Kazushi Yoshitomi

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FREDHOLMNESS OF PSEUDO-DIFFERENTIAL OPERATORS WITH NONREGULAR SYMBOLS

Kazushi Yoshitomi, Tokyo

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Abstract. We establish the Fredholmness of a pseudo-differential operator whose symbol is of class $C^{0,\sigma}$, $0<\sigma<1$, in the spatial variable. Our work here refines the work of H. Abels, C. Pfeuffer (2020).

Keywords: Fredholmness; pseudo-differential operator; nonregular symbol

MSC 2020: 35S05, 47A53, 47G30

1. Introduction

The starting point of our consideration is the result of Abels and Pfeuffer (see [2]), which is recalled below. Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{Z}_+$ the set of all nonnegative integers. Let $n\in\mathbb{N}$. We fix $0<\tau\leqslant 1$, $0\leqslant\varrho$, $\delta\leqslant 1$ and $M\in\mathbb{Z}_+\cup\{\infty\}$. For $m\in\mathbb{R}$ and $\widetilde{m}\in\mathbb{Z}_+$, let $C^{\widetilde{m},\tau}S^m_{\varrho,\delta}(\mathbb{R}^n\times\mathbb{R}^n;M)$ stand for the set of functions $a\colon\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$ such that the following properties are satisfied for all α, $\beta\in\mathbb{Z}_+^n$ with $|\beta|\leqslant\widetilde{m}$ and $|\alpha|\leqslant M$:

(i) $\partial_x^\beta a(x,\cdot)\in C^M(\mathbb{R}^n)$ for all $x\in\mathbb{R}^n$;

(ii) $\partial_x^\beta\partial_\xi^\alpha a\in C^0(\mathbb{R}^n_x\times\mathbb{R}^n_\xi)$;

(iii) $|\partial_\xi^\alpha a(x,\xi)|\leqslant C_\alpha\langle\xi\rangle^{m-\varrho|\alpha|}$ for all $x,\xi\in\mathbb{R}^n$;

(iv) $\|\partial_\xi^\alpha a(\cdot,\xi)\|_{C^{\widetilde{m},\tau}(\mathbb{R}^n)}\leqslant C_\alpha\langle\xi\rangle^{m-\varrho|\alpha|+\delta(\widetilde{m}+\tau)}$ for all $\xi\in\mathbb{R}^n$.

We fix $N\in\mathbb{N}$. Let $\mathcal{L}(\mathbb{C}^N)$ stand for the set of all linear bounded operators on $\mathbb{C}^N$; $\mathcal{L}(\mathbb{C}^N)$ is identified with $\mathbb{C}^{N\times N}$ in the standard way. A function $a\colon\mathbb{R}^n\times\mathbb{R}^n\to\mathcal{L}(\mathbb{C}^N)$ is an element of $C^{\widetilde{m},\tau}S^m_{\varrho,\delta}(\mathbb{R}^n\times\mathbb{R}^n;M;\mathcal{L}(\mathbb{C}^N))$ if $a_{j,k}$, the (j,k)-component of a, belongs to $C^{\widetilde{m},\tau}S^m_{\varrho,\delta}(\mathbb{R}^n\times\mathbb{R}^n;M)$ for all $j,k=1,\ldots,N$. For a symbol $a\in C^{\widetilde{m},\tau}S^m_{\varrho,\delta}(\mathbb{R}^n\times\mathbb{R}^n;M;\mathcal{L}(\mathbb{C}^N))$, we define the associated pseudo-

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differential operator by

$$\begin{aligned}\operatorname{Op}(a)u(x) &= a(x, D_x)u(x) \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi} a(x,\xi)\widehat{u}(\xi)\,\mathrm{d}\xi \quad \forall\, u \in \mathcal{S}(\mathbb{R}^n)^N,\ x \in \mathbb{R}^n.\end{aligned}$$

Here, $\widehat{u}$ stands for the Fourier transform of u and $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space. The main result of [2] reads as follows.

Theorem 1.1 ([2], Theorem 1.1)**.** *Let $\widetilde{m}, N \in \mathbb{N}$, $0 < \tau < 1$, $0 \leqslant \delta < \varrho \leqslant 1$, $m \in \mathbb{R}$, $M \in \mathbb{Z}_+ \cup \{\infty\}$ and $p \in (1,\infty)$ with $p = 2$ if $\varrho \neq 1$. Additionally we choose an arbitrary $\theta \in (0, \min\{(\widetilde{m}+\tau)(\varrho-\delta), 1\})$ and $\widetilde{\varepsilon} \in (0, \min\{(\varrho-\delta)\tau, (\varrho-\delta)(\widetilde{m}+\tau)-\theta, \theta\})$. Moreover let $a \in C^{\widetilde{m},\tau} S^m_{\varrho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M; \mathcal{L}(\mathbb{C}^N))$ be a symbol fulfilling the following properties for some $R > 0$ and $C_0 > 0$:*

(1) *$|\det(a(x,\xi))|\langle\xi\rangle^{-mN} \geqslant C_0$ for all $x, \xi \in \mathbb{R}^n$ with $|x| + |\xi| \geqslant R$;*

(2) *$a(x,\xi) \overset{|x|\to\infty}{\longrightarrow} a(\infty,\xi)$ for all $\xi \in \mathbb{R}^n$.*

Then for all $M \geqslant (n+2) + n \cdot \max\{1/2, 1/p\}$ and $s \in \mathbb{R}$ with

$$(1-\varrho)\frac{n}{2} - (1-\delta)(\widetilde{m}+\tau) + \theta + \widetilde{\varepsilon} < s < \widetilde{m} + \tau$$

the operator

$$a(x, D_x)\colon\ H^{m+s}_p(\mathbb{R}^n)^N \to H^s_p(\mathbb{R}^n)^N$$

is a Fredholm operator.

Theorem 1.1 does not cover the case where $\widetilde{m} = 0$ because of a technical reason. So we shall handle this case. Our main result is stated as follows.

Theorem 1.2. *Let $N \in \mathbb{N}$, $0 \leqslant \tau < 1$, $0 < \sigma < 1$, $m \in \mathbb{R}$, $M \in \mathbb{Z}_+ \cup \{\infty\}$ and $p \in (1,\infty)$. Let $a \in C^{0,\sigma} S^m_{1,\tau}(\mathbb{R}^n \times \mathbb{R}^n; M; \mathcal{L}(\mathbb{C}^N))$ be a symbol satisfying the following properties for some $R > 0$ and $C_0 > 0$:*

(1) *$|\det(a(x,\xi))|\langle\xi\rangle^{-mN} \geqslant C_0$ for all $x, \xi \in \mathbb{R}^n$ with $|x| + |\xi| \geqslant R$;*

(2) *$a(x,\xi) \overset{|x|\to\infty}{\longrightarrow} a(\infty,\xi)$ for all $\xi \in \mathbb{R}^n$.*

Then for all $M > (n+1) + n \cdot \max\{1/2, 1/p\}$ and $s \in \mathbb{R}$ with

$$-(1-\tau)\sigma < s < \sigma$$

the operator

$$a(x, D_x)\colon\ H^{m+s}_p(\mathbb{R}^n)^N \to H^s_p(\mathbb{R}^n)^N$$

is a Fredholm operator.

