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Luděk Zajíček

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Abstract. Let $X_1,\dots,X_n$ be Banach spaces and f a real function on $X = X_1\times\cdots\times X_n$. Let A_f be the set of all points $x\in X$ at which f is partially Fréchet differentiable but is not Fréchet differentiable. Our results imply that if $X_1,\dots,X_{n-1}$ are Asplund spaces and f is continuous (respectively Lipschitz) on X, then A_f is a first category set (respectively a σ-upper porous set). We also prove that if X, Y are separable Banach spaces and $f\colon X\to Y$ is a Lipschitz mapping, then there exists a σ-upper porous set $A\subset X$ such that f is Fréchet differentiable at every point $x\in X\setminus A$ at which it is Fréchet differentiable along a closed subspace of finite codimension and Gâteaux differentiable. A number of related more general results are also proved.

Keywords: Fréchet differentiability; partial Fréchet differentiability; first category set; Asplund space; σ-porous set

Classification: 46G05, 46T20

1. Introduction

If f is a real function on $\mathbb{R}^n$, denote by A_f the set of all points $x\in\mathbb{R}^n$ at which f has all (finite) partial derivatives $f_1'(x),\dots,f_n'(x)$ but it is not (Fréchet) differentiable. Of course, A_f can be nonempty and Stepanoff's examples in [16] show that A_f can have positive measure for a continuous function on $\mathbb{R}^2$ (he constructed even such a function which is everywhere partially differentiable and also a continuous function on $\mathbb{R}^2$ which is partially differentiable almost everywhere but it is nowhere differentiable).

However, the situation is different, if we consider "topological smallness" instead of "measure smallness", as the following result shows.

Theorem C. *Let $G\subset\mathbb{R}^n$ be an open set and $f\colon G\to\mathbb{R}$ a continuous function. Then*

$$A_f := \{x\in G\colon f_1'(x)\in\mathbb{R},\dots,f_n'(x)\in\mathbb{R} \text{ and } f'(x) \text{ does not exist}\}$$

is a first category (=meagre) set.

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For $n = 2$, this result follows from Gorlenko's 1977 article [6]. For general n, it was proved in [10] (even for $f: G \to Y$, where Y is a separable Banach space) by K. S. Lau and C. E. Weil in 1978.

A related remarkable result from [15] gives that the conclusion of Theorem C holds if f is *an arbitrary* function which is partially differentiable everywhere in G.

D. N. Bessis and F. H. Clarke in [1] in 1999 proved the following "Lipschitz version" of Theorem C.

Theorem L. *Let $G \subset \mathbb{R}^n$ be an open set and $f: G \to \mathbb{R}$ a Lipschitz function. Then*

$$A_f := \{x \in G\colon\ f_1'(x) \in \mathbb{R}, \dots, f_n'(x) \in \mathbb{R} \text{ and } f'(x) \text{ does not exist}\}$$

is a σ-porous set.

In the present article, σ-porosity is "σ-upper porosity", see Definition 2.1 below, cf. [19], i.e. it is considered in "Denjoy–Dolzhenko sense". Note that if $A \subset \mathbb{R}^n$ is σ-porous, then it is both of the first category and Lebesgue null, but the opposite implication does not hold. So Theorem L does not follow from Theorem C and the Rademacher theorem.

In the present article we prove some generalizations of Theorem C and Theorem L in the infinite-dimensional setting. Namely, let $X_1, \dots, X_n$ and Y be Banach spaces, G an open subset of $X := X_1 \times \dots \times X_n$ (equipped with the maximum norm), and $f: G \to Y$ a mapping. Denote by A_f the set of all points $x \in G$ at which all partial Fréchet derivatives of f exist but the Fréchet derivative $f'(x)$ does not exist.

In Section 4 we prove in Theorem 4.5 that A_f is a first category set whenever f is continuous and

(1.1) the spaces of continuous linear mappings $\mathcal{L}(X_1, Y), \dots, \mathcal{L}(X_{n-1}, Y)$ are separable.

We obtain this result as an immediate consequence of an easy known result from [13] (Proposition 4.4 below) and Theorem 4.1 (which can be of an independent interest) which says that, under some conditions, Fréchet differentiability along a subspace V generically implies strict differentiability along V.

In Section 5 we prove by a different (more technical) method Theorem 5.2 which implies that, under condition (1.1), the generalization of Theorem L also holds.

In Section 6, using the method of separable reduction, we show in Theorem 6.7 that generalizations of Theorem C and Theorem L hold also under a condition more general than condition (1.1). In particular, we prove that generalizations of Theorem C and Theorem L hold if $Y = \mathbb{R}$ and $X_1, \dots, X_{n-1}$ are Asplund spaces, see Corollary 6.8.

I do not believe that condition (1.1) can be omitted in Theorem 4.5 and/or Theorem 5.2. Unfortunately, I was not able to find any counterexample. So it is still possible that the Banach space generalizations of Theorem C and/or Theorem L hold in the full generality and for this reason I do not discuss all cases in which the validity of these generalizations follow by the methods of the present article, cf. Remark 6.9.

In the proof of Theorem 6.7, we use a result (Proposition 3.3) on the Borel type of the set of all points at which a partial derivative of a continuous (or a slightly more general) function exists. This result which generalizes a proposition on functions on $\mathbb{R}^2$ from [12] and can be of an independent interest is proved in Section 3.

We consider (mainly in Section 5) also related problems where instead of a product space $X := X_1 \times \cdots \times X_n$ we consider an arbitrary Banach space X and instead of Fréchet partial derivatives we consider Fréchet derivatives along subspaces. In this direction, we obtain Propositions 4.8, Proposition 5.3, and Corollary 5.6 which is an immediate consequence of more general Proposition 5.5, which can be of an independent interest. Another consequence of this proposition is Corollary 5.7 which says that if X, Y are separable Banach spaces and $f \colon X \to Y$ is a Lipschitz mapping, then there exists a σ-upper porous set $A \subset X$ such that f is Fréchet differentiable at every point $x \in X \setminus A$ at which it is Fréchet differentiable along a closed subspace of finite codimension and Gâteaux differentiable.

2. Preliminaries

In the following, we consider real nontrivial (i.e. not equal to $\{0\}$) Banach spaces. In any fixed Banach space, we denote the zero vector by 0 and the norm by $|\cdot|$. By a *subspace* Y of a Banach space X, we will mean a Banach subspace of X, i.e. a closed linear subspace $Y \neq \{0\}$. We set $S_X := \{x \in X \colon |x| = 1\}$. By $\operatorname{span} M$ we denote the linear span of $M \subset X$. The equality $X = X_1 \oplus \cdots \oplus X_n$ means that the Banach space X is the topological direct sum of its subspaces $X_1, \dots, X_n$. The symbol $B(x, r)$ will denote the open ball with center x and radius r.

Definition 2.1. Let A be a subset of a Banach space X.

(i) We say that A is porous at a point $x \in X$ if there exists $c > 0$ such that for each $\delta > 0$ there exists $t \in B(x, \delta) \setminus \{x\}$ such that $B(t, c|t-x|) \cap A = \emptyset$.
(ii) A is called a porous set if A is porous at each point $x \in A$.
(iii) A is called a σ-porous set if it is a countable union of porous sets.

If X and Y are Banach spaces, the space of all continuous linear mappings $\varphi\colon X \to Y$ (equipped with the usual norm) will be denoted by $\mathcal{L}(X,Y)$.

The word *"generically"* has the usual sense; it means "at all points except a first category set".

