}$ is the join product of G and $K_{n_2,n_3,\ldots,n_r}$, i.e., $K_{n_1,n_2,\ldots,n_r} = G * K_{n_2,n_3,\ldots,n_r}$. Then, by [14], Theorem 2.1,

$$\operatorname{reg}(J_{K_{n_1,n_2,\ldots,n_r}}) = \max\{\operatorname{reg}(J_G), \operatorname{reg}(J_{K_{n_2,n_3,\ldots,n_r}}), 3\} = \max\{0,3,3\} = 3.$$

□

The following lemma will be used in the proof of the main theorem.

Lemma 5.2. *For* $1 \leqslant m \leqslant n$, $J_{K_n}^t = (J_{K_n}^t \cap A_m^t) + (J_{K_n}^t \cap A_{[m+1,n]}^t)$.

Proof. For any $g \in (\mathcal{G}(J_{K_n}))^t$, let us show that either $g \in A_m^t$ or $g \in A_{[m+1,n]}^t$. Write $g = \prod\limits_{k=1}^{t} f_{a_k b_k} = c_1u_1 + c_2u_2 + \ldots + c_r u_s$, where $u_1, u_2, \ldots, u_s$ are monomials and $c_i \in K^*$, $i = 1,2,\ldots,s$. Let d be the number of elements of $a_1, a_2, \ldots, a_t, b_1, b_2, \ldots, b_t$ which are in $[m]$. Then $u_i \in A_m^d \cap A_{[m+1,n]}^{2t-d}$, $i = 1,2,\ldots,s$. It follows that either $g \in A_m^t$ or $g \in A_{[m+1,n]}^t$. □

Now we can prove our main result.

Theorem 5.3. *Let $G = J_{K_{n_1,n_2,\ldots,n_r}}$ be a complete r-partite graph on $[n]$ with $n > r > 1$. Then, for any $t \geqslant 1$,*

$$\operatorname{reg}(J_G^t) = 2t + 1.$$

P r o o f. When $t = 1$, it is Proposition 5.1. Assume that $t \geqslant 2$. Note that, since the length of the longest induced path in G is 2, $\operatorname{reg}(J_G^t) \geqslant 2t + 1$ by [10], Corollary 3.4. Hence, it is sufficient to show that $\operatorname{reg}(J_G^t) \leqslant 2t + 1$.

Use the same notations as Section 3. Then $J_G^t = J_{K_n}^t \cap P_{s+1}^t \cap P_{s+2}^t \cap \ldots \cap P_r^t$, where $P_i = \Bigl(x_j, y_j\colon\ j \notin \Bigl[1+\sum\limits_{k=1}^{i-1} n_k, \sum\limits_{k=1}^{i} n_k\Bigr]\Bigr)$. Set $Q_i = J_{K_n}^t \cap P_i^t \cap P_{i+1}^t \cap \ldots \cap P_r^t$, $i = s+1, s+2, \ldots, r$. Then $Q_{s+1} = J_G^t$, $Q_i = (J_{K_n}^t \cap P_i^t) \cap Q_{i+1}$, $i = s+1, s+2, \ldots, r-1$, and from

$$\begin{aligned} J_{K_n}^t &\supseteq (J_{K_n}^t \cap P_i^t) + Q_{i+1} = (J_{K_n}^t \cap P_i^t) + (J_{K_n}^t \cap P_{i+1}^t \cap P_{i+2}^t \cap \ldots \cap P_r^t) \\ &\supseteq (J_{K_n}^t \cap P_i^t) + (J_{K_n}^t \cap (P_i')^t) = J_{K_n}^t, \end{aligned}$$

where $P_i' = \Bigl(x_j, y_j\colon\ j \in \Bigl[1+\sum\limits_{k=1}^{i-1} n_k, \sum\limits_{k=1}^{i} n_k\Bigr]\Bigr)$ and the last equality follows by Lemma 5.2, one has $(J_{K_n}^t \cap P_i^t) + Q_{i+1} = J_{K_n}^t$, $i = s+1, s+2, \ldots, r-1$. We have the following short exact sequences:

