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Monadic quasi-modal distributive nearlattices

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Abstract. We prove that there is a one to one correspondence between monadic finite quasi-modal operators on a distributive nearlattice and quantifiers on the distributive lattice of its finitely generated filters, extending the results given in "Calomino I., Celani S., González L. J.: Quasi-modal operators on distributive nearlattices, Rev. Unión Mat. Argent. 61 (2020), 339–352".

Keywords: distributive nearlattice; modal operator; filter

Classification: 06A12, 06D75, 03G25

1. Introduction and preliminaries

A modal algebra is pair $\langle B, \Box\rangle$ such that B is a Boolean algebra and $\Box\colon B \to B$ a map such that $\Box 1 = 1$ and $\Box(a \wedge b) = \Box a \wedge \Box b$ for all $a, b \in B$. It is well know that the variety of modal algebras is the algebraic semantic of classical normal modal logics. A generalization of the notion of modal operator in a Boolean algebra B was studied in [4] where the author introduces a map that sends each element $a \in B$ to an ideal I of B. This type of maps are not operations in the sense of universal algebra, but have some similar properties to modal operators.

The class of distributive nearlattices are a natural generalization of semi-boolean algebras, in the sense of Abbott, see [1], and also of bounded distributive lattices. Several authors have studied these structures from an algebraic, see [12], [19], [8], [9], [10], [2], [15], [3], [16], topological, see [5], [6], [7], and logical, see [13], [14], point of view. In particular, a notion of necessity modal operator on distributive nearlattices was studied in [7]. Later, inspired by [4], in [3] was studied a class of operators on a distributive nearlattice, called *finite quasi-modal operators*, which are in one to one correspondence with possibility modal operators on the distributive lattice of its finitely generated filters. The finite quasi-modal operators are a generalization of the necessity modal operators given in [7]. Following the results given in [3], the main aim of this paper is to prove that there is a one to one correspondence between monadic finite quasi-modal operators on

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a distributive nearlattice and quantifiers on the distributive lattice of its finitely generated filters. Also, the concept of qm-subnearlattice is introduced in the class of quasi-modal distributive nearlattices as a generalization of the $\Box$-subalgebras given in [7].

Let $\mathbf{A} = \langle A, \vee, 1\rangle$ be a join-semilattice with greatest element. A subset U of A is said to be *upper* (*lower*) if for every $x, y \in U$ such that $x \in U$ ($y \in U$) and $x \leq y$, then $y \in U$ ($x \in U$). For each $X \subseteq A$, the upper (lower) set generated by X is $[X) = \{a \in A \colon \exists x \in X (x \leq a)\}$ ($(X] = \{a \in A \colon \exists x \in X (a \leq x)\}$). If $X = \{a\}$, then we will write $[a)$ and $(a]$ instead of $[\{a\})$ and $(\{a\}]$, respectively. A *filter* is a subset F of A such that $1 \in F$, F is upper and if $a, b \in F$, then $a \wedge b \in F$, whenever $a \wedge b$ exists. If X is a subset of A, the least filter containing X is called the *filter generated by* X and will be denoted by $\mathrm{Fig}(X)$. A filter G is said to be *finitely generated* if $G = \mathrm{Fig}(X)$ for some finite subset X of A. If $X = \{a\}$, then $\mathrm{Fig}(\{a\}) = [a) = \{x \in A \colon a \leq x\}$, called the *principal filter of* a. We denote by $\mathrm{Fi}(A)$ and $\mathrm{Fi_f}(A)$ the set of all filters and finitely generated filters of $\mathbf{A}$, respectively. A nonempty subset I of A is called an *ideal* if I is lower and if $a, b \in I$, then $a \vee b \in I$. If X is a subset of A, the least ideal containing X is called the *ideal generated by* X and will be denoted by $\mathrm{Idg}(X)$. Then we have the following characterization of the ideal generated by a subset X of A:

$$\mathrm{Idg}(X) = \{a \in A \colon \exists x_1, \ldots, x_n \in X (a \leq x_1 \vee \ldots \vee x_n)\}.$$

We shall say that a proper ideal P is *prime* if for all $a, b \in A$, $a \wedge b \in P$ implies $a \in P$ or $b \in P$, whenever $a \wedge b$ exists. Denote by $\mathrm{Id}(A)$ and $\mathrm{X}(A)$ the set of all ideals and prime ideals of $\mathbf{A}$, respectively.

In the rest of this section we recall some concepts about distributive nearlattices and quasi-modal operators. The reader is referred to [12], [8], [9], [10], [7], [3].

1.1 Distributive nearlattices.

Definition 1. Let $\mathbf{A}$ be a join-semilattice with greatest element. Then $\mathbf{A}$ is a *distributive nearlattice* if for each $a \in A$, the principal filter $[a)$ is a bounded distributive lattice with respect to the induced order.

Let $\mathbf{A}$ be a distributive nearlattice. For each $a \in A$, the meet operation of the lattice $[a)$ is well defined and is denoted by "$\wedge_a$". Thus, the structure $\langle [a), \vee, \wedge_a, a, 1\rangle$ is a bounded distributive lattice. It should be noted that for all $x, y \in A$, the meet $x \wedge y$ exists in A if and only if x, y have a common lower bound in A. Thus, for all $x, y \in [a)$, the meet of x, y in $[a)$ coincides with their meet in A, that is, $x \wedge_a y = x \wedge y$. This should be kept in mind since we will use it without mention. We can define a ternary operation $m \colon A^3 \to A$ given by $m(x, y, z) = (x \vee z) \wedge (y \vee z)$. The operation m is very useful and characterize the class of distributive nearlattices, for more details see [19], [10], [2]. We introduce the following notation: for each natural number n we define inductively for every $a_1, \ldots, a_n, b \in A$ the element $m^{n-1}(a_1, \ldots, a_n, b)$ as follows:

- $m^0(a_1, b) = m(a_1, a_1, b)$,
- for $n > 1$, $m^{n-1}(a_1, \ldots, a_n, b) = m(m^{n-2}(a_1, \ldots, a_{n-1}, b), a_n, b)$.

