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Practical h-stability behavior of time-varying nonlinear systems

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Abstract. We deal with the problem of practical uniform h-stability for nonlinear time-varying perturbed differential equations. The main aim is to give sufficient conditions on the linear and perturbed terms to guarantee the global existence and the practical uniform h-stability of the solutions based on Gronwall's type integral inequalities. Several numerical examples and an application to control systems with simulations are presented to illustrate the applicability of the obtained results.

Keywords: Gronwall's inequality; perturbed system; practical h-stability

Classification: 34A30, 34A34, 34D10

1. Introduction

One of the main qualitative problem in theory of differential equations is the stability of solutions. Recently, the stability analysis of time-varying systems has been investigated extensively in many areas, see [1], [2], [7], [10], [6], [19]. There are important types of stability of dynamical systems, namely exponential stability and asymptotic stability. Considering asymptotic stability, it is more desirable to consider exponential stability criterion for dynamical systems. It is required that all solutions starting near an equilibrium point not only stay nearly, but tend to equilibrium point very fast with exponential decay rate.

In [21], [22], M. Pinto introduced the notion of h-stability with the intention of obtaining results about stability for weakly stable systems under some perturbations. In [20], the authors extended the study of exponential asymptotic stability to a variety of reasonable systems called h-systems. There have been a number of interesting developments in searching the h-stability criteria for nonlinear differential systems, see [13], but most have been restricted to finding the asymptotic stability conditions. In practice, we may only need to stabilize a system into the region of a phase space where the system may oscillate near the state in which the implementation is still acceptable. This concept is called practical stability, see

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[4], [10], [16], [18], which is very useful for studying the asymptotic behavior of the system in which the origin is not necessarily an equilibrium point. In this case, practical stability is an important concept to analyze the asymptotic behavior of solutions with respect to a small neighborhood of the origin. The investigation of this notion for time-varying perturbed differential equations is established by many researchers, see for example [5] where the authors deal with the problem of global uniform practical exponential stability of a general nonlinear nonautonomous delay equation whose origin is not an equilibrium point of the system, that is what we mean by practical exponential problem. In [17], the author has introduced a new notion of practical stability called practical h-stability. The theory of practical h-stability has gained increasing significance in the last years as is apparent from the large number of publications on the subject, see [12], [11], [8], [14], [9]. For instance, H. Damak et al. in [12] studied the practical h-stability of such time-varying perturbed systems under different conditions on the perturbed term based on nonlinear integral inequalities. In this paper, we extend the most results in [12], [18] to discuss the problem of practical h-stability of differential perturbed equations. The practical h-stability analysis is introduced using the Gronwall inequality approach. Indeed, the Gronwall type integral inequalities play an important role in the area of integral (and differential) equations as a technique to show existence and uniqueness of a solution and to obtain various estimates for the solutions as is given in [3] and [15]. They can be viewed as a type of result which gives a priori bounds for the function which satisfies an integral differential inequality. Then, based on certain integral inequality, we prove global existence and uniqueness of solutions. Also, the global practical uniform h-stability of nonlinear time-varying perturbed systems is investigated by giving some sufficient conditions on the perturbed terms. Furthermore, we give several numerical examples that illustrate the effectiveness of our results.

2. Preliminary results

Throughout this paper we adopt the following notations: $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}^n$ is the n-dimensional Euclidean space with a convenient vector norm $\|\cdot\|$, space $\mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is the space of all continuous functions from $\mathbb{R}_+ \times \mathbb{R}^n$ to $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ is the set of all $n \times n$ matrices whose entries are real valued.

We consider the nonlinear nonautonomous differential system:

$$\dot{x} = F(t, x), \quad x(t_0) = x_0, \qquad t \geq t_0 \geq 0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $F \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is locally Lipschitz in x, uniformly in t.

Let $x(t) = x(t, t_0, x_0)$ denote the solution passing through the point $x_0 \in \mathbb{R}^n$ at time t_0 and assume that it exists for all $t \geq t_0$.

Let us introduce some basic definitions and lemmas which we need in the sequel.

Definition 1 (see [12], [17]). The system (1) is called globally practically uniformly h-stable if there are $c \geq 1$, $r > 0$, and a positive continuous bounded function h on $\mathbb{R}_+$, such that for all $t \geq t_0$ and for all $x_0 \in \mathbb{R}^n$ the solution satisfies the estimation:

$$\|x(t)\| \leq c\|x_0\|h(t)h(t_0)^{-1} + r. \tag{2}$$

Here, $h(t)^{-1} = 1/h(t)$.

Remark 1. The inequality (2) implies that $\|x(t, t_0, x_0)\|$ will be bounded by $r > 0$, that is, the solution is bounded and approach toward a neighborhood of the origin for sufficiently large t when the function h is decreasing towards zero. In particular, Definition 1 generalizes the notion of h-stability where $r = 0$, see [21]. Moreover, the practical h-stability property (2) includes the concepts of practical exponential stability when $h(t) = \mathrm{e}^{-\lambda t}$, with $\lambda > 0$, see [16], and practical polynomial stability when $h(t) = 1/(1+t)^{\gamma}$, with $\gamma > 0$, see [12].