Recall that a Banach space X is called an Asplund space if each continuous convex function on X is generically Fréchet differentiable and that

$$X \text{ is Asplund if and only if } Y^* \text{ is separable for each separable subspace } Y \subset X. \tag{2.1}$$

Let X, Y be Banach spaces, $G \subset X$ an open set, and $f\colon G \to Y$ a mapping. We say that f *is Lipschitz at* $x \in G$ if $\limsup_{y\to x} |f(y)-f(x)|/|y-x| < \infty$. The directional and one-sided directional derivatives of f at $x \in G$ in the direction $v \in X$ are defined respectively by

$$f'(x,v) := \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t} \qquad \text{and} \qquad f'_+(x,v) := \lim_{t\to 0+} \frac{f(x+tv)-f(x)}{t}.$$

Definition 2.2. Let X and Y be Banach spaces, V a closed subspace of X, $G \subset X$ an open set, $a \in G$ and $f\colon G \to Y$ a mapping. We say that f is *Fréchet differentiable at a along V*, if the mapping $g(v) := f(a+v)$, $v \in V \cap (G-a)$, is Fréchet differentiable at $0 \in V$ and set $f'_V(a) := g'(0) \in \mathcal{L}(V,Y)$.

We say that f is *strictly differentiable at a along V* if $f'_V(a)$ exists and

$$\lim_{(x,\tilde{x})\to(a,a),\ 0\neq \tilde{x}-x\in V} \frac{|f(\tilde{x})-f(x)-f'_V(a)(\tilde{x}-x)|}{|\tilde{x}-x|} = 0. \tag{2.2}$$

Remark 2.3.

(i) In the above definition, some authors write " with respect to V " or " in the direction of V " instead of " along V ".

(ii) The standard strict differentiability coincides with strict differentiability along $V := X$. Note that some authors by "strict differentiability" mean (a weaker) "Gâteaux strict differentiability" which is stronger than Gâteaux differentiability.

(iii) Condition (2.2) can be clearly rewritten as

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall v \in V\colon \{x, x+v\} \subset B(a,\delta) \Rightarrow |f(x+v)-f(x)-f'_V(a)v| \le \varepsilon |v|. \tag{2.3}$$

The notions of partial Fréchet differentiability and strict partial differentiability are factually special cases of notions of "directional" and "strict directional" differentiability along a subspace. If $X_1,\dots,X_n$ are Banach spaces, we consider the Banach space $X := X_1 \times \dots \times X_n$ (equipped with the maximum norm). In the following definition, we consider, if $1 \le i \le n$ is given, X_i as a subspace of X (identifying, as usual, $x_i \in X_i$ with $(0, \dots, 0, x_i, 0, \dots, 0) \in X$).

Definition 2.4. Let $X_1, \dots, X_n$ and Y be Banach spaces, $X := X_1 \times \dots \times X_n$, $G \subset X$ an open set, $a \in G$ and $f\colon G \to Y$ a mapping. Then, for $i = 1, \dots, n$,

(i) we set $f_i'(a) := f_{X_i}'(a)$ and call it (if it exists) *partial Fréchet derivative of f at a with respect to the ith variable* and

(ii) we say that f is *partially strictly differentiable at a with respect to the ith variable* if f is strictly differentiable at a along X_i.

3. Borel type of the set of points where a partial derivative exists

If X, Y are Banach spaces, $G \subset X$ an open set and $f\colon G \to Y$ an arbitrary mapping, then, see [18, Theorem 2],

(3.1) the set $D(f)$ of all $x \in G$ at which f is Fréchet differentiable is an $F_{\sigma\delta}$ set.

This result was proved in [18] using a characterization of Fréchet differentiability points (the proof in [11, Corollary 3.5.5] is quite different). Applying this characterization to partial functions, we immediately obtain a characterization, see Lemma 3.2 below, of points at which f has a partial Fréchet derivative. We need the following notation.

Definition 3.1. Let X_1, X_2, Y be Banach spaces, $G \subset X := X_1 \times X_2$ an open set and $f\colon G \to Y$ a mapping.

(i) We denote by $C_1(f)$ the set of all points $x = (x_1, x_2) \in G$ at which f is continuous with respect to the first coordinate (i.e. $f(\cdot, x_2)$ is continuous at x_1).

(ii) For $c > 0$, $\varepsilon > 0$ and $\delta > 0$, denote by $D_1(f, c, \varepsilon, \delta)$ the set of all points $x = (x_1, x_2) \in G$ such that

$$\left| \frac{f(z + kv, x_2) - f(z, x_2)}{k} - \frac{f(z, x_2) - f(z - hv, x_2)}{h} \right| \le \varepsilon \tag{3.2}$$

whenever $v \in X_1$, $|v| = 1$, $h > 0$, $k > 0$, $z \in X_1$, $z \in B(x_1, \delta)$, $z - hv \in B(x_1, \delta)$, $z + kv \in B(x_1, \delta)$ and $\min(h, k) > c|z - x_1|$.

Then, by the definitions, $x = (x_1, x_2) \in D_1(f, c, \varepsilon, \delta)$ if and only if $x_1 \in D(F, c, \varepsilon, \delta)$, where $F := f(\cdot, x_2)$ and $D(F, c, \varepsilon, \delta)$ is as in [18, Definition 3]. Since $f_1'(x)$ exists if and only if $F'(x_1)$ exists, [18, Theorem 1 and Note 2] immediately imply the following result.

Lemma 3.2. *Let X_1, X_2, Y, G and f be as in Definition* 3.1 *and $x \in G$. Then the following conditions are equivalent:*

(i) $f_1'(x)$ *exists.*
(ii) $x \in C_1(f) \cap \bigcap_{c>0} \bigcap_{\varepsilon>0} \bigcup_{\delta>0} D_1(f, c, \varepsilon, \delta)$.
(iii) $x \in C_1(f) \cap \bigcap_{n\in\mathbb{N}} \bigcup_{p\in\mathbb{N}} D_1(f, 1/n, 1/n, 1/p)$.

Using Lemma 3.2, we will prove the following proposition generalizing the corresponding result on real functions in $\mathbb{R}^2$ which was proved in [12] by a quite different elementary method. This proposition will be used in the proofs of Proposition 4.7 and Theorem 6.7.

Proposition 3.3. *Let X_1, X_2, Y be Banach spaces, $G \subset X := X_1 \times X_2$ an open set and $f\colon G \to Y$ a mapping which is continuous with respect to the second variable (i.e. all partial mappings $f(a_1, \cdot)$, $(a_1, a_2) \in G$, are continuous). Then the set $D_1(f) := \{x \in G\colon f_1'(x) \text{ exists}\}$ is an $F_{\sigma\delta}$ set.*

Proof: For each $k \in \mathbb{N}$, set $F_k := \{x \in G\colon \operatorname{dist}(x, X \setminus G) \geq 1/k\}$ if $X \neq G$ and $F_k := X$ if $X = G$. Clearly each F_k is a closed set and $G = \bigcup_{k\in\mathbb{N}} F_k$. So by Lemma 3.2 we obtain

$$D_1(f) = \bigcup_{k\in\mathbb{N}} (F_k \cap D_1(f)) = \bigcup_{k\in\mathbb{N}} \Big(F_k \cap C_1(f) \cap \bigcap_{n\in\mathbb{N}} \bigcup_{p\in\mathbb{N}} D_1\Big(f, \frac{1}{n}, \frac{1}{n}, \frac{1}{p}\Big)\Big). \tag{3.3}$$

Consequently it is sufficient to prove that, for each $k \in \mathbb{N}$, both

$$F_k \cap C_1(f) \text{ is an } F_{\sigma\delta} \text{ set} \tag{3.4}$$

and

$$F_k \cap \bigcap_{n\in\mathbb{N}} \bigcup_{p\in\mathbb{N}} D_1\Big(f, \frac{1}{n}, \frac{1}{n}, \frac{1}{p}\Big) \text{ is an } F_{\sigma\delta} \text{ set.} \tag{3.5}$$

Indeed, (3.4) and (3.5) imply that

$$Z_k := F_k \cap C_1(f) \cap \bigcap_{n\in\mathbb{N}} \bigcup_{p\in\mathbb{N}} D_1\Big(f, \frac{1}{n}, \frac{1}{n}, \frac{1}{p}\Big)$$

is an $F_{\sigma\delta}$ set for each $k \in \mathbb{N}$. Observing that for each $x \in G$ there exists an open neighbourhood $U_x \subset G$ of x such that the set $\{Z_k \cap U_x\colon k \in \mathbb{N}\}$ is finite, we obtain that $D_1(f) \cap U_x$ is an $F_{\sigma\delta}$ set. Thus $D_1(f)$ is an $F_{\sigma\delta}$ set by [9, §30, X, Theorem 1, page 358].

