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REPRESENTATIONS OF A CLASS OF POSITIVELY BASED ALGEBRAS

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Abstract. We investigate the representation theory of the positively based algebra $A_{m,d}$, which is a generalization of the noncommutative Green algebra of weak Hopf algebra corresponding to the generalized Taft algebra. It turns out that $A_{m,d}$ is of finite representative type if $d \leqslant 4$, of tame type if $d = 5$, and of wild type if $d \geqslant 6$. In the case when $d \leqslant 4$, all indecomposable representations of $A_{m,d}$ are constructed. Furthermore, their right cell representations as well as left cell representations of $A_{m,d}$ are described.

Keywords: positively based algebra; indecomposable module; cell module

MSC 2020: 16D80, 16G60

1. Introduction

As a generalization of Hopf algebra, the concept of weak Hopf algebra was introduced by Li in [7]. More precisely, a weak Hopf algebra is a bialgebra with a weak antipode. Su and Yang introduced two classes of the weak Hopf algebra $\mathfrak{w}^s_{n,d}$ $(s = 0, 1)$ based on the generalized Taft algebra $H_{n,d}(q)$ in [15]. The Green rings $r(\mathfrak{w}^s_{n,d})$ of $\mathfrak{w}^s_{n,d}$ are established and it is proved that $r(\mathfrak{w}^1_{n,d})$ is noncommutative as well as $r(\mathfrak{w}^0_{n,d})$ is commutative. Green rings or Green algebras are always positively based algebras. Examples of positively based algebras include the Hecke algebras corresponding to Coxeter groups with respect to the Kazhdan-Lusztig basis. Mazorchuk and Miemietz defined cell 2-representations of finitary 2-categories in [12]. On the level of the Grothendieck group, a cell 2-representation becomes a based module over some finite-dimensional positively based algebras with various nice properties. For exam-

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ple, for the 2-category of Soergel bimodules over the coinvariant algebra of a finite Coxeter group, the Grothendieck group level of a cell 2-representation is exactly the Kazhdan-Lusztig left cell module, see [4], [9], [10], [11].

The aforementioned work motivates us to study indecomposable modules and cell modules of positively based algebras. Of particular interest for us are those noncommutative positively based algebras associated to Green algebras of some finite dimensional bialgebras. On the one hand, it could be helpful to understand their representations of noncommutative Green algebras or noncommutative positively based algebras, and on the other hand, it may help us to recover all classes of original bialgebras. To understand this, in the present paper we first define a new algebra $A_{m,d}$ over some suitable subfields $\mathbb{K}$ of the complex field $\mathbb{C}$. The algebra $A_{m,d}$, which is just the Green algebra of $\mathfrak{w}_{md,d}^1$ if $\mathbb{K} = \mathbb{C}$, can be described by three generators and generating relations controlled by the determinant of some tridiagonal matrices. A new way is provided to show that $A_{m,d}$ is a positively based algebra by avoiding the technique of Green rings. It is observed that $A_{m,d}$ is of finite representation type if $d \leqslant 4$, of tame type if $d = 5$, and of wild type if $d \geqslant 6$. Furthermore, we classify all indecomposable modules of $A_{m,d}$ for $d \leqslant 4$. At last the cells and cell modules of $A_{m,d}$ are constructed. It is pointed out that the right cells and cell modules of $A_{m,d}$ are different from the left ones.

The paper is organized as follows. In Section 2, we introduce the definition of the algebra $A_{m,d}$ by generators and relations. Two sets of the basis of $A_{m,d}$ are constructed. In Section 3, we show that $A_{m,d}$ is a positively based algebra by avoiding the technique of Green rings. In Section 4, the representation type of $A_{m,d}$ is determined. All the indecomposable $A_{m,d}$-modules are constructed when $A_{m,d}$ is of representation-finite type. In Section 5, right cell modules of $A_{m,d}$ as well as their structures are investigated. Also, all the left cell modules of $A_{m,d}$ are listed.

2. Preliminaries

Throughout, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$, stand for the field of complex numbers, real numbers, the ring of integers, and the set of natural numbers, respectively, unless otherwise stated. The symbol $\sharp$ means the number of elements of a set. Representations and modules of an algebra are considered to be the same meanings.

Fixing integers $m, d > 1$ and $n = md$. Suppose that $\mathbb{K}$ is a subfield of $\mathbb{C}$ containing the complex number i and the primitive $2n$th root of the unity

$$\eta = \cos\frac{\pi}{n} + \mathrm{i}\sin\frac{\pi}{n}.$$

For $0 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant d-1$, we always set $\overline{i} = i \pmod{d}$ and

$$\sigma_{i,j} = 2\eta^{\overline{i}m} \cos\frac{j\pi}{d}.$$

Note that $\sigma_{i,\overline{i}} = 1+\eta^{2im}$ when $d \nmid i$ and all $\sigma_{i,j}$, $\cos(k\pi/n)$, $\sin(k\pi/n)$ $(0 \leqslant k \leqslant 2n-1)$ belong to $\mathbb{K}$.

Su and Yang in [14], [15] provided two examples of noncommutative Green rings. One of them is that of a small quantum group, which is of infinite $\mathbb{Z}$-rank with much complicated defining relations. The other one, provided in [15], is the Green rings $r(\mathfrak{w}^s_{md,d})$ $(s = 0, 1)$ of weak Hopf algebras $\mathfrak{w}^s_{md,d}$ $(s = 0, 1)$ based on generalized Taft algebras, which are of finite $\mathbb{Z}$-rank. Obviously, they are positively based algebras. In the sequel, we focus on classifying representations of a more general $\mathbb{K}$-algebra $A_{m,d}$ than the ring $r(\mathfrak{w}^1_{md,d})$.

To introduce the $\mathbb{K}$-algebra $A_{m,d}$, we consider the matrices

$$A_l(x^m, y) = \begin{pmatrix} y & x^m & 0 & \dots & 0 & 0 \\ 1 & y & x^m & \dots & 0 & 0 \\ 0 & 1 & y & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & y & x^m \\ 0 & 0 & 0 & \dots & 1 & y \end{pmatrix}_{l\times l}, \quad \text{where } xy = yx,$$

and put $D_l(x^m, y) = \det(A_l(x^m, y))$. It is well known that

$$D_l(x^m, y) = \sum_{i=0}^{[l/2]} (-1)^i \binom{l-i}{i} x^{mi} y^{l-2i} \quad \text{for } l \geqslant 1,$$

where $[l/2]$ denotes the biggest integer which is not bigger than $l/2$.

Definition 2.1. The $\mathbb{K}$-algebra $A_{m,d}$ is generated by x, y, z with the following relations:
(1) $x^{md} = 1$, $xy = yx$;
(2) $(1 + x^m - y)D_{d-1}(x^m, y) = 0$;
(3) $xz = zx = z$, $yz = 2z$, and $z^2 = z$.

One sees that $A_{m,d}$ is noncommutative and

$$D_l(x^m, y)z = D_l(x^m z, yz) = D_l(1, 2)z = (l+1)z, \tag{2.1}$$

$$zD_l(x^m, y) = zD_l(1, y). \tag{2.2}$$

Now, let $\mathcal{V}_t$ ($0 \leqslant t \leqslant n-1$), $\mathcal{V}$ be $\mathbb{K}$-vector spaces with the bases $\{v_t(k)\colon 1 \leqslant k \leqslant d\}$, $\{v_k\colon 1 \leqslant k \leqslant d\}$, respectively. We introduce two lemmas.

Lemma 2.2. *$\mathcal{V}_t$ is an $A_{m,d}$-module with the following actions of $A_{m,d}$ on $\mathcal{V}_t$.*
(1) *If $d \mid t$, setting*

$$\begin{aligned} v_t(k) \cdot x &= \eta^{2t} v_t(k), \\ v_t(k) \cdot y &= \begin{cases} 2\Big(\cos \dfrac{k\pi}{d}\Big) v_t(k), & 1 \leqslant k \leqslant d-1, \\ 2v_t(d), & k = d, \end{cases} \\ v_t(k) \cdot z &= 0. \end{aligned}$$

(2) *If $d \nmid t$, setting*

$$\begin{aligned} v_t(k) \cdot x &= \eta^{2t} v_t(k), \\ v_t(k) \cdot y &= \begin{cases} \sigma_{t,k} v_t(k), & 1 \leqslant k \leqslant d-1, \\ v_t(\overline{t}) + \sigma_{t,\overline{t}} v_t(d), & k = d, \end{cases} \\ v_t(k) \cdot z &= 0. \end{aligned}$$

Proof. Firstly, we assume that $d \mid t$, and have

$$(v_t(k)\cdot x)\cdot z = (v_t(k)\cdot z)\cdot x = v_t(k)\cdot z, \quad (v_t(k)\cdot y)\cdot z = v_t(k)\cdot 2z, \quad (v_t(k)\cdot z)\cdot z = v_t(k)\cdot z,$$

and

$$((v_t(k) \cdot \underbrace{x) \cdot x \ldots x}_{md}) = \eta^{2tmd} v_t(k) = v_t(k),$$

since $v_t(k) \cdot z = 0$ and $md = n$.

For $1 \leqslant k \leqslant d$, we have

$$(v_t(k) \cdot x) \cdot y = \begin{cases} 2\Big(\eta^{2t} \cos \dfrac{k\pi}{d}\Big) v_t(k) = (v_t(k) \cdot y) \cdot x, & 1 \leqslant k \leqslant d-1, \\ 2\eta^{2t} v_t(d) = (v_t(d) \cdot y) \cdot x, & k = d. \end{cases}$$

If $1 \leqslant k \leqslant d-1$, we have

$$(v_t(k)\cdot(1+x^m-y))\cdot D_{d-1}(x^m, y) = \Big(1+\eta^{2tm} - 2\cos \frac{k\pi}{d}\Big) D_{d-1}\Big(1, 2\cos \frac{k\pi}{d}\Big) v_t(k) = 0.$$

This follows from $D_{d-1}(1, 2\cos(k\pi/d)) = 0$.

If $k = d$, we have

$$v_t(d) \cdot (1 + x^m - y) D_{d-1}(x^m, y) = (1 + \eta^{2tm} - 2) D_{d-1}(1, 2) v_t(d) = 0.$$

Hence, the actions of x, y and z keep the defining relations of $A_{m,d}$.