It is worth mentioning that the condition on M in Theorem 1.2 is milder than that in Theorem 1.1. In order to prove Theorem 1.2 we employ the decomposition of symbols inspired by Nagase in [6], see Theorem 2.2. This, together with several results in [2] and our tricky arguments in Lemmas 3.1 and 3.2, completes the proof. We note that [2] uses the decomposition of symbols due to Taylor (see [7]), which is different from Nagase's. It is also worth mentioning that our proof here is somewhat simpler than that in [2].

The Fredholm properties of (smooth) pseudodifferential operators have been studied by many authors; we refer to [3], [4], [5] and the references therein. In [4] Kohn and Nirenberg show that for the Fredholmness of a pseudo-differential operator the ellipticity of its symbol is necessary. Kumano-go in [5], Chapter 3, Theorem 5.16 shows the Fredholm property of an elliptic pseudo-differential operator whose symbol belongs to a subclass of the Hörmander class and is slowly varying. In [3], Section 19.2, Hörmander studies the Fredholm index of an elliptic pseudo-differential operator on a compact manifold.

We organize this paper as follows. In Section 2 we prepare several notations and recall known results which are utilized in the proof of Theorem 1.2. In Section 3 we complete the proof of Theorem 1.2.

2. Preliminaries

We prepare several notations. We use the standard conventions on multi-indices. For a variable $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, we write

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \ldots \partial_{x_n}^{\alpha_n} \quad \text{and} \quad D_x^\alpha = (-i)^{|\alpha|} \partial_x^\alpha,$$

where $\partial_{x_j} = \partial/\partial x_j$, $j = 1, 2, \ldots, n$. We put

$$\langle x \rangle = \sqrt{1 + |x|^2}.$$

We define the Hölder spaces as follows. For $0 < s \leqslant 1$, let $C^{0,s}(\mathbb{R}^n)$ stand for the set of functions $f\colon \mathbb{R}^n \to \mathbb{C}$ such that

$$\|f\|_{C^{0,s}(\mathbb{R}^n)} \equiv \sup_{x \in \mathbb{R}^n} |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^s} < \infty.$$

Let $\widetilde{m} \in \mathbb{Z}_+$. A function $f\colon \mathbb{R}^n \to \mathbb{C}$ belongs to $C^{\widetilde{m},s}(\mathbb{R}^n)$ if we have $\partial_x^\alpha f \in C^{0,s}(\mathbb{R}^n)$ for every $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leqslant \widetilde{m}$. For $f \in C^{\widetilde{m},s}(\mathbb{R}^n)$, we define $\|f\|_{C^{\widetilde{m},s}(\mathbb{R}^n)} = \sum\limits_{|\alpha| \leqslant \widetilde{m}} \|\partial_x^\alpha f\|_{C^{0,s}(\mathbb{R}^n)}$.

For $s \in \mathbb{R}$ and $1 < p < \infty$, let $H^s_p(\mathbb{R}^n)$ be the set of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\langle D_x\rangle^s f \in L^p(\mathbb{R}^n)$, where $\langle D_x\rangle^s = \mathrm{Op}(\langle\xi\rangle^s)$. For $f \in H^s_p(\mathbb{R}^n)$, we put $\|f\|_{H^s_p(\mathbb{R}^n)} = \|\langle D_x\rangle^s f\|_{L^p(\mathbb{R}^n)}$. We call $H^s_p(\mathbb{R}^n)$ the Bessel potential space.

Next, we recall the definition of a Fredholm operator and the Atkinson theorem.

Definition 2.1. Let X, Y be Banach spaces. A linear bounded operator $T\colon X \to Y$ is called a *Fredholm operator* if $\operatorname{Ker} T$ is finite-dimensional and $T(X)$ is finite-codimensional, that is, there is a finite-dimensional subspace Z of Y such that $Y = T(X) \dot{+} Z$.

Theorem 2.1 (Atkinson). *Let X, Y be Banach spaces. A linear bounded operator $T\colon X \to Y$ is a Fredholm operator if and only if there exist linear bounded operators $S_j\colon Y \to X$, $j = 1, 2$, and compact linear operators $K_1\colon X \to X$ and $K_2\colon Y \to Y$ such that $S_1T = I_X + K_1$ and $TS_2 = I_Y + K_2$. Here, I_X and I_Y are identity operators on X and Y, respectively.*

We recall the decomposition of symbols due to Nagase, which plays a key role in the proof of Theorem 1.2. Let $p(x,\xi)$ be a bounded continuous function on $\mathbb{R}^n \times \mathbb{R}^n$. We choose a $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\int \varphi(y)\,\mathrm{d}y = 1$. Fix $0 < \delta < 1$ and put

$$q(x,\xi) = \int \varphi(y)p(x - \langle\xi\rangle^{-\delta}y, \xi)\,\mathrm{d}y. \tag{2.1}$$

Theorem 2.2 ([6], Theorem 1). *Let $M \in \mathbb{Z}_+ \cup \{\infty\}$. Assume that for all $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \leqslant M$,*

$$|D^\alpha_\xi p(x,\xi)| \leqslant C_\alpha \langle\xi\rangle^{-|\alpha|}, \tag{2.2}$$

$$|D^\alpha_\xi p(x,\xi) - D^\alpha_\xi p(y,\xi)| \leqslant C_\alpha |x-y|^\sigma \langle\xi\rangle^{-|\alpha|+\sigma\tau}, \tag{2.3}$$

where $0 < \sigma \leqslant 1$ and $0 \leqslant \tau < 1$, and set $r(x,\xi) = p(x,\xi) - q(x,\xi)$.