First we will prove (3.4). To this end, fix an arbitrary $k \in \mathbb{N}$ and denote for $m, j \in \mathbb{N}$

$$C_{m,j} := \Big\{ x = (x_1, x_2) \in G \colon |f(t, x_2) - f(\tau, x_2)| \le \frac{1}{m} \\ \text{whenever } |t - x_1| < \frac{1}{j},\ |\tau - x_1| < \frac{1}{j} \Big\},$$

and observe that

$$C_1(f) = \bigcap_{m=1}^{\infty} \bigcup_{j=k}^{\infty} C_{m,j}, \qquad F_k \cap C_1(f) = \bigcap_{m=1}^{\infty} \bigcup_{j=k}^{\infty} (F_k \cap C_{m,j}).$$

So it is sufficient to prove that $F_k \cap C_{m,j}$ is a closed set whenever $j \ge k$. To this end, fix arbitrary $m, j \in \mathbb{N}$ with $j \ge k$ and suppose that $(x_1^i, x_2^i) \in F_k \cap C_{m,j}$, $i \in \mathbb{N}$, and $(x_1^i, x_2^i) \to (x_1, x_2)$, $i \to \infty$. Then $(x_1, x_2) \in F_k$ since F_k is closed. To prove $(x_1, x_2) \in C_{m,j}$, consider arbitrary $t, \tau \in X_1$ with $|t - x_1| < 1/j$, $|\tau - x_1| < 1/j$. Then, for all sufficiently large i, we have $|t - x_1^i| < 1/j$, $|\tau - x_1^i| < 1/j$, and consequently $|f(t, x_2^i) - f(\tau, x_2^i)| \le 1/m$. Since $(x_1, x_2) \in F_k$ and $j \ge k$, we obtain that $(t, x_2) \in G$ and $(\tau, x_2) \in G$. Since f is continuous with respect to the second variable, $f(t, x_2^i) \to f(t, x_2)$, $f(\tau, x_2^i) \to f(\tau, x_2)$, and therefore $|f(t, x_2) - f(\tau, x_2)| \le 1/m$. So we obtain that $(x_1, x_2) \in C_{m,j}$ and we are done.

To prove (3.5), fix an arbitrary $k \in \mathbb{N}$. Since clearly

$$D_1\Big(f, \frac{1}{n}, \frac{1}{n}, \frac{1}{p_1}\Big) \subset D_1\Big(f, \frac{1}{n}, \frac{1}{n}, \frac{1}{p_2}\Big) \qquad \text{whenever } p_1 \le p_2,$$

we have

$$F_k \cap \bigcap_{n \in \mathbb{N}} \bigcup_{p \in \mathbb{N}} D_1\Big(f, \frac{1}{n}, \frac{1}{n}, \frac{1}{p}\Big) = \bigcap_{n=1}^{\infty} \bigcup_{p=k}^{\infty} \Big(F_k \cap D_1\Big(f, \frac{1}{n}, \frac{1}{n}, \frac{1}{p}\Big)\Big).$$

So it is sufficient to prove that $F_k \cap D_1(f, 1/n, 1/n, 1/p)$ is a closed set whenever $p \ge k$. To this end, fix arbitrary $n, p \in \mathbb{N}$ with $p \ge k$ and suppose that $x^i = (x_1^i, x_2^i) \in F_k \cap D_1(f, 1/n, 1/n, 1/p)$, $i \in \mathbb{N}$, and $x^i = (x_1^i, x_2^i) \to x = (x_1, x_2)$, $i \to \infty$. Then $(x_1, x_2) \in F_k$ since F_k is closed. To prove that $x = (x_1, x_2) \in D_1(f, 1/n, 1/n, 1/p)$, consider arbitrary $v \in X_1$ with $|v| = 1$, reals $h > 0$, $k > 0$ and $z \in X_1$ such that $z \in B(x_1, 1/p)$, $z - hv \in B(x_1, 1/p)$, $z + kv \in B(x_1, 1/p)$ and $\min(h, k) > (1/n)|z - x_1|$. Since $(x_1, x_2) \in F_k$ and $p \ge k$, we obtain that $(z, x_2) \in G$, $(z + kv, x_2) \in G$ and $(z - hv, x_2) \in G$. Our aim is to prove that

$$\Big| \frac{f(z + kv, x_2) - f(z, x_2)}{k} - \frac{f(z, x_2) - f(z - hv, x_2)}{h} \Big| \le \frac{1}{n}. \tag{3.6}$$

Since $x_1^i \to x_1$, for all sufficiently large i we have $z \in B(x_1^i, 1/p)$, $z - hv \in B(x_1^i, 1/p)$, $z + kv \in B(x_1^i, 1/p)$ and $\min(h,k) > (1/n)|z - x_1^i|$, which together with $x^i \in D_1(f, 1/n, 1/n, 1/p)$ implies

$$\left|\frac{f(z+kv, x_2^i) - f(z, x_2^i)}{k} - \frac{f(z, x_2^i) - f(z-hv, x_2^i)}{h}\right| \leq \frac{1}{n}. \tag{3.7}$$

Since f is continuous with respect to the second variable, we have $f(z, x_2^i) \to f(z, x_2)$, $f(z+kv, x_2^i) \to f(z+kv, x_2)$, $f(z-hv, x_2^i) \to f(z-hv, x_2)$, and consequently (3.7) implies (3.6). So we obtain that $x \in D_1(f, 1/n, 1/n, 1/p)$ and we are done. □

4. Case of continuous mappings

4.1 Strict directional differentiability via Fréchet directional differentiability. A well-known theorem, see e.g. [18, Theorem B, page 476], asserts that Fréchet differentiability at a point x of an *arbitrary* mapping $f\colon X \to Y$, where X, Y are arbitrary Banach spaces, generically implies strict differentiability of f at x. The following result which is a partial generalization of this theorem will be applied in the proof of Theorem 4.5.

Theorem 4.1. *Let X, Y be Banach spaces and let V be a subspace of X such that the space $\mathcal{L}(V,Y)$ is separable. Let $G \subset X$ be an open set and $f\colon G \to Y$ a continuous mapping. Then the set A of all $a \in G$ such that $f'_V(a)$ exists and f is not strictly differentiable at a along V is a first category set.*

Proof: Choose a countable dense subset Φ of $\mathcal{L}(V,Y)$ and consider an arbitrary point $a \in A$. By (2.3) we can choose $n \in \mathbb{N}$ such that

$$\begin{aligned} &\forall\,\delta > 0\ \exists\, x \in X\ \exists\, v \in V\colon \{x, x+v\} \subset B(a,\delta),\\ &\qquad\qquad |f(x+v) - f(x) - f'_V(a)v| > \frac{4}{n}|v|. \end{aligned} \tag{4.1}$$

Further choose $p \in \mathbb{N}$ such that

$$|f(a+v) - f(a) - f'_V(a)v| \leq \frac{1}{n}|v| \qquad \text{whenever } v \in V,\ |v| \leq \frac{1}{p} \tag{4.2}$$

and choose $\varphi \in \Phi$ such that

$$|f'_V(a) - \varphi| \leq \frac{1}{n}. \tag{4.3}$$

Denote, for each $n \in \mathbb{N}$, $p \in \mathbb{N}$ and $\varphi \in \Phi$, by $A_{n,p,\varphi}$ the set of all $a \in A$ for which the conditions (4.1), (4.2) and (4.3) hold. Then $A = \bigcup\{A_{n,p,\varphi}\colon\ n, p \in \mathbb{N},\ \varphi \in \Phi\}$ and so it is sufficient to show that all sets $A_{n,p,\varphi}$ are nowhere dense.