Secondly, for the case when $d \nmid t$, the proof is similar. Therefore, $\mathcal{V}_t$ is an $A_{m,d}$-module. □

Lemma 2.3. *$\mathcal{V}$ is an indecomposable $A_{m,d}$-module with the following actions of $A_{m,d}$ on $\mathcal{V}$.*

$$v_k \cdot x = v_k,$$
$$v_k \cdot y = \begin{cases} 2\Big(\cos\dfrac{k\pi}{d}\Big)v_k, & 1 \leqslant k \leqslant d-1, \\ 2v_k, & k = d, \end{cases}$$
$$v_k \cdot z = \begin{cases} 0, \quad 1 \leqslant k \leqslant d-1, \\ 2\Big(\sin\dfrac{\pi}{d}\Big)v_1 + 2\Big(\sin\dfrac{2\pi}{d}\Big)v_2 + \ldots + 2\Big(\sin\dfrac{(d-1)\pi}{d}\Big)v_{d-1} + v_d, \quad k = d. \end{cases}$$

P r o o f. The proof of the statement about the actions is similar to the proof of Lemma 2.2. Also, $\mathcal{V}$ is indecomposable.

Indeed, we suppose that $\mathcal{V} = M_1 \oplus M_2$ with both M_1 and M_2 nonzero. If there exists w, $0 \neq w \in M_1$, such that $w = \sum\limits_{k=1}^{d} b_k v_k$ with $b_d \neq 0$, then $\omega \cdot z = b_d v_d \cdot z \in M_1$ and $v_d \cdot zy$, $v_d \cdot zy^2, \ldots, v_d \cdot zy^{d-1}$ belong to M_1. More precisely, we have that

$$\gamma_1 = v_d \cdot z = 2\Bigg(\sum_{j=1}^{d-1} \sin\frac{j\pi}{d}\Bigg)v_j + v_d,$$
$$\gamma_k = \gamma_{k-1} \cdot y = 2^k\Bigg(\sum_{j=1}^{d-1} \sin\frac{j\pi}{d}\cos^{k-1}\frac{j\pi}{d}\Bigg)v_j + 2^{k-1}v_d,$$
$$\gamma_d = \gamma_{d-1} \cdot y = 2^d\Bigg(\sum_{j=1}^{d-1} \sin\frac{j\pi}{d}\cos^{d-1}\frac{j\pi}{d}\Bigg)v_j + 2^{d-1}v_d$$

belong to M_1.

It is easy to see that the matrix

$$\begin{pmatrix} 2\sin\dfrac{\pi}{d} & 2\sin\dfrac{2\pi}{d} & \ldots & 2\sin\dfrac{(d-1)\pi}{d} & 1 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 2^k\sin\dfrac{\pi}{d}\cos^{k-1}\dfrac{\pi}{d} & 2^k\sin\dfrac{2\pi}{d}\cos^{k-1}\dfrac{2\pi}{d} & \ldots & 2^k\sin\dfrac{(d-1)\pi}{d}\cos^{k-1}\dfrac{(d-1)\pi}{d} & 2^{k-1} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 2^d\sin\dfrac{\pi}{d}\cos^{d-1}\dfrac{\pi}{d} & 2^d\sin\dfrac{2\pi}{d}\cos^{d-1}\dfrac{2\pi}{d} & \ldots & 2^d\sin\dfrac{(d-1)\pi}{d}\cos^{d-1}\dfrac{(d-1)\pi}{d} & 2^{d-1} \end{pmatrix}$$

is invertible and we get that $v_k \in M_1$ for $1 \leqslant k \leqslant d$. Hence, $M_1 = \mathcal{V}$ and we get a contradiction in this case.

If there is no $0 \neq w \in M_1$ such that $w = \sum\limits_{k=1}^{d} b_k v_k$, $b_d \neq 0$, then we have $0 \neq v_d \in M_2$. One sees that $M_2 = \mathcal{V}$ in a similar way and get a contradiction. It concludes that $\mathcal{V}$ is indecomposable. $\square$

Proposition 2.4. *The set* $\{x^iy^j, zy^l\colon\ 0\leqslant i\leqslant n-1,\ 0\leqslant j,\ l\leqslant d-1\}$ *forms a basis of* $A_{m,d}$.

P r o o f. Noting that $(1+x^m)D_{d-1}(x^m,y)=yD_{d-1}(x^m,y)$ by the defining relations of $A_{m,d}$, we have

$$\begin{aligned}(1+x^m)\biggl(\sum_{i=0}^{[(d-1)/2]}(-1)^i\binom{d-1-i}{i}x^{mi}y^{d-1-2i}\biggr)\\ =y\biggl(\sum_{i=0}^{[(d-1)/2]}(-1)^i\binom{d-1-i}{i}x^{mi}y^{d-1-2i}\biggr).\end{aligned}$$

One sees that y^d can be represented by linear combinations of $\{x^iy^j\colon\ 0\leqslant j\leqslant d-1\}$. Hence, any element in $A_{m,d}$ is spanned by $\{x^iy^j, zy^l\colon\ 0\leqslant i\leqslant n-1,\ 0\leqslant j,l\leqslant d-1\}$.

In fact,

$$\{x^iy^j, zy^l\colon\ 0\leqslant i\leqslant n-1,\ 0\leqslant j,l\leqslant d-1\}$$

is still linearly independent.

Indeed, suppose that

$$\sum_{i=0}^{n-1}\sum_{j=0}^{d-1}a_{i,j}x^iy^j+\sum_{l=0}^{d-1}b_lzy^l=0. \tag{2.3}$$

For convenience, we set $a_j=\sum\limits_{i=0}^{n-1}a_{i,j}\eta^{2ti}$ and $\alpha=(a_0,\dots,a_{d-1})^\top$. We also set

$$A=\begin{pmatrix}1 & 2\cos\dfrac{\pi}{d} & \dots & 2^{d-1}\cos^{d-1}\dfrac{\pi}{d}\\ \vdots & \vdots & \dots & \vdots\\ 1 & 2\cos\dfrac{k\pi}{d} & \dots & 2^{d-1}\cos^{d-1}\dfrac{k\pi}{d}\\ \vdots & \vdots & \dots & \vdots\\ 1 & 2\cos\dfrac{(d-1)\pi}{d} & \dots & 2^{d-1}\cos^{d-1}\dfrac{(d-1)\pi}{d}\\ 1 & 2 & \dots & 2^{d-1}\end{pmatrix}\quad \text{if } d\mid t$$

and

$$A=\begin{pmatrix}1 & \sigma_{t,1} & (\sigma_{t,1})^2 & \dots & (\sigma_{t,1})^{d-1}\\ \vdots & \vdots & \vdots & \dots & \vdots\\ 1 & \sigma_{t,\bar t} & (\sigma_{t,\bar t})^2 & \dots & (\sigma_{t,\bar t})^{d-1}\\ \vdots & \vdots & \vdots & \dots & \vdots\\ 1 & \sigma_{t,d-1} & (\sigma_{t,d-1})^2 & \dots & (\sigma_{t,d-1})^{d-1}\\ 0 & 1 & 2\sigma_{t,\bar t} & \dots & (d-1)(\sigma_{t,\bar t})^{d-2}\end{pmatrix}\quad \text{if } d\nmid t.$$

It is easy to see that such A is invertible since $\det(A) \neq 0$. For example, in the case when $d \nmid t$, we have

$$|A| = (-1)^{\bar{t}+1} \prod_{k \neq \bar{t}} (\sigma_{t,k} - \sigma_{t,\bar{t}}) \begin{vmatrix} 1 & \sigma_{t,1} & \dots & (\sigma_{t,1})^{d-2} \\ \vdots & \vdots & \dots & \vdots \\ 1 & \sigma_{t,d-1} & \dots & (\sigma_{t,d-1})^{d-2} \\ 1 & \sigma_{t,\bar{t}} & \dots & (\sigma_{t,\bar{t}})^{d-2} \end{vmatrix} \neq 0.$$

Therefore, $\sum\limits_{i=0}^{n-1} a_{i,j}\eta^{2ti} = 0$ for $0 \leqslant j \leqslant d-1$ and $d \nmid t$.

In the case when $d \mid t$ or $d \nmid t$, acting on $\{v_t(1), v_t(2), \dots, v_t(d)\}$ by both the sides of (2.3), respectively, we have $A\alpha = 0$. Consequently, $\alpha = 0$ and hence

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & \dots & \vdots \\ 1 & \eta^{2k} & \dots & (\eta^{2k})^{n-1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & \eta^{2(n-1)} & \dots & (\eta^{2(n-1)})^{n-1} \end{pmatrix} \begin{pmatrix} a_{0,j} \\ \vdots \\ a_{k,j} \\ \vdots \\ a_{n-1,j} \end{pmatrix} = 0$$

for any $0 \leqslant j \leqslant d-1$. Therefore, $a_{i,j} = 0$ for $0 \leqslant i \leqslant n-1$, $0 \leqslant j \leqslant d-1$ and we get that

$$\sum_{l=0}^{d-1} b_l z y^l = 0. \tag{2.4}$$

Acting on v_d of $\mathcal{V}$ by both the sides of (2.4), we have

$$\begin{pmatrix} 2\sin\frac{\pi}{d} & 4\sin\frac{\pi}{d}\cos\frac{\pi}{d} & \dots & 2^d\sin\frac{\pi}{d}\cos^{d-1}\frac{\pi}{d} \\ \vdots & \vdots & \dots & \vdots \\ 2\sin\frac{k\pi}{d} & 4\sin\frac{k\pi}{d}\cos\frac{k\pi}{d} & \dots & 2^d\sin\frac{k\pi}{d}\cos^{d-1}\frac{k\pi}{d} \\ \vdots & \vdots & \dots & \vdots \\ 2\sin\frac{(d-1)\pi}{d} & 4\sin\frac{(d-1)\pi}{d}\cos\frac{(d-1)\pi}{d} & \dots & 2^d\sin\frac{(d-1)\pi}{d}\cos^{d-1}\frac{(d-1)\pi}{d} \\ 1 & 2 & \dots & 2^{d-1} \end{pmatrix} \begin{pmatrix} b_0 \\ \vdots \\ b_k \\ \vdots \\ b_{d-2} \\ b_{d-1} \end{pmatrix} = 0.$$

It is easy to see that the determinant of the coefficient matrix is nonzero, so $b_l = 0$ for $0 \leqslant l \leqslant d-1$. Hence,

$$\{x^i y^j, zy^l \colon\ 0 \leqslant i \leqslant n-1,\ 0 \leqslant j,l \leqslant d-1\}$$

is linearly independent. The above statements imply that

$$\{x^i y^j, zy^l \colon\ 0 \leqslant i \leqslant n-1,\ 0 \leqslant j,l \leqslant d-1\}$$

is a basis of $A_{m,d}$. $\square$

Now we set

$$w(0,0)=1, \quad w(l,0)=D_l(x^m,y), \quad w(l,i)=x^{-i}w(l,0), \quad w_r=zw(r,0)$$

with $0\leqslant l,\, r\leqslant d-1$, $i\in\mathbb{Z}_n$ and we have:

Proposition 2.5. *The set* $\{w(l,i),w_r\colon\ 0\leqslant l,\, r\leqslant d-1,\ i\in\mathbb{Z}_n\}$ *forms a basis of* $A_{m,d}$.