Then, there exist positive constants C_α and $C_{\alpha,\beta}$ such that for any α, β with $|\alpha| \leqslant M$ we have

$$|D^\beta_x D^\alpha_\xi q(x,\xi)| \leqslant C_{\alpha,\beta}\langle\xi\rangle^{-|\alpha|+\delta|\beta|}, \tag{2.4}$$

$$|D^\alpha_\xi r(x,\xi)| \leqslant C_\alpha \langle\xi\rangle^{-|\alpha|-(\delta-\tau)\sigma}. \tag{2.5}$$

We recall from [2] several materials needed in the proof of Theorem 1.2. Let N, $M \in \mathbb{Z}_+ \cup \{\infty\}$ and m, $\tau \in \mathbb{R}$. Let $\mathcal{A}^{m,N}_{\tau,M}(\mathbb{R}^n \times \mathbb{R}^n)$ stand for the set of all continuous functions $a\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that for all α, $\beta \in \mathbb{Z}^n_+$ with $|\alpha| \leqslant N$, $|\beta| \leqslant M$ we have

(i) $\partial^\alpha_\eta \partial^\beta_y a(y,\eta) \in C^0(\mathbb{R}^n_y \times \mathbb{R}^n_\eta)$;

(ii) $|\partial^\alpha_\eta \partial^\beta_y a(y,\eta)| \leqslant C_{\alpha,\beta}(1+|\eta|)^m(1+|y|)^\tau$ for all y, $\eta \in \mathbb{R}^n$, where all derivatives are well defined in the sense of distributions. We choose a $\chi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ for which $\chi(0,0) = 1$. For $a \in \mathcal{A}^{m,N}_{\tau,M}(\mathbb{R}^n \times \mathbb{R}^n)$, we define

$$\text{Os-}\iint \mathrm{e}^{-\mathrm{i}y\cdot\eta} a(y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta = \lim_{\varepsilon\to 0}\iint \chi(\varepsilon y, \varepsilon\eta)\mathrm{e}^{-\mathrm{i}y\cdot\eta}a(y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta \tag{2.6}$$

whenever the limit exists. It is called the *oscillatory integral*.

Theorem 2.3 ([2], Corollary 2.7). *Let $m, \tau \in \mathbb{R}$ and $N \in \mathbb{Z}_+ \cup \{\infty\}$ be such that there is some $l' \in \mathbb{N}$ with $N \geqslant l' > n + \tau$. Moreover, let $l \in \mathbb{N}$ with $l > n + m$. Additionally, let a_j, $a \in C^0(\mathbb{R}^n \times \mathbb{R}^n)$, $j \in \mathbb{Z}_+$ be such that for all α, $\beta \in \mathbb{Z}^n_+$ with $|\alpha| \leqslant N$ and $|\beta| \leqslant l$ the derivatives $\partial^\alpha_\eta \partial^\beta_y a_j$, $\partial^\alpha_\eta\partial^\beta_y a$ exist in the classical sense and*

▷ $|\partial^\alpha_\eta\partial^\beta_y a_j(y,\eta)| \leqslant C_{\alpha,\beta}\langle\eta\rangle^m\langle y\rangle^\tau$ *for all* η, $y \in \mathbb{R}^n$, $j \in \mathbb{Z}_+$;
▷ $|\partial^\alpha_\eta\partial^\beta_y a(y,\eta)| \leqslant C_{\alpha,\beta}\langle\eta\rangle^m\langle y\rangle^\tau$ *for all* η, $y \in \mathbb{R}^n$;
▷ $\partial^\alpha_\eta\partial^\beta_y a_j(y,\eta) \overset{j\to\infty}{\longrightarrow} \partial^\alpha_\eta\partial^\beta_y a(y,\eta)$ *for all* η, $y \in \mathbb{R}^n$.

Then

$$\lim_{j\to\infty} \text{Os-}\iint \mathrm{e}^{-\mathrm{i}y\cdot\eta}a_j(y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta = \text{Os-}\iint \mathrm{e}^{-\mathrm{i}y\cdot\eta}a(y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta.$$

For results analogous to this theorem, see [1], Corollary 3.10 and [5], Chapter 1, Theorem 6.6.

For x, $x' \in \mathbb{R}^n$, we put

$$\langle x; x'\rangle = \sqrt{1 + |x|^2 + |x'|^2}.$$

Definition 2.2. Let $\widetilde{m} \in \mathbb{Z}_+$, $0 < \tau < 1$, m_1, $m_2 \in \mathbb{R}$, $0 \leqslant \delta, \varrho \leqslant 1$ and M_1, $M_2 \in \mathbb{Z}_+ \cup \{\infty\}$. Then a continuous function $a\colon \mathbb{R}^n_x \times \mathbb{R}^n_\xi \times \mathbb{R}^n_{x'} \times \mathbb{R}^n_{\xi'} \to \mathbb{C}$ belongs to the nonsmooth double symbol-class $C^{\widetilde{m},\tau}S^{m_1,m_2}_{\varrho,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; M_1, M_2)$ if

(i) $\partial^\alpha_\xi\partial^{\beta'}_{x'}\partial^{\alpha'}_{\xi'} a \in C^{\widetilde{m},\tau}(\mathbb{R}^n_x)$ and $\partial^\beta_x\partial^\alpha_\xi\partial^{\beta'}_{x'}\partial^{\alpha'}_{\xi'} a \in C^0(\mathbb{R}^n_x \times \mathbb{R}^n_\xi \times \mathbb{R}^n_{x'} \times \mathbb{R}^n_{\xi'})$;
(ii)

$$\begin{aligned}|\partial^\beta_x\partial^\alpha_\xi\partial^{\beta'}_{x'}\partial^{\alpha'}_{\xi'} a(x,\xi,x',\xi')| \\ \leqslant C_{\alpha,\beta,\beta',\alpha'}(x)\widetilde{C}_{\alpha,\beta,\beta',\alpha'}(x')\langle\xi\rangle^{m_1-\varrho|\alpha|+\delta|\beta|}\langle\xi'\rangle^{m_2-\varrho|\alpha'|}\langle\xi;\xi'\rangle^{\delta|\beta'|};\end{aligned}$$

(iii)

$$\|\partial^\alpha_\xi\partial^{\beta'}_{x'}\partial^{\alpha'}_{\xi'} a(\cdot,\xi,x',\xi')\|_{C^{\widetilde{m},\tau}(\mathbb{R}^n)} \leqslant C_{\alpha,\beta',\alpha'}\langle\xi\rangle^{m_1-\varrho|\alpha|+\delta(\widetilde{m}+\tau)}\langle\xi'\rangle^{m_2-\varrho|\alpha'|}\langle\xi;\xi'\rangle^{\delta|\beta'|}$$

for all $x,\xi,x',\xi' \in \mathbb{R}^n$ and arbitrary $\beta,\alpha,\beta',\alpha' \in \mathbb{Z}^n_+$ with $|\beta| \leqslant \widetilde{m}$, $|\alpha| \leqslant M_1$ and $|\alpha'| \leqslant M_2$. Here the constants $C_{\alpha,\beta,\beta',\alpha'}(x)$, $C_{\alpha,\beta',\alpha'}$ and $\widetilde{C}_{\alpha,\beta,\beta',\alpha'}(x')$ are bounded and independent of $\xi, x', \xi' \in \mathbb{R}^n$ or $\xi, x, \xi' \in \mathbb{R}^n$, respectively.

We will utilize the following assertions on double symbols.

