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ON THE CLASS NUMBER OF THE MAXIMAL REAL SUBFIELDS OF A FAMILY OF CYCLOTOMIC FIELDS

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Abstract. For any square-free positive integer $m \equiv 10 \pmod{16}$ with $m \geqslant 26$, we prove that the class number of the real cyclotomic field $\mathbb{Q}(\zeta_{4m}+\zeta_{4m}^{-1})$ is greater than 1, where ζ_{4m} is a primitive $4m$th root of unity.

Keywords: maximal real subfield of cyclotomic field; real quadratic field; class number

MSC 2020: 11R29, 11R18, 11R11

1. Introduction

Let $m > 1$ be a positive integer, we use ζ_m to denote a primitive mth root of unity, and $H(m)$ to denote the class number of the maximal real subfield $K_m = \mathbb{Q}(\zeta_m+\zeta_m^{-1})$ of the cyclotomic field $\mathbb{Q}(\zeta_m)$. Further, let $h(m)$ denote the class number of the quadratic field $\mathbb{Q}(\sqrt{m})$.

The study of the class number of the maximal real subfield of a cyclotomic field is important. There are many beautiful results in this context; we mention some of them. In 1965, Ankeny, Chowla and Hasse in [1] proved that if $m = (2nq)^2 + 1$ is a prime, where q is a prime and $n > 1$ is an integer, then $H(m) > 1$. Later, in 1974, Yamaguchi in [9] relaxed the primality condition on m and took m as a square-free integer and proved $H(4m) > 1$. Lang in [6] proved $H(p) > 1$ for the prime $p = ((2n+1)q)^2 + 4$, where q is an odd prime and $n \geqslant 1$ is an integer. There are many more such results in that direction, we refer the interested reader to consult [4], [5], [7], [8] for more details.

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All results have been proved for the quadratic expression m, but in this article, we prove the same result in this direction for the linear expression $m = 16t + 10$, where $t \geqslant 1$ is an integer. Precisely, we prove the following theorem.

Theorem 1.1. *Let $t \geqslant 1$ be an integer such that $m = 16t + 10$ is a square-free integer. Then $H(4m) > 1$. Moreover, $2 \mid H(4m)$.*

It is well known that the class number of a cyclotomic field is divisible by the class number of its maximal real subfield. So, we have an immediate consequence of Theorem 1.1.

Corollary 1.2. *Let m be a square-free positive integer as in Theorem 1.1. Then the class number of the cyclotomic field $\mathbb{Q}(\zeta_{4m})$ is divisible by 2.*

2. Preliminaries

We begin this section by stating the theorem of Yamaguchi, see [9].

Theorem 2.1. *Let m be a square-free positive integer such that $\varphi(m) > 4$. Then $h(m) \mid H(4m)$, where $\varphi(m)$ stands for Euler's function.*

Analogously to the results of Gica (see [2]), we prove the following.

Proposition 2.2. *If $m \equiv 10 \pmod{16}$ is a positive integer then there exists a prime $p \equiv 3$ or $5 \pmod 8$ such that $p < m$ and m is a quadratic residue modulo p.*

The above proposition plays an important role in proving the main theorem.

Proof. Since $m \equiv 10 \pmod{16}$ is a positive integer, this implies that there exists a nonnegative integer t such that $m = 16t + 10$.

Consider an element $a = \frac{1}{2}(m - 4) = 8t + 3$. Clearly, $a < m$ and $a \equiv 3 \pmod 8$, it follows that a has a prime factor p such that $p \equiv 3$ or $5 \pmod 8$ and $p < m$. Furthermore, $a \equiv 0 \pmod p$ implies $m \equiv 2^2 \pmod p$ and thus, m is a quadratic residue modulo p. Hence, we have proved the proposition. □

Hasse in [3] proved that if the class number of the real quadratic field $\mathbb{Q}(\sqrt{D})$ is one, then $D = p$, $2p$ or qr, where p, q and r are primes, and $p \equiv 1 \pmod 4$, $q \equiv r \equiv 3 \pmod 4$.

However, there are real quadratic fields of the form $\mathbb{Q}(\sqrt{2p})$, with primes $p \equiv 1 \pmod 4$, whose class number is greater than one. In this regard, we prove a more general proposition.

Proposition 2.3. *Let $m = 16t + 10$ be a square-free integer, where t is a positive integer. Then the class number $h(m)$ of the real quadratic field $F = \mathbb{Q}(\sqrt{m})$ is greater than one.*

By Dirichlet's theorem on primes in arithmetic progression there are infinitely many primes p of the form $8t + 5$, hence from Proposition 2.3 it follows that there are infinitely many fields $\mathbb{Q}(\sqrt{2p})$ with prime $p \equiv 1 \pmod 4$, whose class number is greater than one.

P r o o f of Proposition 2.3. Let p be a prime as in Proposition 2.2. Then p splits in the ring of integers $\mathcal{O}_F$ of F. Thus, we obtain $p\mathcal{O}_F = \mathfrak{P}_1\mathfrak{P}_2$, where $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are distinct prime ideals in F above p. We claim that the prime ideal $\mathfrak{P}_1$ is not principal.

If prime ideal $\mathfrak{P}_1$ is principal then by taking its norm we obtain the integers x and y such that

$$x^2 - my^2 = \pm p. \tag{1}$$

Reading the above equation (1) modulo 8 leads to a contradiction and thus $\mathfrak{P}_1$ is a nonprincipal ideal. The ideal class containing $\mathfrak{P}_1$ is nontrivial. Hence, Proposition 2.3 follows. □

We observe that $8t + 5$ has a prime divisor q such that $q \equiv 3$ or $5 \pmod 8$. This prime q divides m, it implies that q is ramified in F.

Let $\mathfrak{Q}$ be the prime ideal of $\mathcal{O}_F$ above q. By using similar arguments, as used to show that $\mathfrak{P}_1$ is nonprincipal, we can show that $\mathfrak{Q}$ is nonprincipal. This implies that $h(m)$ is divisible by 2, as $\mathfrak{Q}^2$ is principal.

If the prime ideals $\mathfrak{Q}$ and $\mathfrak{P}_1$ do not belong to the same ideal class then $h(m) \geqslant 4$.

3. Proof of Theorem 1.1

If $8t + 5$ is a prime then $\varphi(m) = 8t + 4$ which is greater than 4 as $t \geqslant 1$.

On the other hand, if $8t+5$ is not a prime then it has at least two prime divisors, say p_1 and p_2, and then $\varphi(m) \geqslant \varphi(p_1p_2) \geqslant 8$. From Theorem 2.1, it follows that $h(m) \mid H(4m)$. In the proof of Proposition 2.3, we have seen that $h(m) > 1$. Thus, $H(4m) > 1$. Moreover, $H(4m)$ is divisible by 2 as $2 \mid h(m)$. This completes the proof of the theorem. □

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