Proof. We now assume that $0\leqslant l,r\leqslant d-1$, $i\in\mathbb{Z}_n$, then

$$w(l,i)=x^{-i}D_l(x^m,y)=x^{-i}\sum_{k=0}^{[l/2]}(-1)^k\binom{l-k}{k}x^{mk}y^{l-2k}=x^{-i}y^l+(*),$$

$$w_r=zD_r(x^m,y)=z\sum_{k=0}^{[r/2]}(-1)^k\binom{r-k}{k}x^{mk}y^{r-2k}=zy^r+(**).$$

Assuming that

$$\sum_{l=0}^{d-1}\sum_{i\in\mathbb{Z}_n}a_{l,i}w(l,i)+\sum_{r=0}^{d-1}b_rw_r=0,$$

we have

$$\sum_{l=0}^{d-1}\sum_{i\in\mathbb{Z}_n}a_{l,i}(x^{-i}y^l+(*))+\sum_{r=0}^{d-1}b_r(zy^r+(**))=0.$$

It follows that

$$\sum_{l=0}^{d-1}\sum_{i\in\mathbb{Z}_n}a_{l,i}(x^{-i}y^l)+\sum_{r=0}^{d-1}b_r(zy^r)=0$$

and $a_{l,i}=b_r=0$ for all $0\leqslant l,\, r\leqslant d-1$, $i\in\mathbb{Z}_n$ by Proposition 2.4.

Moreover, notice that

$$\sharp\{w(l,i),w_r\}_{0\leqslant l,r\leqslant d-1,i\in\mathbb{Z}_n}=\sharp\{x^iy^j,zy^l\}_{0\leqslant i\leqslant n-1,0\leqslant j,l\leqslant d-1}.$$

Consequently, $\{w(l,i),w_r\colon\ 0\leqslant l,\, r\leqslant d-1,\ i\in\mathbb{Z}_n\}$ is another basis of $A_{m,d}$. $\square$

3. The positive basis of $A_{m,d}$

In this section, we show that $\{w(l,i),w_r\colon\ 0\leqslant l,\, r\leqslant d-1,\ i\in\mathbb{Z}_n\}$ is a positive basis of $A_{m,d}$. To see this, let us review some concepts and some basic results now.

Let A be an n-dimensional $\mathbb{K}$-algebra. The basis $\mathfrak{B}=\{a_i\colon\ i\in I\}$ of A will be called *positive* if all structure constants of A with respect to this basis are nonnegative real numbers, that is,

$$a_i\cdot a_j=\sum_{k=1}^{n}\gamma_{i,j}^{(k)}a_k,$$

holds for all $i, j \in I$, where $\gamma_{i,j}^{(k)} \in \mathbb{R} \geqslant 0$ for all i, j, k. An algebra with a positive basis is called a *positively based algebra.* For example, the group algebra $\mathbb{K}G$ is a positively based algebra when G is a finite group and the positive basis is $\mathfrak{B} = \{g\colon\ g \in G\}$ with the structure constants being one or zero.

For $1 < d \in \mathbb{N}$, we set $[0, d-1] = \{0, \ldots, d-1\}$. Consider a free abelian group generated by the elements (u, i), where $(u, i) \in [0, d-1] \times \mathbb{Z}_n$. Suppose that this group is equipped with an extra multiplicative structure, making it a commutative ring, which is subject to the following relations, where $+$ denotes the addition law, $\cdot$ the multiplication law, and $(u, i) = 0$ if $u < 0$.

$$(0, i) \cdot (0, j) = (0, i + j), \tag{3.1}$$

$$(1, 0) \cdot (l, j) = (l + 1, j) + (l - 1, j - m), \tag{3.2}$$

$$(1, 0) \cdot (d - l, j) = (d - 1, j) + (d - 1, j - m). \tag{3.3}$$

Lemma 3.1. *The following formulas hold.*

(1) $(u, i) \cdot (v, j) = \sum\limits_{k=0}^{\min(u,v)} (u + v - 2k, i + j - mk)$ *for* $u + v \leqslant d - 1$,

(2) $(u, i) \cdot (v, j) = \sum\limits_{k=0}^{t} (d - 1, i + j - mk) + \sum\limits_{k=t+1}^{\min(u,v)} (u + v - 2k, i + j - mk)$ *for* $u + v \geqslant d$, *where* $t = u + v - (d - 1)$.

P r o o f. The proof is similar to that of [3], Proposition 3.1, where we replace (i, u) by (u, i), $i + 1$ by $i - m$, $i + j + 1$ by $i + j - m$, m by d, $i + j + l$ by $i + j - ml$, $j + 1$ by $j - m$, and $i + j + l + 1$ by $i + j - ml - m$.

The proof is finished. See also [16], Lemma 3.1. □

We can now investigate the structure constants of $\{w(l, i), w_r\colon\ 0 \leqslant l,\ r \leqslant d - 1,\ i \in \mathbb{Z}_n\}$ of $A_{m,d}$. Let

$$\mathfrak{B} = \{w(l, i), w_r\colon\ 0 \leqslant l,\ r \leqslant d - 1,\ i \in \mathbb{Z}_n\}.$$

Lemma 3.2. *Let* $0 \leqslant u, v, l, r, r' \leqslant d - 1$, $i, j \in \mathbb{Z}_n$, *then take for the basis of* $A_{m,d}$

(1) *if* $u + v \leqslant d - 1$, *then*

$$w(u, i) \cdot w(v, j) = \sum_{k=0}^{\min(u,v)} w(u + v - 2k, i + j - mk),$$

(2) *if* $u + v \geqslant d$, *set* $t = u + v - (d - 1)$, *then*

$$w(u, i) \cdot w(v, j) = \sum_{k=0}^{t} w(d - 1, i + j - mk) + \sum_{k=t+1}^{\min(u,v)} w(u + v - 2k, i + j - mk),$$

(3) $w(l, i) \cdot w_r = (l + 1) w_r$,

(4) *if* $r+l \leqslant d-1$, *then*

$$w_r \cdot w(l,i) = \sum_{k=0}^{\min(r,l)} w_{r+l-2k},$$

(5) *if* $r+l \geqslant d$, *set* $t = r+l-(d-1)$, *then*

$$w_r \cdot w(l,i) = \sum_{k=t+1}^{\min(r,l)} w_{r+l-2k} + (t+1)w_{d-1},$$

(6) $w_r \cdot w_{r'} = (r+1)w_{r'}$.

P r o o f. It is easy to see that

$$w(1,0) = D_1(x^m, y) = y, \quad w(l+1,0) = yw(l,0) - x^m w(l-1,0),$$

where $1 \leqslant l \leqslant d-2$. It is obvious that

$$w(0,i) \cdot w(0,j) = x^{-i}w(0,0) \cdot x^{-j}w(0,0) = x^{-i-j}w(0,0) = w(0,i+j).$$

Hence, (3.1) hold.

If $1 \leqslant l \leqslant d-2$, we have

$$\begin{aligned} w(1,0) \cdot w(l,j) &= x^{-j}w(1,0) \cdot w(l,0) = x^{-j}yw(l,0) \\ &= x^{-j}yw(l,0) - x^{m-j}w(l-1,0) + x^{m-j}w(l-1,0) \\ &= x^{-j}w(l+1,0) + x^{m-j}w(l-1,0) \\ &= w(l+1,j) + w(l-1,j-m). \end{aligned}$$

If $l = d-1$, then

$$\begin{aligned} w(1,0) \cdot w(d-1,j) &= x^{-j}w(1,0) \cdot w(d-1,0) = x^{-j}yw(d-1,0) \\ &= x^{-j}(1+x^m)w(d-1,0) \\ &= x^{-j}w(d-1,0) + x^{m-j}w(d-1,0) \\ &= w(d-1,j) + w(d-1,j-m). \end{aligned}$$

It follows that (3.2) and (3.3) hold.

Now, the proof of the statements (1) and (2) is obvious by Lemma 3.1.