Theorem 2.4 ([2], Theorem 3.10). *Let $0 < s < 1$, $\widetilde{m} \in \mathbb{Z}_+$ and $m_1, m_2 \in \mathbb{R}$. Additionally we choose $N_1, N_2 \in \mathbb{Z}_+ \cup \{\infty\}$ such that there is an $l \in \mathbb{N}$ with $N_1 \geqslant l > n$. Moreover, we put $\widetilde{N} = \min\{N_1 - (n+1), N_2\}$. Furthermore, let*

$$\mathcal{B} \subset C^{\widetilde{m},s} S^{m_1,m_2}_{\varrho,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N_1, N_2)$$

be bounded. If we define for each $a \in \mathcal{B}$ and $\theta \in [0,1]$ the function $a^\theta_L \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ by

$$a^\theta_L(x,\xi) = \text{Os-}\iint \mathrm{e}^{-\mathrm{i}y\cdot\eta} a(x, \theta\eta + \xi, x+y, \xi)\,\mathrm{d}y\,\mathrm{d}\eta \quad \forall\, x, \xi \in \mathbb{R}^n,$$

we get with $m \equiv m_1 + m_2$ that $a^\theta_L \in C^{\widetilde{m},s} S^m_{\varrho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; \widetilde{N})$ for all $a \in \mathcal{B}$ and $\theta \in [0,1]$, and the existence of a constant C_α, independent of $a \in \mathcal{B}$ and $\theta \in [0,1]$, such that for all α, $\beta \in \mathbb{Z}^n_+$ with $|\alpha| \leqslant \widetilde{N}$ and $|\beta| \leqslant \widetilde{m}$

$$\|\partial^\alpha_\xi a^\theta_L(\cdot,\xi)\|_{C^{\widetilde{m},s}(\mathbb{R}^n)} \leqslant C_\alpha \langle\xi\rangle^{m-\varrho|\alpha|+\delta(\widetilde{m}+s)} \quad \forall\, \xi \in \mathbb{R}^n$$

and

$$|\partial^\alpha_\xi \partial^\beta_x a^\theta_L(x,\xi)| \leqslant C_{\alpha,\beta}(x) \langle\xi\rangle^{m-\varrho|\alpha|+\delta|\beta|} \quad \forall\, x, \xi \in \mathbb{R}^n,$$

where $C_{\alpha,\beta}(x)$ is bounded and independent of $a \in \mathcal{B}$, $\xi \in \mathbb{R}^n$ and $\theta \in [0,1]$. This implies the boundedness of $\{a^\theta_L \colon\ a \in \mathcal{B},\ \theta \in [0,1]\} \subset C^{\widetilde{m},s} S^m_{\varrho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; \widetilde{N})$.

Theorem 2.5 ([2], Theorem 4.5). *Let $0 \leqslant \delta \leqslant \varrho \leqslant 1$, $m_1, m_2 \in \mathbb{R}$, $M_1, M_2 \in \mathbb{Z}_+ \cup \{\infty\}$ with $M_1 > n+1$, $\widetilde{m} \in \mathbb{Z}_+$ and $0 < \tau < 1$. For $a \in C^{\widetilde{m},\tau} S^{m_1,m_2}_{\varrho,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; M_1, M_2)$ we define*

$$a_L(x,\xi) = (2\pi)^{-n} \cdot \text{Os-}\iint \mathrm{e}^{-\mathrm{i}y\cdot\eta} a(x, \xi+\eta, x+y, \xi)\,\mathrm{d}y\,\mathrm{d}\eta \quad \forall\, x, \xi \in \mathbb{R}^n.$$

Additionally we set for all $\theta \in [0,1]$ and $\gamma \in \mathbb{Z}^n_+$ with $|\gamma| \leqslant M_1 - (n+1)$

$$r_{\gamma,\theta}(x,\xi) = (2\pi)^{-n} \cdot \text{Os-}\iint \mathrm{e}^{-\mathrm{i}y\cdot\eta} \partial^\gamma_\eta D^\gamma_y a(x, \xi+\theta\eta, x+y, \xi)\,\mathrm{d}y\,\mathrm{d}\eta \quad \forall\, x, \xi \in \mathbb{R}^n.$$

Moreover we put $\widetilde{M_k} = \min\{M_1 - k - (n+1), M_2\}$ for all $k \leqslant M_1 - (n+1)$. Then we get for all $N \leqslant M_1 - (n+1)$ that

$$a_L(x,\xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!} \partial^\alpha_\eta D^\alpha_y a(x, \xi+\eta, x+y, \xi)\big|_{\eta=y=0} + R_N(x,\xi),$$

where

$$R_N(x,\xi) \equiv N \cdot \sum_{|\gamma|=N} \int_0^1 \frac{(1-\theta)^{N-1}}{\gamma!} r_{\gamma,\theta}(x,\xi)\,\mathrm{d}\theta \in C^{\widetilde{m},\tau} S^{m_1+m_2-(\varrho-\delta)\cdot N}_{\varrho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; \widetilde{M}_N)$$

and

$$\{r_{\gamma,\theta}(x,\xi)\colon\ \theta \in [0,1]\} \subset C^{\widetilde{m},\tau} S^{m_1+m_2-(\varrho-\delta)\cdot N}_{\varrho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; \widetilde{M}_N)$$

is bounded.

We need the following implication concerning the action of a pseudo-differential operator on the Bessel potential spaces.

Theorem 2.6 ([2], Theorem 3.2). *Let $m \in \mathbb{R}$, $0 \leqslant \delta \leqslant \varrho \leqslant 1$ with $\varrho > 0$, $1 < p < \infty$ and $M \in \mathbb{Z}_+ \cup \{\infty\}$ with $M > \max\{n/2, n/p\}$. Additionally let $\widetilde{m} \in \mathbb{Z}_+$ and $0 < \tau \leqslant 1$ be such that $\widetilde{m} + \tau > (1-\varrho)/(1-\delta) \cdot (n/2)$ if $\varrho < 1$ and $\widetilde{m} \in \mathbb{Z}_+$, $\tau > 0$ if $\varrho = 1$, respectively. Moreover let $\mathcal{B} \subset C^{\widetilde{m},\tau} S^{m-k_p}_{\varrho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ be bounded with $k_p \equiv (1-\varrho)n|1/2 - 1/p|$ and let $(1-\varrho)n/p - (1-\delta)(\widetilde{m}+\tau) < s < \widetilde{m}+\tau$. Then there is some $C_s > 0$, independent of $a \in \mathcal{B}$, such that*

$$\|a(x,D_x)f\|_{H^s_p(\mathbb{R}^n)} \leqslant C_s \|f\|_{H^{s+m}_p(\mathbb{R}^n)} \quad \forall\, a \in \mathcal{B} \quad \text{and} \quad f \in H^{s+m}_p(\mathbb{R}^n).$$

The following result concerning the compactness of a pseudo-differential operator is due to Marschall.