To this end, suppose to the contrary that for some fixed n, p, φ, the set $A_{n,p,\varphi}$ is not nowhere dense. Then there exist $z \in A_{n,p,\varphi}$ and $\omega > 0$ such that $A_{n,p,\varphi}$ is dense in $B(z, \omega)$. Now observe that, using (4.2) and (4.3), we easily obtain that

$$(4.4)\quad |f(a+v) - f(a) - \varphi(v)| \leq \frac{2}{n}|v| \qquad \text{whenever } a \in A_{n,p,\varphi},\ v \in V,\ |v| \leq \frac{1}{p}.$$

Applying (4.1) to $a := z$ with $\delta := \min(\omega, 1/2p)$, we can choose $x \in B(z, \delta)$ and $v \in V$ such that $x + v \in B(z, \delta)$ and

$$(4.5)\qquad |f(x+v) - f(x) - f'_V(z)v| > \frac{4}{n}|v|.$$

Clearly $|v| < 1/p$. By the choice of z, ω and δ, there exist $a_k \in A_{n,p,\varphi}$, $k \in \mathbb{N}$, such that $a_k \to x$, and therefore $a_k + v \to x + v$. Since $|v| < 1/p$, using (4.4) to $a := a_k$, $k \in \mathbb{N}$, and continuity of f, we easily obtain $|f(x+v) - f(x) - \varphi(v)| \leq (2/n)|v|$. This inequality together with $|f'_V(z) - \varphi| \leq 1/n$ implies $|f(x+v) - f(x) - f'_V(z)v| \leq (3/n)|v|$ which contradicts (4.5). □

Remark 4.2. Theorem 4.1 can be applied e.g. in the following cases:

(i) V is finite-dimensional and Y is separable.

(ii) V is a separable Asplund space and Y is finite-dimensional.

(iii) $V = C(K)$ for a countable compact set and Y is separable with the Radon–Nikodým property.

(iv) V is a closed subspace of c_0 and Y is separable with the Radon–Nikodým property.

(v) $V = l_p$, $Y = l_q$, $1 \leq q < p < \infty$.

Indeed, in all these cases, the space $\mathcal{L}(V, Y)$ is separable. The most natural cases (i) and (ii) are almost obvious. For the cases (iii) and (iv) see [11, pages 114–115]. In the well-known case (v) Pitt's theorem, see, e.g., [5, Proposition 6.25], says that $\mathcal{L}(V, Y)$ coincides with the space $\mathcal{K}(V, Y)$ of compact operators which is separable (e.g. by well-known result [17, Fact 5.4, page 20]).

For some other cases involving classical Banach spaces see [17, Example 5.5].

Remark 4.3. I do not believe that the assumption that $\mathcal{L}(V, Y)$ is separable can be omitted in Theorem 4.1, but I do not know any counterexample.

4.2 Fréchet differentiability of continuous functions via partial Fréchet differentiability. The main result of the present section (Theorem 4.5 below) is an almost immediate consequence of Theorem 4.1 and the following known result (see [13, Proposition 2.57], where "strict differentiability" is called "circa-differentiability").

Proposition 4.4 ([13]). *Let $X_1, \dots, X_n$ and Y be Banach spaces, $X := X_1 \times \dots \times X_n$, $G \subset X$ an open set, $a \in G$ and $f\colon G \to Y$ a mapping. Let f be partially Fréchet differentiable at a with respect to the nth variable and let f be strictly differentiable at a with respect to the jth variable for each $1 \leq j \leq n-1$. Then f is Fréchet differentiable at a.*

Theorem 4.5. *Let $X_1, \dots, X_n$ and Y be Banach spaces, $X := X_1 \times \dots \times X_n$, $G \subset X$ an open set, and let $f\colon G \to Y$ be a continuous mapping. Suppose that the spaces $\mathcal{L}(X_1, Y), \dots, \mathcal{L}(X_{n-1}, Y)$ are separable. Then there exists a first category set $A \subset G$ such that, for all $x \in G \setminus A$, the following implication holds:*

$$f \text{ has all Fréchet partial derivatives at } x \Rightarrow f \text{ is Fréchet differentiable at } x.$$

Proof: Let A_i, $1 \leq i \leq n-1$, be the set of all $x \in G$ such that $f'_{X_i}(x)$ exists and f is not strictly differentiable at x along X_i (recall that we identify X_i with a subspace of X by the usual way). By Theorem 4.1 each A_i is a first category set and consequently $A := A_1 \cup \dots \cup A_{n-1}$ is also a first category set. If $x \in G \setminus A$ and f has all Fréchet partial derivatives at x, then f is strictly differentiable at x with respect to the jth variable for each $1 \leq j \leq n-1$ and so f is Fréchet differentiable at x by Proposition 4.4. □

Remark 4.6. Probably the most interesting is the case of a real function f (i.e. $Y = \mathbb{R}$). Then we assume that the dual space $(X_i)^*$ is separable (i.e. X_i is a separable Asplund space) for each $1 \leq i \leq n-1$. Using the method of separable reduction, we will show that the result holds also if X_i, $1 \leq i \leq n-1$, are general Asplund spaces, see Corollary 6.8 below. A number of other concrete applications of Theorem 4.5 can be easily obtained using the facts from Remark 4.2.

As an easy consequence of Theorem 4.5 (and Proposition 3.3) we obtain the following result on generic Fréchet differentiability of functions whose all partial functions are generically Fréchet differentiable.

Proposition 4.7. *Let $X_1, \dots, X_n$ and Y be Banach spaces, $X := X_1 \times \dots \times X_n$, and let $f\colon X \to Y$ be continuous. Suppose that all X_i, $1 \leq i \leq n$, are separable and the spaces $\mathcal{L}(X_1, Y), \dots, \mathcal{L}(X_{n-1}, Y)$ are separable. Let each partial function*

$$f(x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_n), \qquad 1 \leq i \leq n,$$

be generically Fréchet differentiable on X_i. Then f is generically Fréchet differentiable.

Proof: For each $1 \leq i \leq n$, denote

$$P_i := \{x \in X\colon f'_i(x) \text{ does not exist}\}.$$

Identifying by the natural way X with $Y_i \times X_i$, where $Y_i := X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$, observe that, by the assumptions, the set $\{t \in X_i : (y,t) \in P_i\}$ is a first category set in X_i for each $y \in Y_i$. Since P_i is a Borel set by Proposition 3.3, it has the Baire property and consequently the Kuratowski–Ulam theorem, see, e.g., [8, Theorem 8.41], implies that P_i is a first category set. Thus f has all Fréchet partial derivatives outside the first category set $P_1 \cup \cdots \cup P_n$ and consequently f is generically Fréchet differentiable by Theorem 4.5. □

Similarly as Theorem 4.5, we obtain the following result.

Proposition 4.8. *Let X, Y be Banach spaces, $G \subset X$ an open set and $f\colon G \to Y$ a continuous mapping. Let V_1 be a subspace of X such that the space $\mathcal{L}(V_1, Y)$ is separable. Then there exists a first category set $A \subset G$ such that, for each $x \in G \setminus A$, the following assertion holds:*

(∗) *If f is Fréchet differentiable at x along V_1 and along some topological complement V_2^x of V_1, then f is Fréchet differentiable at x.*

Proof: Let A be the set of all $x \in G$ such that $f'_{V_1}(x)$ exists and f is not strictly differentiable at x along V_1. By Theorem 4.1, A is a first category set. Now fix an arbitrary $x \in G \setminus A$ and suppose that f is Fréchet differentiable at x along V_1 and along some topological complement V_2^x of V_1. Further identify X with $\widetilde{X} := V_1 \times V_2^x$ by the canonical isomorphism. Then f (considered on $\widetilde{X}$) is strictly differentiable at x along V_1 (considered as a subspace of $\widetilde{X}$) and it is Fréchet differentiable along V_2^x. So Proposition 4.4 implies that f is Fréchet differentiable at x. □

5. Lipschitz case

Lemma 5.1. *Let X, Y be Banach spaces, $G \subset X$ an open set and $f\colon G \to Y$ an arbitrary mapping. Let V_1 be a subspace of X such that $\mathcal{L}(V_1, Y)$ is separable. Then there exists a σ-porous set $A \subset G$ such that the following assertion holds for each $x \in G \setminus A$:*

(∗) *Let f be Lipschitz at x and Fréchet differentiable at x along V_1, and let V^x and V_2^x be subspaces of X such that $V^x = V_1 \oplus V_2^x$ and f is Fréchet differentiable at x along V_2^x. Then f is Fréchet differentiable at x along V^x.*

Proof: First choose a countable dense subset Φ of $\mathcal{L}(V_1, Y)$. Further denote by A the set of all $x \in G$ for which assertion (∗) does not hold.