(3) By (2.1), we have

$$\begin{aligned} w(l,i) \cdot w_r &= x^{-i}w(l,0)zw(r,0) = D_l(x^m,y)zw(r,0) \\ &= D_l(1,2)zw(r,0) = (l+1)zw(r,0) = (l+1)w_r. \end{aligned}$$

(4) If $r+l \leqslant d-1$, then

$$\begin{aligned} w_r \cdot w(l,i) &= zw(r,0)w(l,i) \\ &= \sum_{k=0}^{\min(r,l)} zw(r+l-2k, i-mk) = \sum_{k=0}^{\min(r,l)} zx^{mk-i}w(r+l-2k,0) \\ &= \sum_{k=0}^{\min(r,l)} zw(r+l-2k,0) = \sum_{k=0}^{\min(r,l)} w_{r+l-2k}. \end{aligned}$$

(5) If $r+l \geqslant d$, set $t = r+l-(d-1)$, then

$$\begin{aligned} w_r \cdot w(l,i) &= zw(r,0)w(l,i) \\ &= \sum_{k=0}^{t} zw(d-1, i-mk) + \sum_{k=t+1}^{\min(r,l)} zw(r+l-2k, i-mk) \\ &= (t+1)w_{d-1} + \sum_{k=t+1}^{\min(r,l)} zw(r+l-2k, i-mk) \\ &= (t+1)w_{d-1} + \sum_{k=t+1}^{\min(r,l)} w_{r+l-2k}. \end{aligned}$$

(6) Using (2.1), we have

$$w_r \cdot w_{r'} = zw(r,0)zw(r',0) = z(r+1)zw(r',0) = (r+1)zw(r',0) = (r+1)w_{r'}.$$

□

Theorem 3.3. *The algebra $A_{m,d}$ is a positively based algebra.*

Proof. By Lemma 3.2, it is easy to see that

$$\mathfrak{B} = \{w(l,i), w_r \colon\ 0 \leqslant l,\ r \leqslant d-1,\ i \in \mathbb{Z}_n\}$$

is a positive basis of $A_{m,d}$. The result follows. □

4. The representations of $A_{m,d}$

In this section, the field $\mathbb{K}$ is assumed to be algebraic closed. Firstly we determine the representation type of $A_{m,d}$ and then construct all the indecomposable $A_{m,d}$-modules in the case that $A_{m,d}$ is of finite representation type. For this purpose, we set

$$e_i = \frac{1}{n}\sum_{k=0}^{n-1} \eta^{-2ik} x^k$$

for $0 \leqslant i \leqslant n-1$. It is well known that $\{e_0, e_1, \ldots, e_{n-1}\}$ is the set of central idempotents of $A_{m,d}$. Also, $ze_i = e_i z = 0$ for $1 \leqslant i \leqslant n-1$, $(e_0 - z)z = z(e_0 - z) = 0$, and $(e_0 - z)^2 = e_0 - z$.

$$A_{m,d} = zA_{m,d} \oplus (e_0 - z)A_{m,d} \oplus e_1 A_{m,d} \oplus \ldots \oplus e_{n-1} A_{m,d}.$$

A straightforward verification shows that

$$(1 - z)A_{m,d} = (e_0 - z)A_{m,d} \oplus e_1 A_{m,d} \oplus \ldots \oplus e_{n-1} A_{m,d}$$

is isomorphic to $r(H_{n,d}) \otimes \mathbb{K}$ in [8].

Lemma 4.1. *$zA_{m,d} \cong \mathcal{V}$ is a d-dimensional indecomposable projective module.*

Proof. It is easy to see that $A_{m,d} = zA_{m,d} \oplus (1-z)A_{m,d}$ and $zA_{m,d}$ is projective. Noting that

$$zw(l, i) = w_l, \quad zw_r = w_r \quad \text{for } 0 \leqslant l,\ r \leqslant d-1,\ i \in \mathbb{Z}_n,$$

we have that $\{w_0, w_1, w_2, \ldots, w_{d-1}\}$ is a basis of $zA_{m,d}$.

Let $\omega_k = w_{k-1}$ for $1 \leqslant k \leqslant d$, then the actions of $A_{m,d}$ on $zA_{m,d}$ can be written as

$$\begin{aligned}
&\omega_k \cdot x = \omega_k \quad (1 \leqslant k \leqslant d),\\
&\omega_1 \cdot y = \omega_2, \quad \omega_k \cdot y = \omega_{k-1} + \omega_{k+1} \quad (2 \leqslant k \leqslant d-1), \quad \omega_d \cdot y = 2\omega_d,\\
&\omega_k \cdot z = k\omega_1 \quad (1 \leqslant k \leqslant d).
\end{aligned}$$

With respect to this basis, the matrix B of y is of the form

$$\begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 & 0\\
1 & 0 & 1 & \ldots & 0 & 0 & 0\\
0 & 1 & 0 & \ldots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \ldots & 1 & 0 & 1\\
0 & 0 & 0 & \ldots & 0 & 0 & 2
\end{pmatrix}_{d \times d}.$$

The eigenvalue λ_k and its eigenvector α_k of B $(1 \leqslant k \leqslant d)$ are

$$\lambda_k = \begin{cases} 2\cos\dfrac{k\pi}{d}, & 1 \leqslant k \leqslant d-1,\\ 2, & k = d, \end{cases}$$

$$\alpha_k = \begin{cases} \Bigl(\sin\dfrac{k\pi}{d}, \sin\dfrac{2k\pi}{d}, \ldots, \sin\dfrac{(d-1)k\pi}{d}, \dfrac{(-1)^k}{2}\cot\dfrac{k\pi}{2d}\Bigr), & 1 \leqslant k \leqslant d-1,\\ (0, 0, \ldots, 0, 1), & k = d. \end{cases}$$

Now we assume that

$$
\begin{aligned}
\omega'_k = {} & \sin\frac{k\pi}{d}\omega_1 + \sin\frac{2k\pi}{d}\omega_2 + \ldots + \sin\frac{(d-1)k\pi}{d}\omega_{d-1} \\
& + \frac{(-1)^k}{2}\cot\frac{k\pi}{2d}\omega_d \quad (1 \leqslant k < d-1), \\
\omega'_d = {} & \omega_d.
\end{aligned}
$$

Then

$$
\begin{aligned}
& \omega'_k \cdot x = \omega'_k, \quad (1 \leqslant k \leqslant d), \\
& \omega'_k \cdot y = 2\cos\frac{k\pi}{d}\omega'_k \quad (1 \leqslant k \leqslant d-1), \quad \omega'_d \cdot y = 2\omega'_d, \\
& \omega'_k \cdot z = 0 \quad (1 \leqslant k \leqslant d-1), \\
& \omega'_d \cdot z = 2\sin\frac{\pi}{d}\omega'_1 + 2\sin\frac{2\pi}{d}\omega'_2 + \ldots + 2\sin\frac{(d-1)\pi}{d}\omega'_{d-1} + \omega'_d.
\end{aligned}
$$

We show the actions of z on the basis $\{\omega'_k\colon 1 \leqslant k \leqslant d\}$. By the defining relations of $A_{m,d}$, we have $(\lambda_k - 2)\omega'_k \cdot z = 0$. Hence, $\omega'_k \cdot z = 0$ if $k \neq d$. Furthermore,

$$
\omega'_d \cdot z = d\omega_1 = a_1\omega'_1 + a_2\omega'_2 + \ldots + a_d\omega'_d.
$$

Assume that

$$
(\omega'_1, \omega'_2, \omega'_3, \ldots, \omega'_d)^\top = X(\omega_1, \omega_2, \omega_3, \ldots, \omega_d)^\top,
$$

where

$$
X = \begin{pmatrix} M & \alpha \\ 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix}
\sin\frac{\pi}{d} & \sin\frac{2\pi}{d} & \ldots & \sin\frac{(d-1)\pi}{d} \\
\vdots & \vdots & \ldots & \vdots \\
\sin\frac{k\pi}{d} & \sin\frac{2k\pi}{d} & \ldots & \sin\frac{k(d-1)\pi}{d} \\
\vdots & \vdots & \ldots & \vdots \\
\sin\frac{(d-1)\pi}{d} & \sin\frac{2(d-1)\pi}{d} & \ldots & \sin\frac{(d-1)^2\pi}{d}
\end{pmatrix},
$$

and

$$
\alpha = \begin{pmatrix}
-\frac{1}{2}\cot\frac{\pi}{2d} \\
\vdots \\
\frac{(-1)^k}{2}\cot\frac{k\pi}{2d} \\
\vdots \\
\frac{(-1)^{d-1}}{2}\cot\frac{(d-1)\pi}{2d}
\end{pmatrix}.
$$

It is easy to see that M is invertible with the inverse $M^{-1} = M/d$. Hence, so is X, and

$$X^{-1} = \begin{pmatrix} M^{-1} & -M^{-1}\alpha \\ 0 & 1 \end{pmatrix} = \frac{2}{d}\begin{pmatrix} M & -M\alpha \\ 0 & \dfrac{d}{2} \end{pmatrix}.$$

Consequently, we have

$$(a_1, a_2, \ldots, a_{d-1}, a_d) = (d, 0, \ldots, 0, 0)X^{-1} = \Big(2\sin\frac{\pi}{d}, 2\sin\frac{2\pi}{d}, \ldots, 2\sin\frac{(d-1)\pi}{d}, 1\Big).$$

Hence,

$$\omega'_d \cdot z = 2\sin\frac{\pi}{d}\omega'_1 + 2\sin\frac{2\pi}{d}\omega'_2 + \ldots + 2\sin\frac{(d-1)\pi}{d}\omega'_{d-1} + \omega'_d.$$

By Lemma 2.3, $zA_{m,d} \cong \mathcal{V}$ is indecomposable and projective. $\square$

One sees that the top S_1 of $\mathcal{V}$ is one dimensional with the basis $\{u_1\}$ and the action of $A_{m,d}$ on S_1 is

$$u_1 \cdot x = u_1, \quad u_1 \cdot y = 2u_1, \quad u_1 \cdot z = u_1.$$

For $0 \leqslant i \leqslant md - 1$ with $d \mid i$ and $0 \leqslant j \leqslant d - 1$, let $S_{i,j}$ be a one dimensional simple $A_{m,d}$-module with the basis $\{v_{i,j}\}$, on which the action of $A_{m,d}$ is

$$v_{i,j} \cdot x = \eta^{2i} v_{i,j}, \quad v_{i,j} \cdot y = 2\Big(\cos\frac{j\pi}{d}\Big)v_{i,j}, \quad v_{i,j} \cdot z = 0.$$

For $0 \leqslant i \leqslant md-1$ with $d \nmid i$ and $1 \leqslant j \leqslant d-1$, let $S_{i,j}$ be one dimensional simple $A_{m,d}$-modules with the basis $\{v_{i,j}\}$, on which the action of $A_{m,d}$ is

$$v_{i,j} \cdot x = \eta^{2i} v_{i,j}, \quad v_{i,j} \cdot y = \sigma_{i,j} v_{i,j}, \quad v_{i,j} \cdot z = 0.$$

Let P_i be the two dimensional module with the basis $\{v_i^1, v_i^2\}$, on which the action of $A_{m,d}$ is

$$\begin{aligned} v_i^1 \cdot x &= \eta^{2i} v_i^1, \quad & v_i^1 \cdot y &= (1 + \eta^{2im})v_i^1, \quad & v_i^1 \cdot z &= 0, \\ v_i^2 \cdot x &= \eta^{2i} v_i^2, \quad & v_i^2 \cdot y &= v_i^1 + (1 + \eta^{2im})v_i^2, \quad & v_i^2 \cdot z &= 0, \end{aligned}$$

where $0 \leqslant i \leqslant md - 1$ and $d \nmid i$.