Lemma 2.1 ([2], Lemma 4.2). *Let $m \in \mathbb{R}$, $0 \leqslant \delta \leqslant 1$, $1 < p < \infty$, $\widetilde{m} \in \mathbb{Z}_+$ and $0 < \tau < 1$. Moreover, let $M \in \mathbb{N} \cup \{\infty\}$ with $M > n \cdot \max\{1/2, 1/p\}$. Additionally let $a \in C^{\widetilde{m},\tau} S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ be such that*

$$\lim_{|x|+|\xi|\to\infty} (1+|\xi|)^{-m} a(x,\xi) = 0.$$

Then for $-(1-\delta)(\widetilde{m}+\tau) < s < \widetilde{m}+\tau$

$$a(x,D_x)\colon\ H^{s+m}_p(\mathbb{R}^n) \to H^s_p(\mathbb{R}^n) \text{ is compact.}$$

For $0 \leqslant \varrho, \delta \leqslant 1$ and $M \in \mathbb{Z}_+ \cup \{\infty\}$, let $S^m_{\varrho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ be the set of functions $a\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that the following properties are fulfilled for all $\alpha, \beta \in \mathbb{Z}^n_+$ with $|\alpha| \leqslant M$:

(i) $\partial^\beta_x a(x,\cdot) \in C^M(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$;

(ii) $\partial^\beta_x \partial^\alpha_\xi a \in C^0(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$;

(iii) $|\partial^\beta_x \partial^\alpha_\xi a(x,\xi)| \leqslant C_{\alpha,\beta} \langle\xi\rangle^{m-\varrho|\alpha|+\delta|\beta|}$ for all x, $\xi \in \mathbb{R}^n$.

We will utilize the embedding result, see [2], (3.1):

$$S^{m}_{\varrho,\delta}(\mathbb{R}^n\times\mathbb{R}^n;M)\subset C^{\widetilde{m},s}S^{m}_{\varrho,\delta}(\mathbb{R}^n\times\mathbb{R}^n;M) \tag{2.7}$$

for all $0<s\leqslant 1$, $\widetilde{m}\in\mathbb{Z}_+$, $m\in\mathbb{R}$, $M\in\mathbb{Z}_+\cup\{\infty\}$ and $0\leqslant\varrho,\delta\leqslant 1$.

Let $a\in C^{n+1}(\mathbb{R}^n)$ be such that, for any $\alpha\in\mathbb{Z}^n_+$ with $|\alpha|\leqslant n+1$, there is a $C_\alpha>0$ such that $|\partial^\alpha_y a(y)|\leqslant C_\alpha$ for all $y\in\mathbb{R}^n$. We have

$$(2\pi)^{-n}\cdot\text{Os-}\iint \mathrm{e}^{\mathrm{i}(x-y)\cdot\eta}a(y)\,\mathrm{d}y\,\mathrm{d}\eta=a(x) \tag{2.8}$$

for every $x\in\mathbb{R}^n$. For the proof, see [1], Example 3.11.

We will make use of the following implication.

Lemma 2.2 ([5], Chapter 1, Lemma 6.5). *Let $f(t)$ be a C^2-function defined on $I=[0,1]$. Then there exists a constant $M>0$ independent of f such that*

$$\Bigl(\max_I|f'(t)|\Bigr)^2\leqslant M\Bigl(\max_I|f(t)|\Bigr)\Bigl\{\max_I|f(t)|+\max_I|f''(t)|\Bigr\}.$$

3. Proof of Theorem 1.2

We use the following notations. A function $a\colon\mathbb{R}^n\times\mathbb{R}^n\to\mathcal{L}(\mathbb{C}^N)$ is an element of $S^m_{\varrho,\delta}(\mathbb{R}^n\times\mathbb{R}^n;M;\mathcal{L}(\mathbb{C}^N))$ if $a_{j,k}$ belongs to the class $S^m_{\varrho,\delta}(\mathbb{R}^n\times\mathbb{R}^n;M)$ for all $j,k=1,\dots,N$, where $a_{j,k}$ is the (j,k)-component of a. A matrix version of the class $C^{\widetilde{m},\tau}S^{m_1,m_2}_{\varrho,\delta}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n;M_1,M_2)$ and that of the class $\mathcal{A}^{m,N}_{\tau,M}(\mathbb{R}^n\times\mathbb{R}^n)$ are defined similarly. For $a\in\mathcal{A}^{m,N}_{\tau,M}(\mathbb{R}^n\times\mathbb{R}^n;\mathcal{L}(\mathbb{C}^N))$, we define Os-$\iint \mathrm{e}^{-\mathrm{i}y\cdot\eta}a(y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta$ by (2.6) whenever the limit exists.

First, we prove the statement of Theorem 1.2 for $m=0$. Let $N\in\mathbb{N}$, $0\leqslant\tau<1$, $0<\sigma<1$ and $p\in(1,\infty)$. Let $M\in\mathbb{Z}_+\cup\{\infty\}$ be such that $M>(n+1)+n\cdot\max\{1/2,1/p\}$. Let $p\in C^{0,\sigma}S^0_{1,\tau}(\mathbb{R}^n\times\mathbb{R}^n;M;\mathcal{L}(\mathbb{C}^N))$ be a symbol satisfying the following properties for some $R>0$ and $C_0>0$:

(i) $|\det(p(x,\xi))|\geqslant C_0$ for all $x,\xi\in\mathbb{R}^n$ with $|x|+|\xi|\geqslant R$;

(ii) $p(x,\xi)\overset{|x|\to\infty}{\longrightarrow}p(\infty,\xi)$ for all $\xi\in\mathbb{R}^n$.

We pick a $\delta\in(\tau,1)$ arbitrarily. We define $q(x,\xi)$ by (2.1) and put $r(x,\xi)=p(x,\xi)-q(x,\xi)$. We have

$$r(x,\xi)=\int\varphi(\langle\xi\rangle^\delta y)(p(x,\xi)-p(x-y,\xi))\,\mathrm{d}y\cdot\langle\xi\rangle^{n\delta}. \tag{3.1}$$

Let us prove the following assertion.

Lemma 3.1.