Consider now the linear time-varying system:

$$\dot{x} = A(t)x, \quad x(t_0) = x_0, \qquad t \geq t_0 \geq 0, \tag{3}$$

where $A(\cdot) \in \mathbb{R}^{n\times n}$ is a matrix whose elements are continuous functions on $\mathbb{R}_+$ and its norm is defined by:

$$\|A\| = \max_{\|x\|\leq 1} \|Ax\|.$$

In the case of linear time-varying systems, the stability status of the equilibrium can be ascertained, in principle at least, by studying the state transition matrix $\phi(t, t_0)$ associated with $A(\cdot)$, such that the general solution of system (3) is given by:

$$x(t) = \phi(t, t_0)x_0, \qquad t \geq t_0 \geq 0,$$

where $\phi(t, t_0)$ is the unique solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t, t_0) = A(t)\phi(t, t_0), \qquad \forall\, t \geq t_0 \geq 0,$$
$$\phi(t_0, t_0) = I_n, \qquad \forall\, t_0 \geq 0,$$

with the aid of this explicit characterization of solutions of system (3), it is possible to derive some useful conditions for the stability of the origin.

The next lemma introduces the global uniform h-stability of system (3) in terms of $\phi(t,t_0)$. It is very effective to present the h-stability property for linear time-varying systems.

Lemma 1 (see [22]). *The system* (3) *is globally uniformly h-stable if and only if there exist $c\geq 1$ and a positive continuous bounded function h on $\mathbb{R}_+$, such that for all $t\geq t_0$*

$$\|\phi(t,t_0)\|\leq ch(t)h(t_0)^{-1}.$$

3. Integral inequalities

To achieve our goal, we need the following lemmas.

Lemma 2 (Nonlinear generalization of Gronwall's inequality). *Let u be a nonnegative function that satisfies the integral inequality*

$$u(t)\leq\lambda+\int_{t_0}^{t}(\gamma(s)u(s)+\beta(s)u^{\alpha}(s))\,\mathrm{d}s,\qquad 0\leq\alpha<1,\ \forall\, t\geq t_0\geq 0,$$

where $\lambda\geq 0$, γ and β are nonnegative continuous functions on $\mathbb{R}_+$. Then, for all $t\geq t_0$,

$$u(t)\leq\Bigg[\lambda^{1-\alpha}\exp\left((1-\alpha)\int_{t_0}^{t}\gamma(s)\,\mathrm{d}s\right)\\ +(1-\alpha)\int_{t_0}^{t}\beta(s)\exp\left((1-\alpha)\int_{s}^{t}\gamma(r)\,\mathrm{d}r\right)\mathrm{d}s\Bigg]^{1/(1-\alpha)}.$$

Proof: Denote by v a solution of the integral equation:

$$v(t)=\lambda+\int_{t_0}^{t}[\gamma(s)v(s)+\beta(s)v^{\alpha}(s)]\,\mathrm{d}s,\qquad t\geq t_0\geq 0.$$

Then,

$$v'(t)=\gamma(t)v(t)+\beta(t)v^{\alpha}(t),\quad v(t_0)=\lambda,\qquad t\geq t_0\geq 0.$$

It follows that,

$$\frac{v'(t)}{v^{\alpha}(t)}-\gamma(t)\,\frac{1}{v^{\alpha-1}(t)}=\beta(t),\qquad t\geq t_0\geq 0.$$

Let

$$z(t)=v^{1-\alpha}(t),$$

which implies that

$$z'(t)=\frac{1-\alpha}{v^{\alpha}(t)}\,v'(t).$$

Therefore,

$$z'(t)=(1-\alpha)\gamma(t)z(t)+(1-\alpha)\beta(t),\qquad z(t_0)=\lambda^{1-\alpha}.$$

Hence, we have

$$z(t) = \lambda^{1-\alpha} \exp\left((1-\alpha)\int_{t_0}^{t} \gamma(s)\,\mathrm{d}s\right) + (1-\alpha)\int_{t_0}^{t} \beta(s) \exp\left((1-\alpha)\int_{s}^{t} \gamma(r)\,\mathrm{d}r\right) \mathrm{d}s.$$

This yields

$$\begin{aligned} v(t) = \Bigg[&\lambda^{1-\alpha} \exp\left((1-\alpha)\int_{t_0}^{t} \gamma(s)\,\mathrm{d}s\right) \\ &+ (1-\alpha)\int_{t_0}^{t} \beta(s) \exp\left((1-\alpha)\int_{s}^{t} \gamma(r)\,\mathrm{d}r\right) \mathrm{d}s\Bigg]^{1/(1-\alpha)}. \end{aligned}$$

The proof is completed. □

Lemma 3 (Gronwall–Bellman type integral inequality, see [18])**.** *Let ψ, γ and υ be nonnegative continuous functions on $\mathbb{R}_+$ for which the following inequality holds:*

$$\psi(t) \le \lambda + \int_{t_0}^{t} [\gamma(s)\psi(s) + \upsilon(s)]\,\mathrm{d}s, \qquad \forall\, t \ge t_0 \ge 0,$$

where λ is nonnegative constant. Then,

$$\psi(t) \le \lambda \exp\left(\int_{t_0}^{t} \gamma(s)\,\mathrm{d}s\right) + \kappa \exp\left(\int_{t_0}^{t} \left[\gamma(s) + \frac{1}{\kappa}\upsilon(s)\right] \mathrm{d}s\right), \qquad \forall\, t \ge t_0,\ \forall\, \kappa > 0.$$

4. Practical h-stability results

In this section, we are interested in the study of systems of the following form:

$$\dot{x} = A(t)x + H(t)x + \varpi(t,x), \qquad t \ge t_0 \ge 0, \tag{4}$$

where $A(t), H(t) \in \mathbb{R}^{n\times n}$ are matrices whose elements are continuous functions on $\mathbb{R}_+$ and $\varpi \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is locally Lipschitz in x, uniformly in t. We think of this system as a perturbation of the linear system (3). We will pay our full attention to the perturbed term $\varpi(t,x)$, because in a typical situation, we do not know it but we know some information about it, like knowing the upper bound of $\|\varpi(t,x)\|$. The trivial question posed here is: If the linear time-varying system presents one of types of stability, what can we say about the stability behavior of the perturbed system (4)? Our study is limited to the most interesting case which is, $\varpi(t,0) \ne 0$, that is, if the origin will not be an equilibrium point of the perturbed system, we can no longer study h-stability of the origin as an equilibrium point, nor should we expect the solution of the perturbed system to approach the origin as $t \to \infty$. The best we can hope is the bound and the asymptotic behavior of solutions, if the perturbation terms are small in some sense, then $\|x(t)\|$ will be small for sufficiently large t.