The module P_i is indecomposable and projective with the top and socle $S_{i,\bar{i}}$, where $0 \leqslant i \leqslant md - 1$ and $d \nmid i$.

Now, we are able to classify all the simple modules and indecomposable projective modules of $A_{m,d}$. We keep the notations. By the above discussion and Theorem 5.1 of [8] we have:

Lemma 4.2. *The complete list of simple (and pairwise nonisomorphic) $A_{m,d}$-modules is as follows.*

(1) *One nonprojective simple module S_1 with the projective cover $\mathcal{V}$,*
(2) *$(n-m)$ nonprojective simple modules $S_{i,\bar{i}}$ with the projective cover P_i, where $d \nmid i$,*
(3) *$nd - 2(n-m)$ projective simple modules $S_{i,j}$ with $j \neq \bar{i}$ and $d \nmid i$, and $S_{i,j}$ with $d \mid i$.*

One the other hand, it is easy to see that

$$(e_0 - z)w(l,i) = (e_0 - z)w(l,0),$$
$$(e_0 - z)w_l = e_0 w_l - z w_l = z w_l - z w_l = 0 \quad \text{for } 0 \leqslant l \leqslant d-1.$$

It follows that $(e_0 - z)A_{m,d}$ is d-dimensional with the basis $\{\omega_j := (e_0 - z)w(j-1,0)\colon 1 \leqslant j \leqslant d\}$. The actions of $A_{m,d}$ on $(e_0 - z)A_{m,d}$ can be written as

$$\omega_j \cdot x = \omega_j \quad (1 \leqslant j \leqslant d),$$
$$\omega_1 \cdot y = \omega_2 \quad \omega_j \cdot y = \omega_{j-1} + \omega_{j+1} \quad (2 \leqslant j \leqslant d-1), \quad \omega_d \cdot y = 2\omega_d,$$
$$\omega_j \cdot z = 0 \quad (1 \leqslant j \leqslant d).$$

Similarly to the proof of Lemma 4.1, we can get another basis $\{\omega'_j\colon 1 \leqslant j \leqslant d\}$ of $(e_0 - z)A_{m,d}$ such that

$$\omega'_j \cdot x = \omega'_j \quad (1 \leqslant j \leqslant d),$$
$$\omega'_j \cdot y = 2\Big(\cos\frac{j\pi}{d}\Big)\omega'_j \quad (1 \leqslant j \leqslant d-1), \quad \omega'_d \cdot y = 2\omega'_d,$$
$$\omega'_j \cdot z = 0 \quad (1 \leqslant j \leqslant d).$$

Hence, we get:

Lemma 4.3. $(e_0 - z)A_{m,d} = \bigoplus_{j=0}^{d-1} S_{0,j}$.

Let Q_j $(1 \leqslant j \leqslant d-1)$ be a $\mathbb{K}$-vector space with the basis $\{q_j, q_d\}$. The action of $A_{m,d}$ on Q_j is given by

$$q_j \cdot x = q_j, \quad q_j \cdot y = 2\Big(\cos\frac{j\pi}{d}\Big)q_j, \quad q_j \cdot z = 0,$$
$$q_d \cdot x = q_d, \quad q_d \cdot y = 2q_d, \quad q_d \cdot z = q_j + q_d.$$

Then Q_j is an indecomposable 2-dimensional $A_{m,d}$-module, which can be viewed as the quotient module of the indecomposable projective $A_{m,d}$-module $\mathcal{V}$. It is easy to see that

$$\mathrm{Hom}_{A_{m,d}}(e_iA_{m,d}, e_0A_{m,d}) = 0, \quad \mathrm{Hom}_{A_{m,d}}(e_0A_{m,d}, e_iA_{m,d}) = 0$$

for $1 \leqslant i \leqslant n-1$. Indeed, noting that the idempotents e_i are central and orthogonal in $A_{m,d}$, we have

$$\mathrm{Hom}_{A_{m,d}}(e_iA_{m,d}, e_0A_{m,d}) = e_0A_{m,d}e_i = e_0e_iA_{m,d} = 0$$

by [1], Lemma 4.2.

Lemma 4.4.
(1) *If* $1 \leqslant j \leqslant d-1$, *then* $\mathrm{Hom}_{A_{m,d}}(S_{0,j}, \mathcal{V}) \neq 0$,
(2) $\mathrm{Hom}_{A_{m,d}}(\mathcal{V}, S_{0,j}) = 0$ *for* $0 \leqslant j \leqslant d-1$, *and* $\mathrm{Hom}_{A_{m,d}}(S_{0,0}, \mathcal{V}) = 0$.

Proof. (1) As is shown in Lemma 4.1, the socle of $\mathcal{V}$ is $\bigoplus\limits_{j=1}^{d-1} S_{0,j}$. Hence,

$$\mathrm{Hom}_{A_{m,d}}(S_{0,j}, \mathcal{V}) \neq 0 \quad \text{and} \quad \dim \mathrm{Hom}_{A_{m,d}}(S_{0,j}, \mathcal{V}) = 1$$

for $1 \leqslant j \leqslant d-1$.

(2) It is obvious that

$$\mathrm{Hom}_{A_{m,d}}(S_{0,0}, \mathcal{V}) = 0.$$

On the other hand,

$$\mathrm{Hom}_{A_{m,d}}(\mathcal{V}, S_{0,j}) = \mathrm{Hom}_{A_{m,d}}(zA_{m,d}, S_{0,j}) = S_{0,j}z = 0.$$

The proof is finished. □

Accordingly, $A_0 = \mathcal{V} \oplus \bigoplus\limits_{j=1}^{d-1} S_{0,j}$ is a block of $A_{m,d}$. As is shown in [8], the indecomposable modules of other blocks of $A_{m,d}$ are listed in Lemma 4.2. Therefore, to determine the representation type of $A_{m,d}$, it is sufficient to consider the case of the block A_0.

Lemma 4.5. *The quiver of the block* $A_0 = \mathcal{V} \oplus \bigoplus\limits_{j=1}^{d-1} S_{0,j}$ *is*

$$\begin{tikzcd}
 & & 0 \arrow[dll, "\alpha_1"'] \arrow[dl, "\alpha_2"] \arrow[dr, "\alpha_{d-2}"'] \arrow[drr, "\alpha_{d-1}"] & & \\
1 & 2 & \cdots & d-2 & d-1.
\end{tikzcd}$$

P r o o f. We have shown that $S_{0,j}$, $1 \leqslant j \leqslant d-1$, are projective modules of $A_{m,d}$. Therefore,

$$\operatorname{Ext}_A^1(S_{0,j}, S_1) = 0.$$

On the other hand, there is an extension of $S_{0,j}$ by S_1,

$$0 \to S_{0,j} \to Q_j \to S_1 \to 0.$$

Suppose that

$$0 \to S_{0,j} \to Q'_j \to S_1 \to 0$$

is another extension of $S_{0,j}$ by S_1. We may assume that Q'_j has the basis $\{q'_j, q'_d\}$ with the actions

$$\begin{aligned} &q'_j \cdot x = q'_j, \quad q'_j \cdot y = 2\Big(\cos\frac{j\pi}{d}\Big)q'_j, \quad q'_j \cdot z = 0,\\ &q'_d \cdot x = q'_d, \quad q'_d \cdot y = 2q'_d, \qquad\qquad\quad q'_d \cdot z = \lambda q'_j + q'_d \quad (\lambda \neq 0). \end{aligned}$$

Then $h\colon Q_j \to Q'_j$, given by $q_j \mapsto \lambda q'_j$, $q_d \mapsto q'_d$ is an isomorphism. Hence, we have

$$\dim \operatorname{Ext}_A^1(S_1, S_{0,j}) = 1.$$

Consequently, the quiver of A_0 is as shown, where the points $0, 1, \ldots, d-1$ are corresponding to S_1, $S_{0,1}$, $\ldots$, $S_{0,d-1}$, respectively. □

By the above lemmas and the well-known classification theorem of representation type, we have:

Theorem 4.6. *We have the following statements.*
(1) $A_{m,d}$ *is of the finite representation type for* $d \leqslant 4$*;*
(2) $A_{m,d}$ *is of the tame type for* $d = 5$*;*
(3) $A_{m,d}$ *is of the wild type for* $d \geqslant 6$*.*

Here we list all indecomposable modules of $A_{m,d}$ when they are of finite representation type. We firstly establish some indecomposable $A_{m,d}$-modules which will be needed.

For an arbitrary pair of integers $1 \leqslant i$, $j \leqslant d-1$, let $Q_{i,j}$ be a $\mathbb{K}$-vector space spanned by the basis $\{q_i, q_j, q_d\}$ with the action of $A_{m,d}$ as follows.

$$\begin{aligned} &q_i \cdot x = q_i, \quad q_i \cdot y = 2\Big(\cos\frac{i\pi}{d}\Big)q_i, \quad q_i \cdot z = 0,\\ &q_j \cdot x = q_j, \quad q_j \cdot y = 2\Big(\cos\frac{j\pi}{d}\Big)q_j, \quad q_j \cdot z = 0,\\ &q_d \cdot x = q_d, \quad q_d \cdot y = 2q_d, \qquad\qquad\quad v_3 \cdot z = q_i + q_j + q_d. \end{aligned}$$

It is straightforward to check that $Q_{i,j}$ is an indecomposable 3-dimensional $A_{m,d}$-module.