$$\begin{aligned}
&0 \to \frac{S}{J_G^t} \to \frac{S}{J_{K_n}^t \cap P_{s+1}^t} \oplus \frac{S}{Q_{s+2}} \to \frac{S}{J_{K_n}^t} \to 0,\\
&0 \to \frac{S}{Q_{s+2}} \to \frac{S}{J_{K_n}^t \cap P_{s+2}^t} \oplus \frac{S}{Q_{s+3}} \to \frac{S}{J_{K_n}^t} \to 0,\\
&\qquad\vdots\\
&0 \to \frac{S}{Q_{r-2}} \to \frac{S}{J_{K_n}^t \cap P_{r-2}^t} \oplus \frac{S}{Q_{r-1}} \to \frac{S}{J_{K_n}^t} \to 0,\\
&0 \to \frac{S}{Q_{r-1}} \to \frac{S}{J_{K_n}^t \cap P_{r-1}^t} \oplus \frac{S}{J_{K_n}^t \cap P_r^t} \to \frac{S}{J_{K_n}^t} \to 0.
\end{aligned}$$

Since $\operatorname{reg}(S/J_{K_n}^t) = 2t - 1$ and $\operatorname{reg}(S/(J_{K_n}^t \cap P_i^t)) \leqslant 2t$, $i = s+1$, $s+2, \ldots, r$, by Corollary 4.12, $\operatorname{reg}(S/J_G^t) \leqslant 2t$ follows by applying Regularity Lemma 2.1 to the above exact sequences. □

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References

[1] *W. Bruns, U. Vetter*: Determinantal Rings. Lecture Notes in Mathematics 1327. Springer, Berlin, 1988. zbl MR doi

[2] *A. Conca*: Gröbner bases of powers of ideals of maximal minors. J. Pure Appl. Algebra *121* (1997), 223–231. zbl MR doi

[3] *H. Dao, C. Huneke, J. Schweig*: Bounds on the regularity and projective dimension of ideals associated to graphs. J. Algebr. Comb. *38* (2013), 37–55. zbl MR doi

[4] *D. Eisenbud*: Commutative Algebra: With a View Toward Algebraic Geometry. Graduate Texts in Mathematics 150. Springer, New York, 1995. zbl MR doi

[5] *V. Ene, G. Rinaldo, N. Terai*: Powers of binomial edge ideals with quadratic Gröbner bases. Nagoya Math. J. *246* (2022), 233–255. zbl MR doi

[6] *J. Herzog, T. Hibi*: Monomial Ideals. Graduate Texts in Mathematics 260. Springer, London, 2011. zbl MR doi

[7] *J. Herzog, T. Hibi, F. Hreinsdóttir, T. Kahle, J. Rauh*: Binomial edge ideals and conditional independence statements. Adv. Appl. Math. *45* (2010), 317–333. zbl MR doi

[8] *J. Herzog, T. Hibi, H. Ohsugi*: Binomial Ideals. Graduate Texts in Mathematics 279. Springer, Cham, 2018. zbl MR doi

[9] *L. T. Hoa, N. D. Tam*: On some invariants of a mixed product of ideals. Arch. Math. *94* (2010), 327–337. zbl MR doi

[10] *A. V. Jayanthan, A. Kumar, R. Sarkar*: Regularity of powers of quadratic sequences with applications to binomial ideals. J. Algebra *564* (2020), 98–118. zbl MR doi

[11] *M. Ohtani*: Graphs and ideals generated by some 2-minors. Commun. Algebra *39* (2011), 905–917. zbl MR doi

[12] *M. Ohtani*: Binomial edge ideals of complete multipartite graphs. Commun. Algebra *41* (2013), 3858–3867. zbl MR doi

[13] *S. Saeedi Madani, D. Kiani*: Binomial edge ideals of graphs. Electron. J. Combin. *19* (2012), Article ID 44, 6 pages. zbl MR doi

[14] *S. Saeedi Madani, D. Kiani*: Binomial edge ideals of regularity 3. J. Algebra *515* (2018), 157–172. zbl MR doi

[15] *P. Schenzel, S. Zafar*: Algebraic properties of the binomial edge ideal of a complete bipartite graph. An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat. *22* (2014), 217–237. zbl MR doi

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