We start this section by studying the global practical uniform h-stability of the perturbed system (4) under conditions on the perturbed terms using the Gronwall–Bellman type integral inequality.

Theorem 1. *Consider the time-varying perturbed system* (4)*, where:*

1) *The transition matrix for the linear system* (3) *satisfies*

$$\|\phi(t,s)\| \le ch(t)h(s)^{-1}, \qquad \forall\,(t,s) \in \mathbb{R}_+^2$$

for $c \ge 1$.

2) *There exist* υ *and* ρ *nonnegative continuous functions on* $\mathbb{R}_+$ *verifying*

$$\|\varpi(t,x)\| \le \upsilon(t)\|x\| + \rho(t), \qquad \forall\, x \in \mathbb{R}^n,\ \forall\, t \ge 0, \tag{5}$$

such that $\upsilon \in L^1(\mathbb{R}_+,\mathbb{R}_+)$ *and*

$$\int_0^t h(s)^{-1}\rho(s)\,\mathrm{d}s \le M_1, \qquad M_1 > 0,\ \forall\, t \ge 0. \tag{6}$$

3) *There exists* μ *a nonnegative continuous and integrable function on* $\mathbb{R}_+$ *satisfying*

$$\|H(t)\| \le \mu(t), \qquad \forall\, t \ge 0. \tag{7}$$

Then, for all $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$*, the maximal solution* $x(t)$ *of system* (4)*, such that* $x(t_0) = x_0$*, verifies:*

(i) *The function* x *is unique and defined on* $[t_0,\infty)$.

(ii) *For all* $t \ge t_0$:

$$\|x(t)\| \le \lambda\|x_0\|h(t)h(t_0)^{-1} + r,$$

where λ *and* r *are positive constants.*

Proof: (i) The system (4) can be written as $\dot{x} = F(t,x)$, where

$$F(t,x) = L(t)x + \varpi(t,x),$$

and $L(t) = A(t)+H(t)$. The function $F \in \mathcal{C}(\mathbb{R}_+\times\mathbb{R}^n,\mathbb{R}^n)$ is locally Lipschitz in x, uniformly in t, thus for all $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, there exists a unique solution $x(\cdot)$ of system (4), such that $x(t_0) = x_0$. We prove next that $x(\cdot)$ is defined on $[t_0,\infty)$. Assume that it is not true, that is, there exists $T_{\max} \in (t_0,\infty)$, such that $x(\cdot)$ is defined on $[t_0,T_{\max})$. Then, for all $t \in [t_0,T_{\max})$, we have

$$\|\dot{x}(t)\| \le (N_1+N_2)\|x(t)\| + N_3,$$

where

$$N_1 = \sup_{[t_0, T_{\max}]} \|L(t)\|, \qquad N_2 = \sup_{[t_0, T_{\max}]} |\upsilon(t)|, \qquad N_3 = \sup_{[t_0, T_{\max}]} |\rho(t)|.$$

This yields

$$\left\| \int_{t_0}^{t} \dot{x}(s)\,\mathrm{d}s \right\| \le \int_{t_0}^{t} [(N_1 + N_2)\|x(s)\| + N_3]\,\mathrm{d}s.$$

Thus,

$$\|x(t)\| \le \|x(t_0)\| + \int_{t_0}^{t} [(N_1 + N_2)\|x(s)\| + N_3]\,\mathrm{d}s.$$

Using Lemma 3, we get for all $t \in [t_0, T_{\max})$

$$\|x(t)\| \le \|x(t_0)\| \exp\left(\int_{t_0}^{t} (N_1 + N_2)\,\mathrm{d}s \right) + \exp\left(\int_{t_0}^{t} (N_1 + N_2 + N_3)\,\mathrm{d}s \right) \le N_4,$$

where

$$N_4 = \|x(t_0)\| \exp\left((N_1 + N_2)T_{\max}\right) + \exp\left((N_1 + N_2 + N_3)T_{\max}\right).$$

Consequently, $x(\cdot)$ remains within the compact ball of radius N_4, which contradicts our assumption, that is, if $T_{\max} < \infty$ we should have $\lim_{t\to\infty} \|x(t)\| = \infty$. Therefore, $T_{\max} = \infty$.