For $d = 4$, let $\mathcal{V}'$ be a $\mathbb{K}$-vector space spanned by the basis $\{v_1, v_2, v_3, v_4, v_5\}$, the actions on $\mathcal{V}'$ are

$$\begin{aligned}
&v_i \cdot x = v_i \quad \text{for } 1 \leqslant i \leqslant 5,\\
&v_i \cdot y = 2\Big(\cos\frac{i\pi}{4}\Big)v_i \quad \text{for } 1 \leqslant i \leqslant 3 \quad \text{and} \quad v_4 \cdot y = 2v_4,\ v_5 \cdot y = 2v_5,\\
&v_i \cdot z = 0 \quad \text{for } 1 \leqslant i \leqslant 3 \quad \text{and} \quad v_4 \cdot z = v_1 + v_3 + v_4,\ v_5 \cdot z = v_1 + v_2 + v_5.
\end{aligned}$$

It is straightforward to check that $\mathcal{V}'$ is an indecomposable 5-dimensional $A_{m,4}$-module. We see that

$$\mathcal{V}'/Q_{1,2} \cong Q_3,\ \mathcal{V}'/Q_{1,3} \cong Q_2, \quad \text{and} \quad \mathcal{V}'/S \cong Q_1,$$

where S is the submodule of $\mathcal{V}'$ such that

$$\begin{gathered}
v_2 \cdot x = v_2, \quad v_2 \cdot y = 2\Big(\cos\frac{2\pi}{4}\Big)v_2, \quad v_2 \cdot z = 0,\\
v_3 \cdot x = v_3, \quad v_3 \cdot y = 2\Big(\cos\frac{3\pi}{4}\Big)v_3, \quad v_3 \cdot z = 0,\\
(v_5 - v_4) \cdot x = (v_5 - v_4), \quad (v_5 - v_4) \cdot y = 2(v_5 - v_4),\\
(v_5 - v_4) \cdot z = v_2 - v_3 + (v_5 - v_4),
\end{gathered}$$

which is isomorphic to $Q_{2,3}$.

Theorem 4.7. *The complete list of indecomposable (and pairwise nonisomorphic) $A_{m,d}$-modules for $d \leqslant 4$ follows.*

(1) *$A_{m,2}$ has $4m + 2$ iso-classes of indecomposable modules:*

$$\{S_{i,1}, P_i\colon\ 0 \leqslant i \leqslant 2m-1,\ 2 \nmid i\} \cup \{S_{i,j}\colon\ 0 \leqslant j \leqslant 1,\ 0 \leqslant i \leqslant 2m-1,\ 2 \mid i\} \cup \{\mathcal{V}, S_1\};$$

(2) *$A_{m,3}$ has $9m + 4$ iso-classes of indecomposable modules:*

$$\begin{aligned}\{S_{i,j}, P_i\colon\ 1 \leqslant j \leqslant 2,\ 0 \leqslant i \leqslant 3m-1,\ 3 \nmid i\} \cup \{S_{i,j}\colon\ 0 \leqslant j \leqslant 2,\ 0 \leqslant i \leqslant 3m-1,\ 3 \mid i\}\\
\cup \{\mathcal{V}, S_1, Q_1, Q_2\}.\end{aligned}$$

(3) *$A_{m,4}$ has $16m + 9$ iso-classes of indecomposable modules:*

$$\begin{aligned}\{S_{i,j}, P_i\colon\ 1 \leqslant j \leqslant 3,\ 0 \leqslant i \leqslant 4m-1,\ 4 \nmid i\} \cup \{S_{i,j}\colon\ 0 \leqslant j \leqslant 3,\ 0 \leqslant i \leqslant 4m-1,\ 4 \mid i\}\\
\cup \{\mathcal{V}, \mathcal{V}', S_1, Q_1, Q_2, Q_3, Q_{1,2}, Q_{1,3}, Q_{2,3}\}.\end{aligned}$$

P r o o f. (1) Assume that $d=2$. By Lemma 4.5, the quiver of the block $A_0 = \mathcal{V} \oplus \bigoplus_{j=1}^{d-1} S_{0,j}$ is

$$0 \xrightarrow{\alpha_1} 1$$

It is easy to see that the block A_0 has 3 nonisomorphic indecomposable modules $S_{0,1}$, S_1 and $\mathcal{V}$, where $S_{0,1}$ and S_1 are 1-dimensional modules, $\mathcal{V}$ is 2-dimensional. Consequently, there are $4m+2$ pairwise nonisomorphic indecomposable modules of $A_{m,2}$ by Lemma 4.2.

(2) Assume that $d=3$. By Lemma 4.5, the quiver of the block A_0 is

$$2 \xleftarrow{\alpha_2} 0 \xrightarrow{\alpha_1} 1$$

The associated Auslander-Reiten quiver is

$$\begin{tikzcd}
S_{0,1} \arrow[dr] & & Q_2 \arrow[dr] & \\
& \mathcal{V} \arrow[ur] \arrow[dr] & & S_1 \\
S_{0,2} \arrow[ur] & & Q_1 \arrow[ur] &
\end{tikzcd}$$

It is easy to see that the block A_0 has 6 pairwise nonisomorphic indecomposable modules: S_1, $S_{0,1}$ and $S_{0,2}$, Q_1, Q_2, and $\mathcal{V}$. Consequently, there are $9m+4$ pairwise nonisomorphic indecomposable modules of $A_{m,3}$ by Lemma 4.2.

(4) Assume that $d=4$. By Lemma 4.5, the quiver of the block A_0 is

$$\begin{tikzcd}
1 & 0 \arrow[l, "\alpha_1"'] \arrow[r, "\alpha_2"] \arrow[d, "\alpha^3"] & 3 \\
& 2 &
\end{tikzcd}$$

The associated Auslander-Reiten quiver is

$$\begin{tikzcd}
S_{0,1} \arrow[dr] & & Q_{2,3} \arrow[dr] & & Q_1 \arrow[dr] & \\
S_{0,2} \arrow[r] & \mathcal{V} \arrow[ur] \arrow[r] \arrow[dr] & Q_{1,3} \arrow[r] & \mathcal{V}' \arrow[ur] \arrow[r] \arrow[dr] & Q_2 \arrow[r] & S_1 \\
S_{0,3} \arrow[ur] & & Q_{1,2} \arrow[ur] & & Q_3 \arrow[ur] &
\end{tikzcd}$$

It is easy to see that the block A_0 has 12 pairwise nonisomorphic indecomposable modules: 1-dimensional simple modules S_1, $S_{0,1}$, $S_{0,2}$, $S_{0,3}$; 2-dimensional modules Q_1, Q_2, Q_3; 3-dimensional modules $Q_{1,2}$, $Q_{1,3}$ and $Q_{2,3}$; 4-dimensional module $\mathcal{V}$; and 5-dimensional module $\mathcal{V}'$. Consequently, there are $16m+9$ pairwise nonisomorphic indecomposable modules of $A_{m,4}$ by Lemma 4.2. The proof is completed. $\square$

Remark 4.8. When $d=5$, A_0 is the algebra of the tame type $\widetilde{D}_4$. The classification of indecomposable modules of the tame type $\widetilde{D}_n$ $(n\geqslant 4)$ was studied by many authors in the last decades, see for example [2], [13]. On the other hand, it is hopeless to construct all indecomposable modules of $A_{m,d}$ $(d\geqslant 6)$ of the wild type.

5. The cell modules of $A_{m,d}$

Let A be a positively based algebra with a fixed positive basis $\mathfrak{B}=\{a_i\colon i\in I\}$ with the identity a_1 of A. For $i,j\in I$, set

$$i\star j=\{k\colon \gamma_{i,j}^{(k)}>0\}.$$

This defines an associative multi-valued operation on the set I and turns the latter set into a finite multi-semigroup, see [6], Subsection 3.7. Denote by $i\leqslant_R j$ if there is an $s\in I$ such that $j\in i\star s$. Then $\leqslant_R$ is a partial pre-order on I called a *right pre-order.* Denote by $i\sim_R j$ if $i\leqslant_R j$ and also $j\leqslant_R i$. This defines an equivalence relation on I and the set of associated equivalence classes of $i\sim_R j$ is called *right cells.* Furthermore, the pre-order $i\sim_R j$ induces a genuine partial order on the set of all cells in I. We write $i<_R j$ provided that $i\leqslant_R j$ and $i\nsim_R j$, similarly for $i<_L j$.

Similarly we can also define the left pre-order $i\leqslant_L j$, the equivalence relation $i\sim_L j$ and left cells. Here $i\leqslant_L j$ means that there is an $s\in I$ such that $j\in s\star i$.

Let $\mathcal{R}$ be a right cell in I and $\overline{\mathcal{R}}$ the union of all right cells $\mathcal{R}'$ in I such that $\mathcal{R}'\geqslant\mathcal{R}$. Set $\underline{\overline{\mathcal{R}}}=\overline{\mathcal{R}}\setminus\mathcal{R}$. Consider the $\mathbb{K}$-submodule $M_{\mathcal{R}}$ of the regular A-module A_A, which is spanned by all a_j with $j\in\overline{\mathcal{R}}$. Let $N_{\mathcal{R}}$ be the $\mathbb{K}$-submodule of A_A spanned by all a_j with $j\in\underline{\overline{\mathcal{R}}}$. It is easy to see that both $M_{\mathcal{R}}$ and $N_{\mathcal{R}}$ are A-submodules of A_A by [5], Proposition 1 and $N_{\mathcal{R}}\subset M_{\mathcal{R}}$. It allows us to define the cell A-modules $C_{\mathcal{R}}$ as the quotient $M_{\mathcal{R}}/N_{\mathcal{R}}$. Here $N_{\mathcal{R}}=0$ if $\underline{\overline{\mathcal{R}}}=\emptyset$.

In this section, we focus on describing the right cells and right cell modules of $A_{m,d}$. For this purpose, its positive basis

$$\mathfrak{B}=\{w(l,i),w_r\colon\ 0\leqslant l,\ r\leqslant d-1,\ i\in\mathbb{Z}_n\}$$

is fixed.

Proposition 5.1. *The algebra $A_{m,d}$ has only the following three right cells $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$.*

(1) $\mathcal{R}_1=\{r\colon\ r$ *is the index of* $w_r,\ 0\leqslant r\leqslant d-1\}$;
(2) $\mathcal{R}_2=\{(d-1,i)\colon\ (d-1,i)$ *is the index of* $w(d-1,i),\ i\in\mathbb{Z}_n\}$;
(3) $\mathcal{R}_3=\{(l,i)\colon\ (l,i)$ *is the index of* $w(l,i),\ 0\leqslant l\leqslant d-2,\ i\in\mathbb{Z}_n\}$.

P r o o f. (1) Setting $0 \leqslant r, r' \leqslant d-1$, we have $w_r \cdot w_{r'} = (r+1)w_{r'}$ by Theorem 3.2. It follows that $r' \in r \star r'$, which implies that $r \leqslant_R r'$. Similarly, $r \in r' \star r$ and $r' \leqslant_R r$. The equivalent relation $r \sim_R r'$ holds and $\mathcal{R}_1$ is a right cell.