Then, for each $x \in A$, f is Lipschitz at x and Fréchet differentiable at x along V_1, and we can fix spaces V^x and V_2^x such that $V^x = V_1 \oplus V_2^x$, f is Fréchet differentiable at x along V_2^x and f is not Fréchet differentiable at x along V^x.

Now consider an arbitrary fixed $x \in A$. We can write any $v \in V^x$ in a unique way as $v = v_1^x + v_2^x$ with $v_1^x \in V_1$ and $v_2^x \in V_2^x$ and choose $p \in \mathbb{N}$ such that

$$\max(|v_1^x|, |v_2^x|) \le p|v| \qquad \text{for each } v \in V^x. \tag{5.1}$$

Further, since f is Lipschitz at x, we can choose $l \in \mathbb{N}$ so big that

$$|f(y) - f(x)| \le l|y - x| \qquad \text{whenever } |y - x| \le \frac{1}{l}. \tag{5.2}$$

Define $\Psi^x\colon V^x \to Y$ setting $\Psi^x(v) = f'_{V_1}(x)v_1^x + f'_{V_2^x}(x)v_2^x$, $v \in V^x$. Then $\Psi^x \in \mathcal{L}(V^x, Y)$. Since $f'_{V^x}(x)$ does not exist, we can choose $n \in \mathbb{N}$ such that

$$\limsup_{v \to 0, v \in V^x} \frac{|f(x+v) - f(x) - (f'_{V_1}(x)v_1^x + f'_{V_2^x}(x)v_2^x)|}{|v|} > \frac{7}{n}. \tag{5.3}$$

Further choose $m \in \mathbb{N}$ such that

$$|f(x+h_1) - f(x) - f'_{V_1}(x)h_1| \le \frac{1}{pn}|h_1| \qquad \text{whenever } h_1 \in V_1,\ |h_1| < \frac{1}{m} \tag{5.4}$$

and

$$|f(x+h_2) - f(x) - f'_{V_2^x}(x)h_2| \le \frac{1}{pn}|h_2| \qquad \text{whenever } h_2 \in V_2^x,\ |h_2| < \frac{1}{m}. \tag{5.5}$$

Finally choose $\varphi \in \Phi$ such that

$$|f'_{V_1}(x) - \varphi| \le \frac{1}{pn}. \tag{5.6}$$

For $p, l, n, m \in \mathbb{N}$ and $\varphi \in \Phi$, denote by $A_{p,l,n,m,\varphi}$ the set of all $x \in A$ for which conditions (5.1), (5.2), (5.3), (5.4), (5.5), (5.6) hold. Then

$$A = \bigcup \{A_{p,l,n,m,\varphi}\colon\ p, l, n, m \in \mathbb{N},\ \varphi \in \Phi\}$$

and thus it is sufficient to prove that, for each fixed $p, l, n, m \in \mathbb{N}$ and $\varphi \in \Phi$, the set $A^* := A_{p,l,n,m,\varphi}$ is porous.

Suppose to the contrary that $x \in A^*$ such that A^* is not porous at x is given. Then, by Definition 2.1, we can choose $0 < \delta < 1$ such that

$$B\Big(t, \frac{|t - x|}{2p\,ln}\Big) \cap A^* \ne \emptyset \qquad \text{whenever } t \in B(x, \delta) \setminus \{x\}. \tag{5.7}$$

By (5.3) we can choose $v \in V^x$ such that

$$0 < |v| < \frac{\delta}{pm} < 1 \tag{5.8}$$

and

$$D := |f(x+v) - f(x) - (f'_{V_1}(x)v_1^x + f'_{V_2^x}(x)v_2^x)| > \frac{7}{n}|v|. \tag{5.9}$$

By (5.1) and (5.8) we obtain

$$\max(|v_1^x|, |v_2^x|) \leq p|v| < p \cdot \frac{\delta}{pm} = \frac{\delta}{m} < \frac{1}{m} \tag{5.10}$$

and so (5.5) and (5.1) imply

$$|f(x+v_2^x) - f(x) - f'_{V_2}(x)v_2^x| \leq \frac{1}{pn}|v_2^x| \leq \frac{1}{n}|v|. \tag{5.11}$$

By (5.10) we have $|v_2^x| < \delta$. Further we have $v_2^x \neq 0$, since otherwise $v = v_1^x$ and so (5.4) with (5.10) imply $D \leq (1/n)|v|$ which contradicts (5.9). Thus we can apply (5.7) to $t := x + v_2^x$ and obtain a point $y \in A^*$ such that

$$|(x+v_2^x) - y| < \frac{|v_2^x|}{2p\,ln} \leq \frac{p|v|}{2p\,ln} = \frac{|v|}{2\,ln} \leq \frac{1}{l} \tag{5.12}$$

(we have used also (5.1) and (5.8)). Similarly, since $0 < |v| < \delta$ by (5.8), we obtain by (5.7) a point $z \in A^*$ such that

$$|(x+v) - z| < \frac{|v|}{2p\,ln} \leq \frac{1}{l}. \tag{5.13}$$

Since $y \in A^*$ and $z \in A^*$, by (5.12), (5.13) and (5.2) we obtain

$$|f(x+v_2^x) - f(y)| \leq l \cdot |(x+v_2^x) - y| \leq l \cdot \frac{|v|}{2\,ln} \leq \frac{1}{n}|v| \tag{5.14}$$

and

$$|f(x+v) - f(z)| \leq l \cdot |(x+v) - z| \leq l \cdot \frac{|v|}{2p\,ln} \leq \frac{1}{n}|v|. \tag{5.15}$$

Since $y \in A^*$ and $|v_1^x| < 1/m$ by (5.10), using (5.4) and (5.1) we obtain

$$|f(y+v_1^x) - f(y) - f'_{V_1}(y)v_1^x| \leq \frac{1}{pn}|v_1^x| \leq \frac{1}{n}|v|. \tag{5.16}$$

We have $|(x+v)-(y+v_1^x)| = |(x+v_2^x)-y| \leq (2ln)^{-1}|v|$ by (5.12) and consequently

$$|z-(y+v_1^x)| \leq |(x+v)-z| + |(x+v)-(y+v_1^x)| \leq \frac{|v|}{2p\,ln} + \frac{|v|}{2\,ln} \leq \frac{|v|}{ln} \leq \frac{1}{l} \tag{5.17}$$

by (5.13). Since $z \in A^*$, we obtain by (5.17) and (5.2)

$$|f(z) - f(y + v_1^x)| \le l|z - (y + v_1^x)| \le l \cdot \frac{|v|}{ln} = \frac{1}{n}|v|,$$

which together with (5.15) gives

$$(5.18)\quad |f(x+v) - f(y+v_1^x)| \le |f(x+v) - f(z)| + |f(z) - f(y+v_1^x)| \le \frac{2}{n}|v|.$$