(ii) Assume that $x(t)$ is the solution of system (4), then we have

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^{t} \phi(t, s)[H(s)x(s) + \varpi(s, x(s))]\,\mathrm{d}s,$$

where $\phi(t, t_0)$ is the state transition matrix of system (3). Thus, from the global uniform h-stability of system (3), the conditions (7) and (5), we obtain

$$h(t)^{-1}\|x(t)\| \le c\|x_0\|h(t_0)^{-1} + c\int_{t_0}^{t} h(s)^{-1}[\,\mu(s)\|x(s)\| + \upsilon(s)\|x(s)\| + \rho(s)]\,\mathrm{d}s.$$

Put $\varrho(t) = h(t)^{-1}\|x(t)\|$, then we get

$$\varrho(t) \le c\varrho(t_0) + \int_{t_0}^{t} (c[\upsilon(s) + \mu(s)]\varrho(s) + ch(s)^{-1}\rho(s))\,\mathrm{d}s.$$

Yielding by Lemma 3 that for all $t \ge t_0$

$$\begin{aligned}\varrho(t) &\le c\varrho(t_0) \exp\left(\int_{t_0}^{t} c[\upsilon(s) + \mu(s)]\,\mathrm{d}s \right) \\ &\quad + \kappa \exp\left(c\int_{t_0}^{t} [\upsilon(s) + \mu(s)]\,\mathrm{d}s + \frac{c}{\kappa}\int_{t_0}^{t} h(s)^{-1}\rho(s)\,\mathrm{d}s \right) \\ &\le cM_2\varrho(t_0) + \kappa M_2 \exp\left(\frac{c}{\kappa} M_1 \right),\end{aligned}$$

where $\kappa > 0$ and $M_2 = \exp\left(c\int_0^\infty [\upsilon(s) + \mu(s)]\,\mathrm{d}s\right)$. Therefore,

$$\|x(t)\| \leq cM_2\|x_0\|h(t)h(t_0)^{-1} + \kappa M_2\|h\|_\infty \exp\left(\frac{c}{\kappa}M_1\right),$$

with $\|h\|_\infty = \sup_{t\geq 0}\{h(t)\}$. This yields, for all $t \geq t_0$ and all $x_0 \in \mathbb{R}^n$ the solution $x(t)$ satisfies the following estimation:

$$\|x(t)\| \leq \lambda\|x_0\|h(t)h(t_0)^{-1} + r,$$

with

$$\lambda = cM_2,$$
$$r = \kappa M_2\|h\|_\infty \exp\left(\frac{c}{\kappa}M_1\right).$$

Consequently, the system (4) is globally practically uniformly h-stable. □

Remark 2. If we replace the condition (6) by $\rho(s) \leq h(s)\mathrm{e}^{-\beta s}$ for $s \in \mathbb{R}_+$ and $\beta > 0$, we obtain the global practical uniform h-stability of system (4).

A particular case of Theorem 1 can be given in the following corollary.

Corollary 1. *Consider the perturbed system* (4), *where the linear system* (3) *is globally uniformly h-stable, $H(t)$ satisfies the condition* (7) *and $\varpi(t,x)$ verifies the following assumption:*

$$\|\varpi(t,x)\| \leq \rho(t), \qquad \forall\, t \geq 0,$$

where ρ is a nonnegative continuous function on $\mathbb{R}_+$, and the condition (6) *holds. Then, the system* (4) *is globally practically uniformly h-stable.*

In as follows, we will show the global existence and uniqueness of solutions of system (4). Further, we investigate the global practical uniform h-stability of the system under a general condition on the right-hand side of the system via nonlinear generalization of Gronwall's inequality.

We are now in position to present the following result.

Theorem 2. *Consider the time-varying perturbed system* (4), *where:*

1) *The transition matrix for the linear system* (3) *satisfies*

$$\|\phi(t,s)\| \leq ch(t)h(s)^{-1}, \qquad \forall\,(t,s) \in \mathbb{R}_+^2$$

for $c \geq 1$.

2) *There exist υ and ρ nonnegative continuous functions on $\mathbb{R}_+$ verifying*

$$\|\varpi(t,x)\| \leq \upsilon(t)\|x\|^\alpha + \rho(t), \qquad 0 \leq \alpha < 1,\ \forall\, x \in \mathbb{R}^n,\ \forall\, t \geq 0. \tag{8}$$

such that

$$(9)\quad \int_0^t \rho(s)h(s)^{-1}\,\mathrm{d}s \le M_1, \quad \int_0^t \upsilon(s)h(s)^{-1}\,\mathrm{d}s \le M_2, \qquad M_1, M_2 > 0,\ \forall\, t \ge 0.$$

3) *The perturbation H satisfies the condition* (7).

Then, for all $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, the maximal solution $x(t)$ of system (4)*, such that $x(t_0) = x_0$, verifies:*

(i) *The function x is unique and defined on $[t_0, \infty)$.*

(ii) *For all $t \ge t_0$*

$$\|x(t)\| \le \lambda \|x_0\| h(t)h(t_0)^{-1} + r,$$

where λ and r are positive constants.

Proof: (i) The system (4) can be written as $\dot{x} = F(t, x)$, where

$$F(t, x) = L(t)x + \varpi(t, x),$$

and $L(t) = A(t) + H(t)$. The function $F \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is locally Lipschitz in x, uniformly in t, hence for all $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, there exists a unique solution $x(\cdot)$ of system (4), such that $x(t_0) = x_0$. We show next that $x(\cdot)$ is defined on $[t_0, \infty)$. We suppose that it is not true, that is, there exists $T_{\max} \in (t_0, \infty)$, such that $x(\cdot)$ is defined on $[t_0, T_{\max})$. Thus, for all $t \in [t_0, T_{\max})$ we have

$$\|\dot{x}(t)\| \le L_1 \|x(t)\| + L_2 \|x(t)\|^\alpha + L_3,$$

where

$$L_1 = \sup_{[t_0, T_{\max}]} \|L(t)\|, \qquad L_2 = \sup_{[t_0, T_{\max}]} |\upsilon(t)|, \qquad L_3 = \sup_{[t_0, T_{\max}]} |\rho(t)|.$$