(2) For any $i \in \mathbb{Z}_n$, since $w(d-1,0) \cdot w(0,i) = w(d-1,i)$, we have $(d-1,i) \in (d-1,0) \star (0,i)$ and $(d-1,0) \leqslant_R (d-1,i)$. On the other hand,

$$w(d-1,i) \cdot w(d-1,n-i) = \sum_{k=0}^{d-1} w(d-1,n-mk).$$

Hence, $(d-1,0) \in (d-1,i) \star (d-1,n-i)$ and $(d-1,i) \leqslant_R (d-1,0)$.

Consequently, $\mathcal{R}_2 = \{(d-1,i)\colon\ i \in \mathbb{Z}_n\}$ is a right cell.

(3) If $l \leqslant [\frac{1}{2}(d-1)]$, then $w(0,i) \cdot w(l,0) = w(l,i)$ by Theorem 3.2. Therefore, $(l,i) \in (0,i) \star (l,0)$ and $(0,i) \leqslant_R (l,i)$. On the other hand,

$$w(l,i) \cdot w(l,ml) = \sum_{k=0}^{l} w(2l-2k, i+ml-mk)$$

implies that $(0,i) \in (l,i) \star (l,ml)$ and $(l,i) \leqslant_R (0,i)$.

Consequently, $(0,i) \sim_R (l,i)$ holds for $l \leqslant [(d-1)/2]$.

If $[\frac{1}{2}(d-1)] + 1 \leqslant l \leqslant d-2$, then $w(0,i) \cdot w(l,0) = w(l,i)$. Therefore, $(l,i) \in (0,i) \star (l,0)$ and $(0,i) \leqslant_R (l,i)$. On the other hand,

$$w(l,i) \cdot w(l,ml) = \sum_{k=0}^{t} w(d-1, i+ml-mk) + \sum_{k=t+1}^{l} w(2l-2k, i+ml-mk),$$

where $t = 2l-(d-1)$, implies that $w(0,i) \in w(l,i) \cdot w(l,ml)$ for $k=l$. Therefore,

$$(0,i) \in (l,i) \star (l,ml) \text{ and } (l,i) \leqslant_R (0,i).$$

The equivalent relation $(0,i) \sim_R (l,i)$ still holds for $[\frac{1}{2}(d-1)] + 1 \leqslant l \leqslant d-2$.

In other words, $\mathcal{R}_3 = \{(l,i)\colon\ 0 \leqslant l \leqslant d-2,\ i \in \mathbb{Z}_n\}$ is a right cell. The proof is completed. $\square$

Corollary 5.2. *As the right cells, we have $\mathcal{R}_3 <_R \mathcal{R}_2 <_R \mathcal{R}_1$ in $A_{m,d}$.*

P r o o f. As $w(l,0) \cdot w_r = (l+1)w_r$ for $0 \leqslant r,\ l \leqslant d-1$, it implies that $r \in (l,0) \star r$ and

$$\mathcal{R}_2 <_R \mathcal{R}_1, \quad \mathcal{R}_3 <_R \mathcal{R}_1$$

by Proposition 5.1.

Moreover, for $0 \leqslant l \leqslant d-2$ we have

$$w(l,i) \cdot w(d-1,n-i) = \sum_{k=0}^{l} w(d-1,n-mk).$$

It implies that $(d-1,0) \in (l,i) \star (d-1,n-i)$ and $\mathcal{R}_3 <_R \mathcal{R}_2$. Consequently, we get that $\mathcal{R}_3 <_R \mathcal{R}_2 <_R \mathcal{R}_1$. $\square$

Proposition 5.3. *For the right cells $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$, the corresponding right cell modules $\mathcal{C}_{\mathcal{R}_1}$, $\mathcal{C}_{\mathcal{R}_2}$ and $\mathcal{C}_{\mathcal{R}_3}$ of $A_{m,d}$ are given as*

(1) $\mathcal{C}_{\mathcal{R}_1} = \mathrm{Span}\{w_r\colon\ 0 \leqslant r \leqslant d-1\}$, *where* $N_{\mathcal{R}_1} = \{0\}$*;*

(2) $\mathcal{C}_{\mathcal{R}_2} = \mathrm{Span}\{\overline{w(d-1,i)}\colon\ i \in \mathbb{Z}_n\}$, *where* $\overline{w(l,i)} = w(l,i) + N_{\mathcal{R}_2}$, $N_{\mathcal{R}_2} = \mathrm{Span}\{w_r\colon\ 0 \leqslant r \leqslant d-1\}$*;*

(3) $\mathcal{C}_{\mathcal{R}_3} = \mathrm{Span}\{\overline{w(l,i)}\colon\ 0 \leqslant l \leqslant d-2,\ i \in \mathbb{Z}_n\}$, *where* $\overline{w(l,i)} = w(l,i) + N_{\mathcal{R}_3}$, *and*

$$N_{\mathcal{R}_3} = \mathrm{Span}\{w_r, w(d-1,i)\colon\ 0 \leqslant r \leqslant d-1,\ i \in \mathbb{Z}_n\}.$$

Proof. (1) By Corollary 5.2, it is easy to see that

$$\overline{\mathcal{R}_1} = \mathcal{R}_1 \quad \text{and} \quad \underline{\mathcal{R}_1} = \emptyset.$$

It follows that

$$M_{\mathcal{R}_1} = \mathrm{Span}\{w_r\colon\ 0 \leqslant r \leqslant d-1\} \quad \text{and} \quad N_{\mathcal{R}_1} = \{0\}.$$

Hence,

$$\mathcal{C}_{\mathcal{R}_1} = M_{\mathcal{R}_1}/N_{\mathcal{R}_1} = \mathrm{Span}\{w_r\colon\ 0 \leqslant r \leqslant d-1\}.$$

(2) By Corollary 5.2, it is easy to see that

$$\overline{\mathcal{R}_2} = \mathcal{R}_1 \cup \mathcal{R}_2, \quad \underline{\mathcal{R}_2} = \mathcal{R}_1.$$

It follows that

$$M_{\mathcal{R}_2} = \mathrm{Span}\{w(d-1,i), w_r\colon\ 0 \leqslant r \leqslant d-1,\ i \in \mathbb{Z}_n\}$$

and

$$N_{\mathcal{R}_2} = \mathrm{Span}\{w_r\colon\ 0 \leqslant r \leqslant d-1\}.$$

Hence,

$$\mathcal{C}_{\mathcal{R}_2} = M_{\mathcal{R}_2}/N_{\mathcal{R}_2} = \mathrm{Span}\{\overline{w(d-1,i)}\colon\ i \in \mathbb{Z}_n\}.$$

(3) By Corollary 5.2, it is easy to see that

$$\overline{\mathcal{R}_3} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3, \quad \underline{\overline{\mathcal{R}_3}} = \mathcal{R}_1 \cup \mathcal{R}_2.$$

It follows that

$$M_{\mathcal{R}_3} = \operatorname{Span}\{w(l,i), w_r\colon\ 0 \leqslant l,\, r \leqslant d-1,\, i \in \mathbb{Z}_n\}$$

and

$$N_{\mathcal{R}_3} = \operatorname{Span}\{w_r, w(d-1,i)\colon\ 0 \leqslant r \leqslant d-1,\, i \in \mathbb{Z}_n\}.$$

Hence,

$$\mathcal{C}_{\mathcal{R}_3} = M_{\mathcal{R}_3}/N_{\mathcal{R}_3} = \operatorname{Span}\{\overline{w(l,i)}\colon\ 0 \leqslant l \leqslant d-2,\, i \in \mathbb{Z}_n\}.$$

The result follows. $\square$

In the sequel, we describe the structures of right cell modules $\mathcal{C}_{\mathcal{R}_i}$ $(i=1,2,3)$.

For the right cell module $\mathcal{C}_{\mathcal{R}_1}$, we have:

Theorem 5.4. *$\mathcal{C}_{\mathcal{R}_1}$ is isomorphic to $\mathcal{V}$.*

Proof. Set $\omega_k = w_{k-1}$ for $1 \leqslant k \leqslant d$. The actions of $A_{m,d}$ on $\mathcal{C}_{\mathcal{R}_1}$ are the same as those of $\mathcal{V}$. The result follows from Lemma 2.3 and Lemma 4.1. $\square$

For the right cell $A_{m,d}$-module $\mathcal{C}_{\mathcal{R}_2}$, recall that $\mathcal{C}_{\mathcal{R}_2} = \operatorname{Span}\{w_{d-1,i} := \overline{w(d-1,i)}\colon$ $i \in \mathbb{Z}_n\}$ by Proposition 5.3. The actions of $A_{m,d}$ on $\mathcal{C}_{\mathcal{R}_2}$ now can be given by

$$w_{d-1,i}\cdot x = w_{d-1,i-1}, \quad w_{d-1,i}\cdot y = w_{d-1,i} + w_{d-1,i-m}, \quad w_{l,i}\cdot z = 0.$$

Set

$$w_d(i) = \frac{1}{n}(w_{d-1,0} + \eta^{2i}w_{d-1,1} + \eta^{4i}w_{d-1,2} + \ldots + \eta^{2(n-1)i}w_{d-1,n-1}),$$

where $0 \leqslant i \leqslant n-1$.

A straightforward verification shows that

(1) if $d \mid i$, we have

$$w_d(i)\cdot x = \eta^{2i}w_d(i), \quad w_d(i)\cdot y = 2w_d(i), \quad w_d(i)\cdot z = 0.$$

We get a simple submodule $S_{i,0}$ of $\mathcal{C}_{\mathcal{R}_2}$.

(2) if $d \nmid i$, we have

$$w_d(i)\cdot x = \eta^{2i}w_d(i), \quad w_d(i)\cdot y = (1+\eta^{2im})w_d(i), \quad w_d(i)\cdot z = 0.$$

We get a simple submodule $S_{i,\bar{i}}$ of $\mathcal{C}_{\mathcal{R}_2}$.