Since $y \in A^*$, we obtain by (5.6) (with $x := y$) $|f'_{V_1}(y) - \varphi| \le (pn)^{-1}$, which together with (5.6) gives $|f'_{V_1}(y) - f'_{V_1}(x)| \le 2(pn)^{-1}$. Using also (5.1), we obtain

$$(5.19)\quad |f'_{V_1}(y)v_1^x - f'_{V_1}(x)v_1^x| \le \frac{2}{pn} \cdot |v_1^x| \le \frac{2}{n}|v|.$$

Using (5.11), we obtain the following upper estimate of D (from (5.9))

$$\begin{aligned} D &:= |f(x+v) - f(x) - (f'_{V_1}(x)v_1^x + f'_{V_2^x}(x)v_2^x)| \\ (5.20)\quad &= |(f(x+v) - f(x+v_2^x) - f'_{V_1}(x)v_1^x) + (f(x+v_2^x) - f(x) - f'_{V_2^x}(x)v_2^x)| \\ &\le |f(x+v) - f(x+v_2^x) - f'_{V_1}(x)v_1^x| + \frac{1}{n}|v|. \end{aligned}$$

Further, using (5.16), (5.18), (5.14) and (5.19), we obtain

$$\begin{aligned} &|f(x+v) - f(x+v_2^x) - f'_{V_1}(x)v_1^x| \\ &\qquad \le |f(y+v_1^x) - f(y) - f'_{V_1}(y)v_1^x| + |f(x+v) - f(y+v_1^x)| \\ &\qquad\quad + |f(y) - f(x+v_2^x)| + |f'_{V_1}(y)v_1^x - f'_{V_1}(x)v_1^x| \\ &\qquad \le \frac{1}{n}|v| + \frac{2}{n}|v| + \frac{1}{n}|v| + \frac{2}{n}|v| = \frac{6}{n}|v|. \end{aligned}$$

Thus (5.20) gives $D \le (7/n)|v|$ which contradicts (5.9). □

By induction, we easily infer from Lemma 5.1 the following analogue of Theorem 4.5.

Theorem 5.2. *Let $X_1, \dots, X_n$ and Y be Banach spaces, $X := X_1 \times \cdots \times X_n$, $G \subset X$ an open set and $f\colon G \to Y$ an arbitrary mapping. Let the spaces $\mathcal{L}(X_1, Y), \dots, \mathcal{L}(X_{n-1}, Y)$ be separable. Then there exists a σ-porous set $A \subset G$ such that, for all $x \in G \setminus A$, the following implication holds:*

(∗) *f is Lipschitz at x and has all Fréchet partial derivatives at x $\Rightarrow$ f is Fréchet differentiable at x.*

Proof: We will proceed by induction on n. For $n = 2$ the assertion immediately follows from Lemma 5.1 used for $V_1 := X_1$ and $V_2^x := X_2$.

Now suppose that $n \geq 3$ and "the theorem holds for $n := n-1$". Further suppose that $X_1, \dots, X_n$, Y, G and f which satisfy the assumptions of the theorem are given. Since $\mathcal{L}(X_{n-1}, Y)$ is separable, we can use Lemma 5.1 with $V_1 := X_{n-1}$, and obtain a σ-porous set $A_1 \subset G$ such that, for each $x \in G \setminus A_1$, the following assertion holds:

(α_1) If f is Lipschitz at x and Fréchet differentiable at x along X_{n-1} and along X_n, then f is Fréchet differentiable at x along the space $\widetilde{X}_{n-1} := X_{n-1} \times X_n$.

Now we identify X with $\widetilde{X} := X_1 \times \cdots \times X_{n-2} \times \widetilde{X}_{n-1}$ by the usual way. By the inductive assumption, there exists a σ-porous set $A_2 \subset G$ such that, for each $x \in G \setminus A_2$, the following assertion holds:

(α_2) If f is Lipschitz at x and Fréchet differentiable at x along all spaces $X_1, \dots, X_{n-2}, \widetilde{X}_{n-1}$, then f is Fréchet differentiable at x (in $\widetilde{X} = X$).

Setting $A := A_1 \cup A_2$, and using for each $x \in G \setminus A$ the validity of (α_1) and (α_2), we obtain that implication ($*$) holds for each $x \in G \setminus A$. □

As an immediate consequence of Lemma 5.1, we obtain the following analogue of Proposition 4.8.

Proposition 5.3. *Let X, Y be Banach spaces, $G \subset X$ an open set and $f\colon G \to Y$ an arbitrary mapping. Let V_1 be a subspace of X such that the space $\mathcal{L}(V_1, Y)$ is separable. Then there exists a σ-porous set $A \subset G$ such that, for each $x \in G \setminus A$, the following assertion holds:*

($*$) *If f is Lipschitz at x and Fréchet differentiable at x along V_1 and along some topological complement V_2^x of V_1, then f is Fréchet differentiable at x.*

In connection with Propositions 4.8 and 5.3, it is natural to ask, for which Banach spaces X, Y the following statement holds:

(S) Let $f\colon X \to Y$ be continuous (respectively Lipschitz). Denote by E_f the set of all points $x \in X$ at which there exist a subspace V^x of X and its topological complement W^x such that f is Fréchet differentiable at x both along V^x and W^x but is not Fréchet differentiable at x. Then E_f is a first category set (respectively a σ-porous set).

If $\dim X < \infty$ and $Y = \mathbb{R}$, then the "continuous part" of (S) holds; it easily follows from [7]. Further, if $\dim X < \infty$, then the "Lipschitz part" of (S) easily follows from [14, Theorem 2] (and also from Corollary 5.6 below).

I conjecture that no part of (S) holds if X is an infinite-dimensional space, but I do not know any counterexample.

For the weaker version of (S) (which we obtain demanding that V^x in the definition of E_f is finite-dimensional), see Corollary 5.6 below. It is an immediate consequence of Proposition 5.5 below, which can be of an independent interest. In its proof we use substantially Lemma 5.1 and the following result which immediately follows from [21, Corollary 3.4] (since each σ-directionally porous set is clearly σ-porous).

Proposition 5.4 ([21]). *Let X be a separable Banach space, Y a Banach space, $G \subset X$ an open set, and $f\colon G \to Y$ an arbitrary mapping. Then there exists a σ-porous set $A \subset G$ such that, for each $x \in G \setminus A$, the following assertion holds:*

($*$) *If f is Lipschitz at x and the one-sided directional derivative $f'_+(x,u)$ exists in all directions u from a set $S_x \subset X$ whose linear span is dense in X, then f is Gâteaux differentiable at x.*

Proposition 5.5. *Let X, Y be separable Banach spaces, $G \subset X$ an open set and $f\colon G \to Y$ an arbitrary mapping. Then there exists a σ-porous set $A \subset G$ such that, for each $x \in G \setminus A$, the following assertion holds.*

($**$) *Let f be Lipschitz at x and Fréchet differentiable at x along a subspace M^x of finite codimension, and let there exist a set $S_x \subset X$ such that $\operatorname{span} S_x$ is dense in X and $f'_+(x,u)$ exists for all $u \in S_x$. Then f is Fréchet differentiable at x.*

Proof: For the given f, we define A as the set of all $x \in G$ for which assertion ($**$) does not hold; we will prove that A is σ-porous.

Choose a dense countable set $D \subset X$ and for each point $x \in A$ set

$$k_x := \inf\{k \in \mathbb{N} \cup \{0\}\colon\ f \text{ is Fréchet differentiable at } x \text{ along a subspace } M^x \text{ of codimension } k\},$$

(where we adopt the convention that X has codimension 0). Note that, if $x \in A$, then clearly $1 \le k_x < \infty$. Further, for each $x \in A$, choose a subspace M^x of X of codimension k_x such that f is Fréchet differentiable at x along M^x and then choose a vector $0 \neq v_x \in D \setminus M^x$. For each $k \in \mathbb{N}$ and $v \in D$, set

$$A_{k,v} := \{x \in A\colon\ k_x = k,\ v_x = v\}.$$

Then $A = \bigcup\{A_{k,v}\colon\ k \in \mathbb{N},\ v \in D\}$ and thus it is sufficient to prove that, for each fixed $k \in \mathbb{N}$ and $v \in D$, the set $A_{k,v}$ is σ-porous.