(1) *We have* $\lim_{|x|+|\xi|\to\infty} r(x,\xi)=0.$

(2) *For* $-(1-\delta)\sigma<s<\sigma$*, the operator* $r(x,D_x)\colon\ H_p^s(\mathbb{R}^n)^N\to H_p^s(\mathbb{R}^n)^N$ *is compact.*

Proof. (1) Seeking a contradiction, we assume that $r(x,\xi)$ does not converge to 0 as $|x|+|\xi|\to\infty$. Then, there are an $\varepsilon_0>0$ and a sequence $\{(x_i,\xi_i)\}_{i=1}^{\infty}$ in $\mathbb{R}^n\times\mathbb{R}^n$ such that $\lim_{i\to\infty}(|x_i|+|\xi_i|)=\infty$ and

$$|r(x_i,\xi_i)|\geqslant\varepsilon_0\quad\forall\, i\in\mathbb{N}. \tag{3.2}$$

It follows from Theorem 2.2 that $|r(x,\xi)|\leqslant C_0\langle\xi\rangle^{-(\delta-\tau)}$. This, combined with $\delta\in(\tau,1)$ and (3.2), implies that the sequence $\{\xi_i\}_{i=1}^{\infty}$ is bounded. So, the Bolzano-Weierstrass theorem implies that $\{\xi_i\}_{i=1}^{\infty}$ has a convergent subsequence $\{\xi_{i(l)}\}_{l=1}^{\infty}$. We put $\xi^0=\lim_{l\to\infty}\xi_{i(l)}$. From condition (ii) we have for every $y\in\mathbb{R}^n$

$$\lim_{|x|\to\infty}(p(x,\xi^0)-p(x-y,\xi^0))=0.$$

In addition, we have

$$|\varphi(\langle\xi^0\rangle^{\delta}y)(p(x,\xi^0)-p(x-y,\xi^0))|\leqslant 2C_0|\varphi(\langle\xi^0\rangle^{\delta}y)|,$$

where $C_0=\sup_{(x,\xi)}|p(x,\xi)|$. Since the right-hand side of this inequality is an integrable function on $\mathbb{R}^n$, we infer by the Lebesgue theorem and (3.1) that

$$\lim_{|x|\to\infty} r(x,\xi_0)=0. \tag{3.3}$$

It follows from Theorem 2.2 that $K\equiv\sup_{(x,\xi)}|\nabla_\xi r(x,\xi)|<\infty$. Since $|x_{i(l)}|\to\infty$ as $l\to\infty$, we have from (3.3)

$$\begin{aligned}|r(x_{i(l)},\xi_{i(l)})|&\leqslant|r(x_{i(l)},\xi_{i(l)})-r(x_{i(l)},\xi^0)|+|r(x_{i(l)},\xi^0)|\\&\leqslant K|\xi_{i(l)}-\xi^0|+|r(x_{i(l)},\xi^0)|\to 0\quad(\text{as } l\to\infty).\end{aligned}$$

Since this violates (3.2), we get the conclusion.

(2) It follows from Theorem 2.2 that $q\in S^0_{1,\delta}(\mathbb{R}^n\times\mathbb{R}^n;M;\mathcal{L}(\mathbb{C}^N))$. Combining this with (2.7) we have $q\in C^{0,\sigma}S^0_{1,\delta}(\mathbb{R}^n\times\mathbb{R}^n;M;\mathcal{L}(\mathbb{C}^N))$. But,

$$p\in C^{0,\sigma}S^0_{1,\tau}(\mathbb{R}^n\times\mathbb{R}^n;M;\mathcal{L}(\mathbb{C}^N))\subset C^{0,\sigma}S^0_{1,\delta}(\mathbb{R}^n\times\mathbb{R}^n;M;\mathcal{L}(\mathbb{C}^N)).$$

So, $r\in C^{0,\sigma}S^0_{1,\delta}(\mathbb{R}^n\times\mathbb{R}^n;M;\mathcal{L}(\mathbb{C}^N))$. This, together with assertion (1) and Lemma 2.1, yields the conclusion. □

Put $d(A) = \det A$ for $A \in \mathcal{L}(\mathbb{C}^N)$. We have

$$|\det(p(x,\xi)) - \det(q(x,\xi))| \leqslant \max_{0\leqslant t\leqslant 1} |(\nabla d)(tp(x,\xi) + (1-t)q(x,\xi))||r(x,\xi)|.$$

This, combined with the boundedness of $p(x,\xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ and (1) of Lemma 3.1, implies that $\lim_{|x|+|\xi|\to\infty} |\det(p(x,\xi)) - \det(q(x,\xi))| = 0$. So, we see from condition (i) that there exist $R > 0$ and $C_0 > 0$ such that

$$|\det(q(x,\xi))| \geqslant C_0 \quad \forall\, x,\xi \in \mathbb{R}^n \quad \text{with} \quad |x| + |\xi| \geqslant R. \tag{3.4}$$

Furthermore, it follows from condition (ii) and (1) of Lemma 2.2 that

$$q(x,\xi) \overset{|x|\to\infty}{\longrightarrow} p(\infty,\xi) \quad \forall\, \xi \in \mathbb{R}^n. \tag{3.5}$$

We choose a $\chi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that $\chi(x,\xi) = 0$ for $|x| + |\xi| \leqslant R + 1$ and $\chi(x,\xi) = 1$ for $|x| + |\xi| \geqslant R + 2$. Let

$$w(x,\xi) = \chi(x,\xi)q(x,\xi)^{-1} \quad \forall\, x,\xi \in \mathbb{R}^n.$$

Because of (3.4) and the fact that $q \in S^0_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M; \mathcal{L}(\mathbb{C}^N))$, we obtain $w \in S^0_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M; \mathcal{L}(\mathbb{C}^N))$. Since q, $w \in S^0_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M; \mathcal{L}(\mathbb{C}^N))$, we claim by (2.7) that q, $w \in C^{\widetilde{m},\sigma}S^0_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M; \mathcal{L}(\mathbb{C}^N))$ for all $\widetilde{m} \in \mathbb{Z}_+$. Thus, the functions $q_{i,j}(x,\xi)w_{k,l}(x',\xi')$ and $w_{k,l}(x,\xi)q_{i,j}(x',\xi')$ belong to the class $C^{0,\sigma}S^{0,0}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; M, M)$, where $q_{i,j}$, $1 \leqslant i,j \leqslant N$, and $w_{k,l}$, $1 \leqslant k,l \leqslant N$, stand for the (i,j)-component of q or the (k,l)-component of w, respectively. We put

$$m_1(x,\xi) = (2\pi)^{-n} \cdot \text{Os-}\iint \mathrm{e}^{-\mathrm{i}y\cdot\eta} q(x,\xi+\eta)w(x+y,\xi)\,\mathrm{d}y\,\mathrm{d}\eta$$

and

$$m_2(x,\xi) = (2\pi)^{-n} \cdot \text{Os-}\iint \mathrm{e}^{-\mathrm{i}y\cdot\eta} w(x,\xi+\eta)q(x+y,\xi)\,\mathrm{d}y\,\mathrm{d}\eta.$$

Since $M \geqslant n+1$, we infer from Theorem 2.4 that $m_1, m_2 \in C^{0,\sigma}S^0_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M - (n+1); \mathcal{L}(\mathbb{C}^N))$. We also put

$$r_1(x,\xi) = m_1(x,\xi) - q(x,\xi)w(x,\xi) \quad \text{and} \quad r_2(x,\xi) = m_2(x,\xi) - w(x,\xi)q(x,\xi).$$

Because $q(x,\xi)w(x,\xi)$ and $w(x,\xi)q(x,\xi)$ belong to $S^0_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M; \mathcal{L}(\mathbb{C}^N))$, we have r_1, $r_2 \in C^{0,\sigma}S^0_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M - (n+1); \mathcal{L}(\mathbb{C}^N))$. Since $q_{i,j}(x,\xi)w_{k,l}(x',\xi')$ and $w_{k,l}(x,\xi)q_{i,j}(x',\xi')$ belong to the class $C^{0,\sigma}S^{0,0}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; M, M)$ and since $M \geqslant n+2$, we have by Theorem 2.5 that $r_1, r_2 \in C^{0,\sigma}S^{-(1-\delta)}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M - (n+2); \mathcal{L}(\mathbb{C}^N))$. Let us demonstrate the following assertion.