Hence,

$$\left\| \int_{t_0}^t \dot{x}(s)\,\mathrm{d}s \right\| \le \int_{t_0}^t [L_1 \|x(s)\| + L_2 \|x(t)\|^\alpha + L_3]\,\mathrm{d}s.$$

Therefore,

$$\|x(t)\| \le \|x(t_0)\| + \int_{t_0}^t [L_1 \|x(s)\| + L_2 \|x(t)\|^\alpha + L_3]\,\mathrm{d}s.$$

Using Lemma 2 and the inequality $(a+b)^p \le 2^{p-1}(a^p + b^p)$ for all $a, b \ge 0$ and $p \ge 1$, we obtain for all $t \in [t_0, T_{\max})$

$$\|x(t)\| \le 2^{\alpha/(1-\alpha)} \big(\|x(t_0)\| + L_3 T_{\max}\big) \mathrm{e}^{L_1 T_{\max}} + 2^{\alpha/(1-\alpha)} \big((1-\alpha) L_2 T_{\max}\big)^{1/(1-\alpha)} \mathrm{e}^{L_1 T_{\max}} \le L_4,$$

where

$$L_4 = 2^{\alpha/(1-\alpha)} \big(\|x(t_0)\| + L_3 T_{\max}\big) \mathrm{e}^{L_1 T_{\max}} + 2^{\alpha/(1-\alpha)} \big((1-\alpha) L_2 T_{\max}\big)^{1/(1-\alpha)} \mathrm{e}^{L_1 T_{\max}}.$$

Consequently, $x(\cdot)$ remains within the compact ball of radius L_4, which contradicts our assumption, that is, if $T_{\max} < \infty$ we should have $\lim_{t\to\infty} \|x(t)\| = \infty$. Therefore, $T_{\max} = \infty$.

(ii) Suppose that $x(t)$ is the solution of system (4), then we have

$$x(t) = \phi(t,t_0)x_0 + \int_{t_0}^{t} \phi(t,s)[H(s)x(s) + \varpi(s,x(s))]\,\mathrm{d}s,$$

where $\phi(t,t_0)$ is the state transition matrix of system (3). Thus, from the global uniform h-stability of system (3), the conditions (7) and (8), we obtain

$$\|x(t)\| \le c\|x_0\|h(t)h(t_0)^{-1} + ch(t)\int_{t_0}^{t} \mu(s)h(s)^{-1}\|x(s)\|\,\mathrm{d}s \\ + ch(t)\int_{t_0}^{t} \upsilon(s)h(s)^{-1}\|x(s)\|^{\alpha}\,\mathrm{d}s + ch(t)\int_{t_0}^{t} h(s)^{-1}\rho(s)\,\mathrm{d}s.$$

Hence,

$$h(t)^{-1}\|x(t)\| \le (c\|x_0\|h(t_0)^{-1} + cM_1) + c\int_{t_0}^{t} [\,\mu(s)h(s)^{-1}\|x(s)\| \\ + \upsilon(s)h(s)^{\alpha-1}(h(s)^{-1}\|x(s)\|)^{\alpha}]\,\mathrm{d}s.$$

Let $\varrho(t) = h(t)^{-1}\|x(t)\|$, then

$$\varrho(t) \le (c\varrho(t_0) + cM_1) + c\int_{t_0}^{t} [\,\mu(s)\varrho(s) + \upsilon(s)h(s)^{\alpha-1}\varrho^{\alpha}(s)]\,\mathrm{d}s.$$

By Lemma 2, we get

$$\varrho(t) \le \left[\big(c\varrho(t_0) + cM_1\big)^{1-\alpha}\exp(cM_3(1-\alpha)) \\ + cM_2(1-\alpha)\|h\|_{\infty}^{\alpha}\exp(cM_3(1-\alpha))\right]^{1/(1-\alpha)},$$

with $M_3 = \int_0^{\infty}\mu(s)\,\mathrm{d}s$ and $\|h\|_{\infty} = \sup_{t\ge 0}\{h(t)\}$. Using the inequality $(a+b)^p \le 2^{p-1}(a^p + b^p)$ for all $a, b \ge 0$ and $p \ge 1$, we obtain

$$\varrho(t) \le 2^{\alpha/(1-\alpha)}(c\varrho(t_0) + cM_1)\mathrm{e}^{cM_3} + 2^{\alpha/(1-\alpha)}\big(cM_2(1-\alpha)\|h\|_{\infty}^{\alpha}\big)^{1/(1-\alpha)}\mathrm{e}^{cM_3}.$$

Then,

$$\|x(t)\| \le 2^{\alpha/(1-\alpha)}c\,\mathrm{e}^{cM_3}\|x_0\|h(t)h(t_0)^{-1} + 2^{\alpha/(1-\alpha)}cM_1\|h\|_{\infty}\mathrm{e}^{cM_3} \\ + 2^{\alpha/(1-\alpha)}\big(cM_2(1-\alpha)\|h\|_{\infty}^{\alpha}\big)^{1/(1-\alpha)}\mathrm{e}^{cM_3}\|h\|_{\infty}.$$

This yields, for all $t \ge t_0$ and all $x_0 \in \mathbb{R}^n$ the solution $x(t)$ satisfies:

$$\|x(t)\| \le \lambda\|x_0\|h(t)h(t_0)^{-1} + r,$$

with

$$\lambda = 2^{\alpha/(1-\alpha)} c\, \mathrm{e}^{cM_3},$$
$$r = 2^{\alpha/(1-\alpha)} cM_1 \|h\|_\infty \mathrm{e}^{cM_3} + 2^{\alpha/(1-\alpha)} \big(cM_2(1-\alpha)\|h\|_\infty\big)^{1/(1-\alpha)} \mathrm{e}^{cM_3}.$$

Consequently, the system (4) is globally practically uniformly h-stable. □

Remark 3. If we take $\upsilon(s) \le h(s)^{1-\alpha} \mathrm{e}^{-\beta s}$ for $s \in \mathbb{R}_+$, $\beta > 0$ and $0 \le \alpha < 1$, we get the global practical uniform h-stability of system (4).