In fact, we have:

Theorem 5.5. *$\mathcal{C}_{\mathcal{R}_2}$ is decomposable and* $\mathcal{C}_{\mathcal{R}_2} = \bigoplus\limits_{\substack{0\leqslant i\leqslant n-1\\ d\mid i}} S_{i,0} \oplus \bigoplus\limits_{\substack{0\leqslant i\leqslant n-1\\ d\nmid i}} S_{i,\bar{i}}.$

P r o o f. It is obvious that

$$\sum_{\substack{0\leqslant i\leqslant n-1\\ d|i}} S_{i,0}+\sum_{\substack{0\leqslant i\leqslant n-1\\ d\nmid i}} S_{i,\bar{i}}\subseteq \mathcal{C}_{\mathcal{R}_2}.$$

On the other hand, $S_{i,\bar{i}}$, where $0\leqslant i\leqslant md-1$, are nonisomorphic 1-dimensional simple modules by Lemma 4.2. Therefore the left hand side is in fact direct sum. Comparing the dimension of both the sides, we get

$$\mathcal{C}_{\mathcal{R}_2}=\bigoplus_{\substack{0\leqslant i\leqslant n-1\\ d|i}} S_{i,0}\oplus\bigoplus_{\substack{0\leqslant i\leqslant n-1\\ d\nmid i}} S_{i,\bar{i}}.$$

The proof is completed. □

To consider the structure of $\mathcal{C}_{\mathcal{R}_3}$, recall that

$$\mathcal{C}_{\mathcal{R}_3}=\mathrm{Span}\{w_{l,i}:=\overline{w(l,i)}\colon\ 0\leqslant l\leqslant d-2,\ i\in\mathbb{Z}_n\}$$

by Proposition 5.3. The actions of $A_{m,d}$ on $\mathcal{C}_{\mathcal{R}_3}$ are given by

$$\begin{aligned}
&w_{l,i}\cdot x=w_{l,i-1},\\
&w_{l,i}\cdot y=\begin{cases} w_{1,i}, & l=0,\\ w_{l+1,i}+w_{l-1,i-m}, & 1\leqslant l<d-2,\\ w_{d-3,i-m}, & l=d-2,\end{cases}\\
&w_{l,i}\cdot z=0.
\end{aligned}$$

For each $1\leqslant l\leqslant d-1$ and $0\leqslant i\leqslant n-1$, we set

$$w_l(i)=\frac{1}{n}(w_{l-1,0}+\eta^{2i}w_{l-1,1}+\eta^{4i}w_{l-1,2}+\ldots+\eta^{2(n-1)i}w_{l-1,n-1}).$$

It is noted that $w_d(i)=0$ in $\mathcal{C}_{\mathcal{R}_3}$.

Let $\mathcal{W}_i$ be the vector space spanned by $\{w_1(i),w_2(i),\ldots,w_{d-1}(i)\}$ with the actions of $A_{m,d}$ on $\mathcal{W}_i$ being

$$\begin{aligned}
&w_l(i)\cdot x=\eta^{2i}w_l(i),\\
&w_l(i)\cdot y=\begin{cases} w_2(i), & l=1,\\ \eta^{2im}w_{l-1}(i)+w_{l+1}(i), & 2\leqslant l\leqslant d-2,\\ \eta^{2im}w_{d-2}(i), & l=d-1,\end{cases}\\
&w_l(i)\cdot z=0.
\end{aligned}$$

Lemma 5.6. *$\mathcal{W}_i$ is a submodule of $\mathcal{C}_{\mathcal{R}_3}$.*

P r o o f. The proof which is similar to that of Lemma 2.2 shows that

$$(w_l(i)\cdot x)\cdot z = (w_l(i)\cdot z)\cdot x = w_l(i)\cdot z, \quad (w_l(i)\cdot y)\cdot z = w_l(i)\cdot 2z, \quad (w_l(i)\cdot z)\cdot z = w_l(i)\cdot z,$$

and

$$((w_l(i)\cdot \underbrace{x)\cdot x\ldots x}_{md}) = \eta^{2ni}w_l(i) = w_l(i).$$

For $1 \leqslant l \leqslant d-1$, we have

$$(w_l(i)\cdot x)\cdot y = \begin{cases} \eta^{2i}w_l(i)\cdot y = \eta^{2i}w_{l+1}(i) = (w_l(i)\cdot y)\cdot x, & l=1,\\ \eta^{2(m+1)i}w_{l-1}(i) + \eta^{2i}w_{l+1}(i) = (w_l(i)\cdot y)\cdot x, & 2\leqslant l\leqslant d-2,\\ (\eta^{2(m+1)i})w_{d-2}(i) = (w_{d-1}(i)\cdot y)\cdot x, & l=d-1. \end{cases}$$

If $1 \leqslant l \leqslant d-1$, then

$$(w_l(i)\cdot(1+x^m-y))\cdot D_{d-1}(x^m,y) = w_d(i)\cdot(1+x^m-y)D_{l-1}(x^m,y) = 0.$$

The actions of x, y and z on $\mathcal{W}_i$ keep the defining relations of $A_{m,d}$. Hence, $\mathcal{W}_i$ is a submodule of $\mathcal{C}_{\mathcal{R}_3}$. □

It is not hard to show that the sum of $\mathcal{W}_i$ is direct sum since $\mathcal{W}_i$ is the eigenvector space of the eigenvalue η^{2i} of x. Moreover,

$$\dim_{A_{m,d}}(\mathcal{W}_0\oplus\mathcal{W}_1\oplus\ldots\oplus\mathcal{W}_{n-1}) = (d-1)n = \dim_{A_{m,d}}\mathcal{C}_{\mathcal{R}_3}.$$

Therefore,

$$\mathcal{C}_{\mathcal{R}_3} = \mathcal{W}_0\oplus\mathcal{W}_1\oplus\ldots\oplus\mathcal{W}_{n-1}$$

by Lemma 5.6.

Theorem 5.7. *$\mathcal{C}_{\mathcal{R}_3}$ is decomposable and* $\mathcal{C}_{\mathcal{R}_3} \cong \bigoplus\limits_{i=0}^{n-1}\bigoplus\limits_{j=1}^{d-1} S_{i,j}$.

P r o o f. Recall that

$$\mathcal{C}_{\mathcal{R}_3} = \mathcal{W}_0\oplus\mathcal{W}_1\oplus\ldots\oplus\mathcal{W}_{n-1}.$$

Therefore, it is sufficient to show that

$$\mathcal{W}_i \cong \bigoplus_{j=1}^{d-1} S_{i,j}.$$

To see this statement, we let A_i, B_i, C_i be the matrices of x, y, z acting on $\{w_1(i), w_2(i), \ldots, w_{d-1}(i)\}$. Then A_i is the scalar matrix $\eta^{2i}E$ and C_i is the zero matrix of size $(d-1) \times (d-1)$ on each basis. As for B_i, we have

$$B_i = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ \eta^{2im} & 0 & 1 & \ldots & 0 & 0 \\ 0 & \eta^{2im} & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & \eta^{2im} & 0 \end{pmatrix}_{(d-1)\times(d-1).}$$

It is straightforward to see that B_i has eigenvalues

$$\sigma_j = 2\left(\cos\frac{j\pi}{d}\right) \text{ if } d \mid i \quad \text{and} \quad \sigma_{i,j} \quad \text{if } d \nmid i,$$

where $1 \leqslant j \leqslant d-1$.

For each $\mathcal{W}_i$ $(0 \leqslant i \leqslant n-1)$, whether $d \mid i$ or $d \nmid i$, its submodules

$$v_{i,j} \cdot x = \eta^{2i} v_{i,j}, \quad v_{i,j} \cdot y = \sigma_j v_{i,j}, \quad v_{i,j} \cdot z = 0$$

or

$$v_{i,j} \cdot x = \eta^{2i} v_{i,j}, \quad v_{i,j} \cdot y = \sigma_{i,j} v_{i,j}, \quad v_{i,j} \cdot z = 0$$

are isomorphic to $S_{i,j}$ respectively. Consequently,

$$\mathcal{W}_i \cong \bigoplus_{j=1}^{d-1} S_{i,j} \quad \text{and} \quad \mathcal{C}_{\mathcal{R}_3} \cong \bigoplus_{i=0}^{n-1} \bigoplus_{j=1}^{d-1} S_{i,j}.$$

The proof is completed. □

Remark 5.8. We can also describe all the left cell modules of $A_{m,d}$. More precisely, by the discussion analogous as above, we conclude that there are $d+2$ left cells in $A_{m,d}$:

$$\mathcal{L}_r = \{r\} \quad \text{for } 0 \leqslant r \leqslant d-1, \ \mathcal{R}_2, \text{ and } \mathcal{R}_3.$$

However, there are only three left cell modules $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ up to isomorphism, which are listed as follows.

(1) $\mathcal{C}_1$: it is spanned by $\{w_1\}$, the actions of $A_{m,d}$ are given by

$$x \cdot w_1 = w_1, \quad y \cdot w_1 = 2w_1, \quad z \cdot w_1 = w_1.$$

It is noticed that $\mathcal{C}_{\mathcal{L}_r}$ is isomorphic to $\mathcal{C}_1$ for $0 \leqslant r \leqslant d-1$.

(2) $\mathcal{C}_2$: it is spanned by $\{w(d-1,i)\colon\ i\in\mathbb{Z}_n\}$, the actions of $A_{m,d}$ on $\mathcal{C}_2$ are given by

$$x\cdot w_{d-1,i}=w_{d-1,i-1},\quad y\cdot w_{d-1,i}=w_{d-1,i}+w_{d-1,i-m},\quad z\cdot w_{l,i}=0.$$

(3) $\mathcal{C}_3$: it is spanned by $\{w(l,i)\colon\ 0\leqslant l\leqslant d-2,\ i\in\mathbb{Z}_n\}$, the actions of $A_{m,d}$ on $\mathcal{C}_3$ are given by

$$\begin{aligned}
x\cdot w_{l,i}&=w_{l,i-1},\\
y\cdot w_{l,i}&=\begin{cases} w_{1,i}, & l=0,\\ w_{l+1,i}+w_{l-1,i-m}, & 1\leqslant l<d-2,\\ w_{d-3,i-m}, & l=d-2,\end{cases}\\
z\cdot w_{l,i}&=0.
\end{aligned}$$

Of course, Theorems 5.5 and 5.7 also hold for the left cell modules C_2 and C_3, respectively.

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