Lemma 3.2.

(1) *We have* $\lim\limits_{|x|+|\xi|\to\infty} r_1(x,\xi) = 0$ *and* $\lim\limits_{|x|+|\xi|\to\infty} r_2(x,\xi) = 0$.

(2) *For* $-(1-\delta)\sigma < s < \sigma$, *the operators* $r_1(x,D_x)\colon H^s_p(\mathbb{R}^n)^N \to H^s_p(\mathbb{R}^n)^N$ *and* $r_2(x,D_x)\colon H^s_p(\mathbb{R}^n)^N \to H^s_p(\mathbb{R}^n)^N$ *are compact.*

P r o o f. (1) On account of (3.4) and (3.5), we have

$$w(x,\xi) \overset{|x|\to\infty}{\longrightarrow} p(\infty,\xi)^{-1} \quad \forall\, \xi \in \mathbb{R}^n. \tag{3.6}$$

Seeking a contradiction, we suppose that $r_1(x,\xi)$ does not tend to 0 as $|x|+|\xi| \to \infty$. Then, there are an $\varepsilon_0 > 0$ and a sequence $\{(x_k,\xi_k)\}_{k=1}^{\infty}$ in $\mathbb{R}^n \times \mathbb{R}^n$ such that $\lim\limits_{k\to\infty}(|x_k|+|\xi_k|) = \infty$ and

$$|r_1(x_k,\xi_k)| \geqslant \varepsilon_0 \quad \forall\, k \in \mathbb{N}. \tag{3.7}$$

Since $r_1 \in C^{0,\sigma}S^{-(1-\delta)}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M-(n+2); \mathcal{L}(\mathbb{C}^N))$, there is a constant C_0 for which $|r_1(x,\xi)| \leqslant C_0\langle\xi\rangle^{-(1-\delta)}$ for all $x,\xi \in \mathbb{R}^n$. This, combined with $-(1-\delta) < 0$ and (3.6), implies that the sequence $\{\xi_k\}_{k=1}^{\infty}$ is bounded. It then follows from the Bolzano-Weierstrass theorem that $\{\xi_k\}_{k=1}^{\infty}$ admits a convergent subsequence $\{\xi_{k(l)}\}_{l=1}^{\infty}$. Put $\xi^0 = \lim\limits_{l\to\infty}\xi_{k(l)}$. We have

$$r_1(x_{k(l)},\xi_{k(l)}) = r_1(x_{k(l)},\xi_{k(l)}) - r_1(x_{k(l)},\xi^0) + r_1(x_{k(l)},\xi^0). \tag{3.8}$$

Since $\nabla_\xi r_1(x,\xi)$ is bounded on $\mathbb{R}^n \times \mathbb{R}^n$, we get

$$\lim_{l\to\infty}(r_1(x_{k(l)},\xi_{k(l)}) - r_1(x_{k(l)},\xi^0)) = 0. \tag{3.9}$$

Since $w(x_{k(l)}+\cdot,\xi^0) \in C^\infty_b(\mathbb{R}^n;\mathcal{L}(\mathbb{C}^N))$, we obtain from (2.8)

$$r_1(x_{k(l)},\xi^0) = (2\pi)^{-n}\cdot\mathrm{Os}\text{-}\iint \mathrm{e}^{-\mathrm{i}y\cdot\eta}(q(x_{k(l)},\xi^0+\eta)-q(x_{k(l)},\xi^0))w(x_{k(l)}+y,\xi^0)\,\mathrm{d}y\,\mathrm{d}\eta. \tag{3.10}$$

We note that for any $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \leqslant 1$, there is a constant C_α such that $|\partial^\alpha_\eta q(x_{k(l)},\eta)| \leqslant C_\alpha\langle\eta\rangle^{-|\alpha|}$ for all $\eta \in \mathbb{R}^n$ and $l \in \mathbb{N}$ and that $|\partial^\alpha_y w(x_{k(l)}+y,\xi^0)| \leqslant C_\alpha$ for all $y \in \mathbb{R}^n$ and $l \in \mathbb{N}$. This, together with the Ascoli-Arzela theorem and the diagonalization argument, yields that $\{k(l)\}_{l=1}^{\infty}$ admits a subsequence $\{k(l(m))\}_{m=1}^{\infty}$ such that the sequence $[q(x_{k(l(m))},\xi^0+\eta) - q(x_{k(l(m))},\xi^0)]w(x_{k(l)}+y,\xi_0)$ converges uniformly on each compact subset of $\mathbb{R}^n_y \times \mathbb{R}^n_\eta$. But, (3.5), (3.6) and $\lim\limits_{m\to\infty}|x_{k(l(m))}| = \infty$ yield that

$$\begin{aligned}\lim_{m\to\infty}&[q(x_{k(l(m))},\xi^0+\eta) - q(x_{k(l(m))},\xi^0)]w(x_{k(l)}+y,\xi^0)\\ &= (p(\infty,\xi^0+\eta) - p(\infty,\xi^0))p(\infty,\xi^0)^{-1} \quad \forall\, y,\eta \in \mathbb{R}^n.\end{aligned}$$

So, it holds that

$$\lim_{m\to\infty}[q(x_{k(l(m))},\xi^0+\eta)-q(x_{k(l(m))},\xi^0)]w(x_{k(l)}+y,\xi^0) = (p(\infty,\xi^0+\eta)-p(\infty,\xi^0))p(\infty,\xi^0)^{-1} \tag{3.11}$$

uniformly on each compact subset of $\mathbb{R}^n_y\times\mathbb{R}^n_\eta$.