Remark 4. Theorems 1 and 2 hold for practical uniform exponential stability $h(t) = \mathrm{e}^{-\beta t}$, practical polynomial stability $h(t) = 1/(1+t)^\gamma$ and for other functions like $h(t) = 1/(1+\ln(1+t))$, $h(t) = \mathrm{e}^{-\beta t}/(1+t)^\gamma$, $h(t) = (2+t)/(1+t)$, $h(t) = \mathrm{e}^{-\beta t^2}$ for all $t \in \mathbb{R}_+$ and $\gamma, \beta > 0$.

A direct consequence of Theorem 2 can be formulated as follows.

Corollary 2. *Consider the perturbed system* (4) *with* $H(t)$ *satisfying the condition* (7) *and we assume the following assumption on the perturbation* $\varpi(t,x)$:

$$\|\varpi(t,x)\| \le \upsilon(t)\|x\|^{1/2} + \rho(t), \qquad \forall\, x \in \mathbb{R}^n,\ \forall\, t \ge 0, \tag{10}$$

where υ *and* ρ *are nonnegative continuous functions on* $\mathbb{R}_+$ *verifying* (9). *Then, the system* (4) *is globally practically uniformly* h*-stable.*

5. Application to control systems

We consider the control system:

$$\dot{x} = A(t)x + B(t)u + H(t)x + \varpi(t,x), \tag{11}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input of the system, $A(t), H(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times m}$ are matrices whose elements are continuous functions on $\mathbb{R}_+$. The function $\varpi \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is locally Lipschitz in x, uniformly in t. The matrix H and the function ϖ represent the perturbations terms of the system. We suppose that there exists a gain feedback matrix $K(t) \in \mathbb{R}^{m\times n}$ which provides a stabilizing feedback law $u(t,x) = K(t)x$ for the nominal system $\dot{x} = (A(t) + B(t)K(t))x$. Then, system (11) becomes,

$$\dot{x} = A_K(t)x + H(t)x + \varpi(t,x), \tag{12}$$

where $A_K(t) = A(t) + B(t)K(t)$. The solution of system (12) is given by:

$$x(t) = \phi_{A_K}(t,t_0)x(t_0) + \int_{t_0}^{t} \phi_{A_K}(t,s)\big(H(s)x(s) + \varpi(s,x(s))\big)\,\mathrm{d}s.$$

Then, we have

$$\|x(t)\| \le c\|x(t_0)\|h(t)h(t_0)^{-1} + c\int_{t_0}^{t} h(t)h(s)^{-1}\big(H(s)x(s) + \varpi(s, x(s))\big)\,\mathrm{d}s.$$

Now, we consider system (11) in $2D$, $x = (x_1, x_2) \in \mathbb{R}^2$ is the state of the system and

$$A(t) = \begin{pmatrix} \frac{-t}{1+t^2} & -t^2 \\ \frac{-t}{(1+t^2)^2} & \frac{-t}{1+t^2} \end{pmatrix}, \qquad B(t) = \begin{pmatrix} t \\ \frac{-t}{1+t^2} \end{pmatrix},$$

$$H(t) = \begin{pmatrix} \cos(t)\,\mathrm{e}^{-t} & 0 \\ 0 & \cos(t)\,\mathrm{e}^{-t} \end{pmatrix}$$

and

$$\varpi(t, x) = \begin{pmatrix} \frac{1}{(1+t^2)^2}\frac{x_1^2}{1+\sqrt{x_1^2+x_2^2}} + \frac{\mathrm{e}^{-t}}{(1+t^2)(1+x_1^2)} \\ \frac{t}{(1+t^2)^2}\frac{x_2^2}{1+\sqrt{x_1^2+x_2^2}} \end{pmatrix}.$$

Let $K(t) = (-t/(1+t^2), t)$. Thus,

$$A(t) + B(t)K(t) = \begin{pmatrix} \frac{-2t}{1+t^2} & 0 \\ 0 & \frac{-2t}{1+t^2} \end{pmatrix}.$$

The linear system $\dot{x} = A_K(t)x$ is globally uniformly h-stable with $h(t) = 1/(1+t^2)$, which is positive, continuous and bounded on $\mathbb{R}_+$. Moreover, we have $\|H(t)\| \le \mathrm{e}^{-t} = \gamma(t)$ which is nonnegative, continuous and integrable on $\mathbb{R}_+$ and

$$\|\varpi(t, x)\| \le \frac{1}{(1+t^2)^3}(x_1^2 + x_2^2)^2 + \frac{2\,\mathrm{e}^{-2t}}{(1+t^2)^2}.$$

Put, $v(t) = 1/(1+t^2)^{3/2}$ and $\rho(t) = \sqrt{2}\,\mathrm{e}^{-t}/(1+t^2)$, where v and ρ are nonnegative continuous and integrable on $\mathbb{R}_+$. In addition, condition (6) holds. Therefore, all assumptions of Theorem 1 are satisfied and system (11) is globally practically uniformly $1/(1+t^2)$-stable.