To this end, set $V_1 := \operatorname{span}\{v\}$ and observe that $\mathcal{L}(V_1, Y)$ is separable. Let $\tilde{A} \subset G$ be a σ-porous set which corresponds to V_1 and f by Lemma 5.1; i.e.

(5.21) assertion ($*$) from Lemma 5.1 holds for each $x \in G \setminus \tilde{A}$.

Further, let $\tilde{\tilde{A}}$ be a σ-porous set which corresponds to f by Proposition 5.4.

Now it is sufficient to prove that $A_{k,v} \subset \tilde{A} \cup \tilde{\tilde{A}}$. To prove this inclusion, suppose to the contrary that there exists a point $x \in A_{k,v} \setminus (\tilde{A} \cup \tilde{\tilde{A}})$. Set $V_2^x := M^x$ and $V^x := V_1 + V_2^x$. We know that V_2^x has codimension $1 \leq k_x < \infty$ and $V_1 \cap V_2^x = \{0\}$. Consequently, V^x is closed, $V^x = V_1 \oplus V_2^x$, see, e.g., [5, Exercise 5.27 and Proposition 5.3], and it is easy to see that V^x has codimension $k_x - 1$. Since $x \in A$, we have that the assumptions of assertion $(**)$ hold and so also the assumptions of assertion $(*)$ of Proposition 5.4 hold. So, since $x \in G \setminus \tilde{\tilde{A}}$, by the choice of $\tilde{\tilde{A}}$ we obtain that f is Gâteaux differentiable at x, and consequently, f is Fréchet differentiable along V_1 at x. Since f is Fréchet differentiable at x along the space $V_2^x = M^x$ and $x \in G \setminus \tilde{A}$, we can use (5.21) and obtain that f is Fréchet differentiable along the space V^x of codimension $k_x - 1$ which contradicts the definition of k_x. $\square$

As immediate consequences, we obtain the following results.

Corollary 5.6. *Let X, Y be separable Banach spaces, $G \subset X$ an open set and $f \colon G \to Y$ an arbitrary mapping. Then there exists a σ-porous set $A \subset G$ such that the following assertion holds.*

$(*)$ *Let f be Lipschitz at x and let there exist a finite-dimensional subspace V^x of X and its topological complement W^x of X such that f is Fréchet differentiable at x along V^x and W^x. Then f is Fréchet differentiable at x.*

Corollary 5.7. *Let X, Y be separable Banach spaces, $G \subset X$ an open set and $f \colon G \to Y$ a Lipschitz mapping. Then there exists a σ-upper porous set $A \subset G$ such that f is Fréchet differentiable at every point $x \in G \setminus A$ at which it is Fréchet differentiable along a closed subspace of finite codimension and Gâteaux differentiable.*

Remark 5.8. In some cases, the set A from Corollary 5.7 is necessarily nonempty. To show this, set $X := l_2$, $Y := \mathbb{R}$ and denote by e_n, $n \in \mathbb{N}$, the canonical basis vectors in l_2. Set

$$p_n := \frac{1}{n}(e_1 + e_n), \qquad n \geq 2,$$

$$F := X \setminus \bigcup_{n=2}^{\infty} B\Big(p_n, \frac{1}{2n}\Big) \qquad \text{and}$$

$$f(x) := \operatorname{dist}(x, F), \qquad x \in X.$$

Then f is Lipschitz on $G := X$ and it is easy to check that $0 \in A$, whenever A is as in Corollary 5.7.

6. Results proved by the separable reduction method

In this section, we will use the well-known method of separable reduction. Namely, we will first prove "the separable case" and from it we will obtain the "nonseparable case" using some known results which say that some notions are "separably determined in the sense of rich families". For the following notion of a "rich family", see e.g. [11, page 37] or [3].

Definition 6.1. Let X be a Banach space. A family $\mathcal{F}$ of separable subspaces of X is called a *rich family* if:

(R1) $V_i \in \mathcal{F}$, $i \in \mathbb{N}$, and $V_1 \subset V_2 \subset \dots$, then $\overline{\bigcup\{V_n \colon n \in \mathbb{N}\}} \in \mathcal{F}$;

(R2) for each separable subspace V_0 of X there exists $V \in \mathcal{F}$ such that $V_0 \subset V$.

A basic (easy) fact, see e.g. [11, Proposition 3.6.2], concerning rich families is the following.

Lemma 6.2. *Let X be a Banach space and let $\{\mathcal{F}_n \colon n \in \mathbb{N}\}$ be rich families of separable subspaces of X. Then $\mathcal{F} := \bigcap\{\mathcal{F}_n \colon n \in \mathbb{N}\}$ is also a rich family of separable subspaces of X.*

We will use also the following simple fact which is a reformulation of [20, Lemma 4.4].

Lemma 6.3. *Let $X_1, \dots, X_n$ be Banach spaces and $X := X_1 \times \dots \times X_n$. Let $\mathcal{F}_k$ be a rich family of separable subspaces of X_k, $1 \le k \le n$. Then*

$$\mathcal{F} := \{V_1 \times \dots \times V_n \colon\ V_k \in \mathcal{F}_k,\ 1 \le k \le n\}$$

is a rich family of separable subspaces of X.

Much more difficult is the following result which says that Fréchet differentiability at a point is "separably determined in the sense of rich families".

Theorem 6.4 ([11, Theorem 3.6.10])**.** *Let X, Y be Banach spaces, $G \subset X$ an open set and $f \colon G \to Y$ a mapping. Then there exists a rich family $\mathcal{F}$ of separable subspaces of X such that for every $V \in \mathcal{F}$, f is Fréchet differentiable (with respect to X) at every $x \in V \cap G$, at which its restriction to $V \cap G$ is Fréchet differentiable (with respect to V).*

(In fact, [11, Theorem 3.6.10] is formulated for $G = X$ only, but if we apply this formally weaker theorem to any extension $\tilde{f}$ of f to X, we obtain the assertion of Theorem 6.4.)

The following result on "separable determination of first category sets and σ-upper porous sets" were first proved in [2] and [4] "in the sense of suitable models" and then transferred to the following result in [3].

Theorem 6.5 ([3, Corollary 5]). *Let X be a Banach space and $A \subset X$ a Souslin set. Then there exists a rich family $\mathcal{F}$ of separable subspaces of X such that for every $V \in \mathcal{F}$ we have:*

(i) *A is of the first category in $X \iff A \cap V$ is of the first category in V,*

(ii) *A is σ-porous in $X \iff A \cap V$ is σ-porous in V.*

Recall that every Borel set in X is Souslin. We will use also the following known fact.

Theorem 6.6 ([20, Theorem 4.7]). *Let X, Y be Banach spaces, $G \subset X$ an open set, and let $f\colon G \to Y$ be an arbitrary mapping. Then the following conditions are equivalent:*

(i) *Mapping f is generically Fréchet differentiable.*

(ii) *There exists a rich family $\mathcal{F}$ of separable subspaces of X such that $f|_{V\cap G}$ is generically Fréchet differentiable (with respect to V) on $V \cap G$ for each $V \in \mathcal{F}$.*

Now we will prove, using Theorem 4.5, Theorem 5.2 and the method of separable reduction, the following result which generalizes Theorem 4.5 and partly generalizes Theorem 5.2.

Theorem 6.7. *Let $X_1, \dots, X_n$ and Y be Banach spaces, $X := X_1 \times \cdots \times X_n$ and $G \subset X$ an open set. Suppose that each space $\mathcal{L}(\widetilde{X}_i, \widetilde{Y})$ is separable whenever $\widetilde{X}_i$ is a separable subspace of X_i, $i = 1, \dots, n-1$, and $\widetilde{Y}$ is a separable subspace of Y. Let $f\colon G \to Y$ be a continuous (respectively Lipschitz) mapping.*

Then there exists a first category (respectively σ-porous) set $A \subset G$ such that, for each $x \in G \setminus A$, the following implication holds:

$$f \text{ has all Fréchet partial derivatives at } x \Rightarrow f \text{ is Fréchet differentiable at } x.$$

Proof: We will prove the "continuous part" and the "Lipschitz part" of the theorem together.