Because $q,w\in S^0_{1,\delta}(\mathbb{R}^n\times\mathbb{R}^n;M;\mathcal{L}(\mathbb{C}^N))$, we have, for any α, $\beta\in\mathbb{Z}^n_+$ with $|\alpha|\leqslant n+2$ and $|\beta|\leqslant n+2$, there is a constant $C_{\alpha,\beta}$ such that

$$|\partial^\alpha_\eta\partial^\beta_y[(q(x_{k(l(m))},\xi^0+\eta)-q(x_{k(l(m))},\xi^0))w(x_{k(l(m))}+y,\xi^0)]|\leqslant C_{\alpha,\beta} \quad \forall\,\eta,y\in\mathbb{R}^n \text{ and } m\in\mathbb{N}. \tag{3.12}$$

This, together with (3.11) and repeated use of Lemma 2.2, yields that, for any α, $\beta\in\mathbb{Z}^n_+$ with $|\alpha|\leqslant n+1$ and $|\beta|\leqslant n+1$,

$$\lim_{m\to\infty}\partial^\alpha_\eta\partial^\beta_y[(q(x_{k(l(m))},\xi^0+\eta)-q(x_{k(l(m))},\xi^0))w(x_{k(l)}+y,\xi^0)] = \partial^\alpha_\eta\partial^\beta_y[(p(\infty,\xi^0+\eta)-p(\infty,\xi^0))p(\infty,\xi^0)^{-1}] \tag{3.13}$$

uniformly on each compact subset of $\mathbb{R}^n_y\times\mathbb{R}^n_\eta$. We infer from Theorem 2.3, (2.8), (3.10), (3.12) and (3.13) that

$$\lim_{m\to\infty}r_1(x_{k(l(m))},\xi^0) = (2\pi)^{-n}\cdot\text{Os-}\iint e^{-iy\cdot\eta}(p(\infty,\xi^0+\eta)-p(\infty,\xi^0))p(\infty,\xi^0)^{-1}\,dy\,d\eta=0.$$

This, combined with (3.8) and (3.9), yields that

$$\lim_{m\to\infty}r_1(x_{k(l(m))},\xi_{k(l(m))})=0.$$

Since this violates (3.7), we obtain $\lim_{|x|+|\xi|\to\infty}r_1(x,\xi)=0$. Similarly, we get $\lim_{|x|+|\xi|\to\infty}r_2(x,\xi)=0$.

(2) Combining assertion (1) with the fact that r_1, $r_2\in C^{0,\sigma}S^0_{1,\delta}(\mathbb{R}^n\times\mathbb{R}^n;M-(n+1);\mathcal{L}(\mathbb{C}^N))$ and Lemma 2.1, we arrive at the conclusion. □

Since p, $w\in C^{0,\sigma}S^0_{1,\delta}(\mathbb{R}^n\times\mathbb{R}^n;M;\mathcal{L}(\mathbb{C}^N))$ and $M>n\cdot\max\{1/2,1/p\}$, we infer from Theorem 2.6 that for $-(1-\delta)\sigma<s<\sigma$ the operators $p(x,D_x)\colon H^s_p(\mathbb{R}^n)^N\to H^s_p(\mathbb{R}^n)^N$ and $w(x,D_x)\colon H^s_p(\mathbb{R}^n)^N\to H^s_p(\mathbb{R}^n)^N$ are bounded. Let I stand for the identity matrix of size N. Since the function $q(x,\xi)w(x,\xi)-I$ belongs to the class $C^{0,\sigma}S^0_{1,\delta}(\mathbb{R}^n\times\mathbb{R}^n;M;\mathcal{L}(\mathbb{C}^N))$ and has a compact support, we see from Lemma 2.1

that for $-(1-\delta)\sigma < s < \sigma$ the operator $\mathrm{Op}(qw - I)\colon H^s_p(\mathbb{R}^n)^N \to H^s_p(\mathbb{R}^n)^N$ is compact. We define

$$R_1 = \mathrm{Op}(r_1) + \mathrm{Op}(qw - I) + \mathrm{Op}(r) \circ \mathrm{Op}(w).$$

We see from Lemmas 3.1 and 3.2 and the observation above that for $-(1-\delta)\sigma < s < \sigma$ the operator $R_1\colon H^s_p(\mathbb{R}^n)^N \to H^s_p(\mathbb{R}^n)^N$ is compact. Furthermore, we obtain

$$\begin{aligned}\mathrm{Op}(p) \circ \mathrm{Op}(w) &= (\mathrm{Op}(q) + \mathrm{Op}(r)) \circ \mathrm{Op}(w) = \mathrm{Op}(r_1) + \mathrm{Op}(qw) + \mathrm{Op}(r) \circ \mathrm{Op}(w)\\ &= \mathrm{Id} + R_1.\end{aligned}$$

We also define

$$R_2 = \mathrm{Op}(r_2) + \mathrm{Op}(wq - I) + \mathrm{Op}(w) \circ \mathrm{Op}(r).$$

As in the discussion above, we claim that for $-(1-\delta)\sigma < s < \sigma$ the operator $R_2\colon H^s_p(\mathbb{R}^n)^N \to H^s_p(\mathbb{R}^n)^N$ is compact and that

$$\mathrm{Op}(w) \circ \mathrm{Op}(p) = \mathrm{Id} + R_2.$$

So we infer by the Atkinson theorem that for $-(1-\delta)\sigma < s < \sigma$ the operator $p(x, D_x)\colon H^s_p(\mathbb{R}^n)^N \to H^s_p(\mathbb{R}^n)^N$ is a Fredholm operator. Since we have chosen $\delta \in (\tau, 1)$ arbitrarily, the assertion of Theorem 1.2 for $m = 0$ holds true.

Next, we turn our attention to the general case. Assume that the conditions in Theorem 1.2 are satisfied. Put

$$\widetilde{a}(x,\xi) = a(x,\xi) \cdot \langle \xi \rangle^{-m} I. \tag{3.14}$$

Since $a \in C^{0,\sigma} S^m_{1,\tau}(\mathbb{R}^n \times \mathbb{R}^N; M; \mathcal{L}(\mathbb{C}^N))$, we have $\widetilde{a} \in C^{0,\sigma} S^0_{1,\tau}(\mathbb{R}^n \times \mathbb{R}^N; M; \mathcal{L}(\mathbb{C}^N))$. Thus, Theorem 1.2 for $m = 0$ yields that for $-(1-\tau)\sigma < s < \sigma$ the operator $\widetilde{a}(x, D_x)\colon H^s_p(\mathbb{R}^n)^N \to H^s_p(\mathbb{R}^n)^N$ is a Fredholm operator. Note that (3.14) yields $a(x, D_x) = \widetilde{a}(x, D_x) \circ \mathrm{Op}(\langle \xi \rangle^m I)$. Since $\mathrm{Op}(\langle \xi \rangle^m I)\colon H^{s+m}_p(\mathbb{R}^n)^N \to H^s_p(\mathbb{R}^n)^N$ is a linear isomorphism for any $s \in \mathbb{R}$, we get the conclusion of Theorem 1.2.

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Author's address: Kazushi Yoshitomi, Department of Mathematical Sciences, Tokyo Metropolitan University, Minamiohsawa 1-1, Hachioji, Tokyo 192-0397, Japan, e-mail: `yositomi@tmu.ac.jp`.