Figure 1 shows the time evolution of the states $x(t)$ of the system (11) with the initial states $\big(x_1(0), x_2(0)\big) = (0.2, -0.3)$.

Consider now the dynamical system (11), where the perturbation is modelled by:

$$\varpi(t, x) = \frac{\sqrt{\|x\|}\,\mathrm{e}^{-t}}{1+t^2}(t, 1) + \frac{1}{(1+t^2)^2}(1, 1).$$

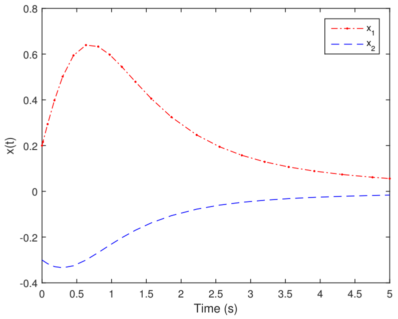

FIGURE 1. Time evolution of the states (x_1, x_2) of system (11). The parameters used are $t_0 = 0$ and $\big(x_1(0), x_2(0)\big) = (0.2, -0.3)$.

Then, we have

$$\begin{aligned} \|\varpi(t,x)\|^2 &= \Big(\frac{\sqrt{\|x\|}t\,\mathrm{e}^{-t}}{1+t^2} + \frac{1}{(1+t^2)^2}\Big)^2 + \Big(\frac{\sqrt{\|x\|}\,\mathrm{e}^{-t}}{1+t^2} + \frac{1}{(1+t^2)^2}\Big)^2 \\ &\leq \frac{2\|x\|\,\mathrm{e}^{-2t}}{(1+t^2)^3} + \frac{4}{(1+t^2)^4}. \end{aligned}$$

Put, $v(t) = \sqrt{2}\,\mathrm{e}^{-t}/(1+t^2)^{3/2}$ and $\rho(t) = 2/(1+t^2)^2$, where v and ρ are nonnegative continuous functions on $\mathbb{R}_+$ which verifies the condition (9). Consequently, Corollary 2 asserts the global practical uniform h-stability of system (11), where $h(t) = 1/(1+t^2)$.

Figure 2 shows the time evolution of the states $x(t)$ of the system (11) with the initial states $\big(x_1(0), x_2(0)\big) = (-0.2, 0.5)$.

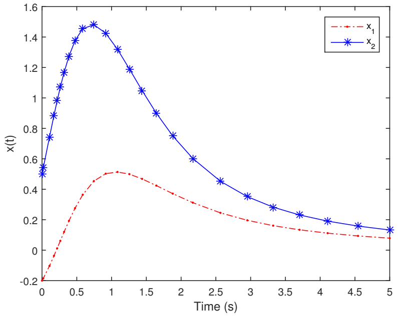

FIGURE 2. The time evolution of the state (x_1, x_2) of system (11). The parameters used are $t_0 = 0$ and $\big(x_1(0), x_2(0)\big) = (-0.2, 0.5)$.

6. Examples and simulations

In this section, we introduce numerical examples to illustrate the applicability of the obtained results. The first one is related to Theorem 1. The second illustrates Corollary 2. All illustrations have been performed with the software Matlab.

Example 1. We consider the first order problem:

$$\dot{x} = -\frac{x\sqrt{t}}{(1+t^{3/2})} + \frac{t \arctan t}{(1+t^2)^3}x + \frac{x}{(2+t)^2} + 2t\,\mathrm{e}^{-t}, \qquad x \in \mathbb{R},\ t \geq 0, \tag{13}$$

which can be written as

$$\dot{x} = A(t)x + H(t)x + \varpi(t,x), \qquad t \geq 0,$$

with

$$A(t) = -\frac{\sqrt{t}}{(1+t^{3/2})}, \qquad H(t) = \frac{t \arctan t}{(1+t^2)^3} \quad \text{and} \quad \varpi(t,x) = \frac{x}{(2+t)^2} + 2t\,\mathrm{e}^{-t}.$$

The solution of the linear time-varying system $\dot{x} = A(t)x$ is given by:

$$x(t) = x(t_0)\Big(\frac{1+t_0^{3/2}}{1+t^{3/2}}\Big)^{2/3}.$$

Then,

$$|x(t)| \leq c|x(t_0)|h(t)h(t_0)^{-1}, \qquad \forall\, t \geq 0,$$

where $c = 1$ and $h(t) = (1/(1+t^{3/2}))^{2/3}$ which is positive, bounded and continuous on $\mathbb{R}_+$. Then, the linear system is globally uniformly h-stable. Moreover, the perturbation $\varpi(t,x)$ satisfies the condition (5) with $\upsilon(t) = 1/(2+t)^2$ and $\rho(t) = 2t\,\mathrm{e}^{-t}$ being nonnegative and continuous functions on $\mathbb{R}_+$, where $\upsilon \in L^1(\mathbb{R}_+)$ and the condition (6) holds. Furthermore, for all $t \geq 0$

$$|H(t)| \leq \frac{\pi t}{2(2+t^2)^3} = \mu(t),$$

such that μ is a nonnegative continuous and integrable function on $\mathbb{R}_+$. This yields, by applying Theorem 1, the global practical uniform h-stability of system (13).

In Figure 3, it can be seen that the trajectory of the system approach to a sufficiently close neighborhood of the origin with the initial state $x(0) = 0$.