In the first step of the proof, we will prove the theorem in the special case when all spaces $X_1, \dots, X_n$ are separable. In this case observe that the space $\widetilde{Y} := \overline{\operatorname{span} f(G)}$ is separable and therefore the spaces $\mathcal{L}(X_1, \widetilde{Y}), \dots, \mathcal{L}(X_{n-1}, \widetilde{Y})$ are separable by the assumptions of the theorem. Further observe that $f\colon G \to Y$ is partially Fréchet differentiable (respectively Fréchet differentiable) at $x \in G$ if and only if $f\colon G \to \widetilde{Y}$ has this property. So, if f is continuous (respectively Lipschitz), then the existence of a first category (respectively σ-porous) set A from the conclusion of the theorem follows from Theorem 4.5 (respectively Theorem 5.2).

In the second step, we will prove the general case using the method of separable reduction. Denote by A the set of all $x \in G$ at which all partial Fréchet derivatives $f_i'(x)$, $i = 1, \dots, n$, exist but f is not Fréchet differentiable. Our aim is to prove that

$$A \text{ is a first category set if } f \text{ is continuous, and} \tag{6.1}$$

$$A \text{ is a } \sigma\text{-porous set if } f \text{ is Lipschitz.} \tag{6.2}$$

Notice that (3.1) and Proposition 3.3 give that A is a Borel set (and hence a Souslin set) in X. Thus Theorem 6.5 implies that there exists a rich family $\mathcal{F}_1$ of separable subspaces of X such that for every $V \in \mathcal{F}_1$ we have that

$$\begin{aligned} &A \text{ is of the first category in } X \text{ whenever} \\ &A \cap V \text{ is of the first category in } V, \text{ and} \end{aligned} \tag{6.3}$$

$$A \text{ is of } \sigma\text{-porous in } X \text{ whenever } A \cap V \text{ is } \sigma\text{-porous in } V. \tag{6.4}$$

By Lemma 6.3, the family

$$\mathcal{F}_2 := \{V_1 \times \cdots \times V_n \colon V_k \text{ is a separable subspace of } X_k,\ 1 \le k \le n\}$$

is a rich family of separable subspaces of X.

By Theorem 6.4 there exists a rich family $\mathcal{F}_3$ of separable subspaces of X such that for every $V \in \mathcal{F}_3$, f is Fréchet differentiable (with respect to X) at every $x \in V \cap G$ at which its restriction to $V \cap G$ is Fréchet differentiable (with respect to V).

Now, by Lemma 6.2, $\mathcal{F} := \mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3$ is a rich family of separable subspaces of X.

Choose an arbitrary $V \in \mathcal{F}$. Since $V \in \mathcal{F}_2$, we have $V = V_1 \times \cdots \times V_n$ where V_k is a separable subspace of X_k, $1 \le k \le n$.

If $A \cap V \neq \emptyset$, set $g := f|_{(V \cap G)}$. Using the definition of A, we easily see that $g_i'(x) = g_{V_i}'(x)$, $i = 1, \dots, n$, exist for each $x \in A \cap V$. Since f is Fréchet nondifferentiable at each $x \in A \cap V$ and $V \in \mathcal{F}_3$, we conclude that g is Fréchet nondifferentiable at each $x \in A \cap V$. Further, by the assumptions of the theorem, the space $\mathcal{L}(\widetilde{X}_i, \widetilde{Y})$ is separable whenever $\widetilde{X}_i$ is a separable subspace of V_i, $i = 1, \dots, n-1$, and $\widetilde{Y}$ is a separable subspace of Y. So, using for g on $G \cap V \subset V = V_1 \times \cdots \times V_n$ the special case of the theorem proved in the first step of the proof, we obtain that $A \cap V$ is a first category (respectively σ-porous) set in V if f is continuous (respectively Lipschitz). Therefore, since $V \in \mathcal{F}_1$, we obtain that both (6.3) and (6.4) hold. $\square$

Using (2.1), we see that Theorem 6.7 has the following interesting consequence.

Corollary 6.8. *Let $X_1, \dots, X_n$ be Banach spaces such that $X_1, \dots, X_{n-1}$ are Asplund spaces, and let $G \subset X := X_1 \times \cdots \times X_n$ be an open set. Let f be a continuous (respectively Lipschitz) real function on G.*

Then there exists a first category (respectively σ-porous) set $A \subset G$ such that, for each $x \in G \setminus A$, the following implication holds:

$$f \text{ has all Fréchet partial derivatives at } x \Rightarrow f \text{ is Fréchet differentiable at } x.$$

Remark 6.9.

- (i) There are also other concrete consequences of Theorem 6.7. For example, using facts from Remark 4.2, it is easy to show that Theorem 6.7 can be used if each X_i, $i = 1, \dots, n-1$, is a Hilbert space (respectively a subspace of $c_0(\Gamma)$) and $Y = l_q(\Gamma)$, $1 \le q < 2$, (respectively Y has the Radon–Nikodým property).
- (ii) Theorem 6.7 can be further slightly generalized to a more complicated general theorem (working with "rich families of $\widetilde{X}_i$") which has further concrete applications. Let us note that the case with $n = 2$, $X_1 = X_2 = l_1$ and $Y = \mathbb{R}$ remains open.
- (iii) I do not know any example excluding the possibility that the assumptions concerning Banach spaces $X_1, \dots, X_n, Y$ can be omitted in Theorem 6.7. For this reason the observations in (i) and (ii) are mentioned without any details.

By the separable reduction method, we prove also the following result on continuous functions whose all partial functions are DC (recall that a function on a Banach space is called DC if it is the difference of two continuous convex functions).

Proposition 6.10. *Let $X_1, \dots, X_n$ be Asplund spaces and f a continuous real function on $X := X_1 \times \cdots \times X_n$. Let each partial function*

$$f(x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_n), \qquad 1 \le i \le n,$$

is DC on X_i. Then f is generically Fréchet differentiable.

Proof: By Lemma 6.3, the family

$$\mathcal{F} := \{V_1 \times \cdots \times V_n \colon V_k \text{ is a separable subspace of } X_k,\ 1 \le k \le n\}$$

is a rich family of separable subspaces of X. Choose an arbitrary $V = V_1 \times \cdots \times V_n \in \mathcal{F}$ and set $g := f|_V$. Then each partial function

$$g(v_1, \dots, v_{i-1}, \cdot, v_{i+1}, \dots, v_n), \qquad 1 \le i \le n,$$

is DC on the Asplund space V_i, see (2.1), and consequently is generically differentiable on V_i. Since the spaces $V_1^*, \dots, V_{n-1}^*$ are separable, we can use Proposition 4.7 and obtain that g is generically differentiable. Therefore f is generically differentiable by Theorem 6.6. □

Remark 6.11.

(i) In the case of a locally Lipschitz f, Proposition 6.10 immediately follows from [20, Corollary 8.1] and also from [20, Corollary 8.4].

(ii) The proof of Proposition 6.10 works if we weaken the assumption that each partial function of f is DC to the assumption that it is a difference of two approximately convex functions. Thus we obtain that, in [20, Corollary 8.1], it is possible to suppose the continuity of f instead of the local Lipschitzness of f.

(iii) Proposition 6.10 could be proved quite similarly as Theorem 6.7, but the present proof based on Theorem 6.6 is shorter.

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L. Zajíček:

Charles University, Faculty of Mathematics and Physics, Sokolovská 83, 186 75 Praha 8 Karlín, Czech Republic

E-mail: zajicek@karlin.mff.cuni.cz

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