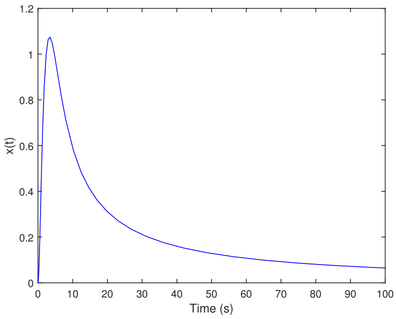

FIGURE 3. Time evolution of the state $x(t)$ of system (13). The parameters used are $t_0 = 0$ and $x(0) = 0$.

Example 2. Consider the following second order problem:

$$\begin{aligned} \dot{x}_1 &= -2tx_1 + 2x_2 + \frac{t}{2}\,\mathrm{e}^{-2t}x_1 + \frac{\mathrm{e}^{-t^2}}{\sqrt{2}(1+t^2)^2}(x_1^2+x_2^2)^{1/4} + \frac{t\,\mathrm{e}^{-2t^2}}{\sqrt{2(1+x_2^2)}}, \\ \dot{x}_2 &= -2x_1 - 2tx_2 + \frac{t\,\mathrm{e}^{-t^2}}{\sqrt{2}(1+t^2)^2}(x_1^2+x_2^2)^{1/4}, \qquad t \geq 0, \end{aligned} \tag{14}$$

where $x \in \mathbb{R}^2$, then the problem (14) can be written as

$$\dot{x} = A(t)x + H(t)x + \varpi(t,x), \qquad t \geq 0,$$

with

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad A(t) = \begin{pmatrix} -2t & 2 \\ -2 & -2t \end{pmatrix}, \qquad H(t) = \begin{pmatrix} \frac{t}{2}\,\mathrm{e}^{-2t} & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\varpi(t,x) = \begin{pmatrix} \varpi_1(t,x) \\ \varpi_2(t,x) \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{e}^{-t^2}}{\sqrt{2}(1+t^2)^2}(x_1^2+x_2^2)^{1/4} + \frac{t\,\mathrm{e}^{-2t^2}}{\sqrt{2(1+x_2^2)}} \\ \frac{t\,\mathrm{e}^{-t^2}}{\sqrt{2}(1+t^2)^2}(x_1^2+x_2^2)^{1/4} \end{pmatrix}.$$

The state transition matrix $\phi(t,t_0)$ of the linear system

$$\begin{aligned} \dot{x}_1 &= -2tx_1 + 2x_2, \\ \dot{x}_2 &= -2x_1 - 2tx_2, \end{aligned} \tag{15}$$

is given by:

$$\phi(t,t_0) = \mathrm{e}^{-(t^2-t_0^2)}\, K(t-t_0),$$

with

$$K(t) = \begin{pmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{pmatrix}.$$

Then, we have

$$\|\phi(t,t_0)\| \leq \mathrm{e}^{-(t^2-t_0^2)}, \qquad \forall\, t \geq 0.$$

Thus, the system (15) is globally uniformly h-stable with $h(t) = \mathrm{e}^{-t^2}$. On the other hand, we have

$$\|\varpi(t,x)\| \leq \frac{\mathrm{e}^{-t^2}}{(1+t^2)^{3/2}}\|x\|^{1/2} + t\,\mathrm{e}^{-2t^2}, \qquad \forall\, x \in \mathbb{R}^2,\ \forall\, t \geq 0.$$

Put, $\upsilon(t) = \mathrm{e}^{-t^2}/(1+t^2)^{3/2}$ and $\rho(t) = t\,\mathrm{e}^{-2t^2}$, that are nonnegative continuous functions on $\mathbb{R}_+$ and the condition (9) holds. Moreover, for all $t \in \mathbb{R}_+$ one has

$$\|H(t)\| = \frac{t}{2}\,\mathrm{e}^{-2t}.$$

Let $\mu(t) = (t/2)\,\mathrm{e}^{-2t}$ that is nonnegative continuous and integrable on $\mathbb{R}_+$. Consequently, by applying Corollary 2 we conclude the global practical uniform h-stability of system (14).

In Figure 4, it can be seen that the trajectories of system (14) practically converge with the initial state $\big(x_1(0), x_2(0)\big) = (2,2)$.

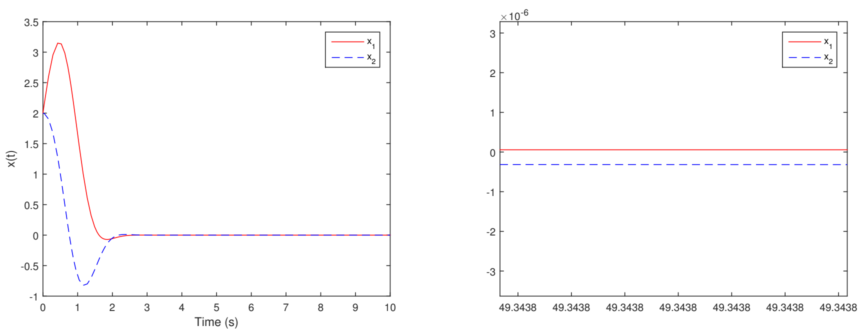

FIGURE 4. Time evolution of the state (x_1, x_2) of system (14). The parameters used are $t_0 = 0$ and $\big(x_1(0), x_2(0)\big) = (2,2)$.

Conclusion

In this study, we have presented some sufficient conditions to ensure the global existence of solutions and the practical uniform h-stability of a class of time-varying perturbed systems based on Gronwall type integral inequalities. This results can be viewed as a generalization of the practical exponential stability.

Some examples and simulations results, as well as an application to control system are given to illustrate the applicability of the main results.

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