Then $m^{n-1}(a_1, \ldots, a_n, b) = (a_1 \vee b) \wedge \ldots \wedge (a_n \vee b)$. In particular, $m^0(a_1, b) = a_1 \vee b$ and $m^1(a_1, a_2, b) = m(a_1, a_2, b)$.

On the other hand, if $\mathbf{A}$ is a distributive nearlattice, then by results given in [12] the structure $\mathrm{Fi}(\mathbf{A}) = \langle \mathrm{Fi}(A), \veebar, \barwedge, \{1\}, A \rangle$ is a bounded distributive lattice, where the least element is $\{1\}$, the greatest element is A, $G \barwedge H = G \cap H$, and $G \veebar H = \mathrm{Fig}(G \cup H)$ for every $G, H \in \mathrm{Fi}(A)$. We have the following characterization of the filter generated by a subset X of A:

$$\mathrm{Fig}(X)^1 = \{a \in A\colon \exists x_1, \ldots, x_n \in [X)(a = x_1 \wedge \ldots \wedge x_n)\}.$$

If $X = \{a_1, \ldots, a_n\}$, then

$$\mathrm{Fig}(X) = [\,a_1) \veebar \ldots \veebar [\,a_n) = \{a \in A\colon\ a = m^{n-1}(a_1, \ldots, a_n, a)\}.$$

Moreover, $\mathrm{Fi_f}(\mathbf{A}) = \langle \mathrm{Fi_f}(A), \veebar, \barwedge, \{1\}, A \rangle$ is a bounded distributive lattice.

Theorem 2 ([17], [9])**.** *Let $\mathbf{A}$ be a distributive nearlattice. Let $I \in \mathrm{Id}(A)$ and $F \in \mathrm{Fi}(A)$ be such that $I \cap F = \emptyset$. Then there exists $P \in \mathrm{X}(A)$ such that $I \subseteq P$ and $P \cap F = \emptyset$.*

1.2 Quasi-modal operators.

Definition 3. Let $\mathbf{A}$ be a distributive nearlattice. A *quasi-modal operator* defined on $\mathbf{A}$ is a map $\nabla\colon A \to \mathrm{Fi}(A)$ such that:

(1) $\nabla 1 = \{1\}$,
(2) $\nabla(a \wedge b) = \nabla a \veebar \nabla b$, whenever $a \wedge b$ exists.

A *finite quasi-modal operator* defined on $\mathbf{A}$ is a quasi-modal operator such that $\nabla a \in \mathrm{Fi_f}(A)$ for every $a \in A$. A pair $\langle \mathbf{A}, \nabla \rangle$ is a *quasi-modal distributive nearlattice*, or *qm-distributive nearlattice* for short, if $\mathbf{A}$ is a distributive nearlattice and ∇ is a quasi-modal operator on $\mathbf{A}$. Analogously, a pair $\langle \mathbf{A}, \nabla \rangle$ is a *finite quasi-modal distributive nearlattice*, or *fqm-distributive nearlattice* for short, if $\mathbf{A}$ is a distributive nearlattice and ∇ is a finite quasi-modal operator on $\mathbf{A}$.

Remark 4. It is easy to prove that the condition (2) of Definition 3 is equivalent to the equation $\nabla m(a, b, c) = \nabla(a \vee c) \veebar \nabla(b \vee c)$ for every $a, b, c \in A$.

[1] Note that in the class of distributive nearlattices it is also valid $\mathrm{Fig}(X) = \{a \in A : \exists x_1, \ldots, x_n \in [X)(a \geq x_1 \wedge \ldots \wedge x_n)\}$. However, in this paper we will work with the equality, following the line of research proposed in [13], [14], [3].

Example 5. A *necessity modal operator* on a distributive nearlattice $\mathbf{A}$ is a monotone map $\Box\colon A \to A$ such that $\Box 1 = 1$ and $\Box(a \wedge b) = \Box a \wedge \Box b$, whenever $a \wedge b$ exists, see [7]. If for each $a \in A$ we put $\nabla_{\Box}(a) = [\Box a)$, then $\Box$ induces a finite quasi-modal operator $\nabla_{\Box}$. Conversely, if $\langle \mathbf{A}, \nabla \rangle$ is a fqm-distributive nearlattice such that for each $a \in A$ the filter ∇a is principal, then the map $\Box_{\nabla}\colon A \to A$ given by $\Box_{\nabla}(a) = b$ if and only if $\nabla a = [b)$ defines a necessity modal operator on $\mathbf{A}$. Thus, finite quasi-modal operators are a natural generalization of necessity modal operators.

Example 6. Let $\mathbf{A}$ be a distributive nearlattice. We consider the map $\nabla\colon A \to \mathrm{Fi}(A)$ given by

$$\nabla a = \begin{cases} \{1\} & \text{if } a = 1, \\ \underline{\bigvee}\{F \in \mathrm{Fi}(A)\colon a \notin F\} & \text{if } a < 1. \end{cases}$$

Let $a, b \in A$ be such that $a \wedge b$ exists and $F \in \mathrm{Fi}(A)$. Note that $a \wedge b \notin F$ if and only if $a \notin F$ or $b \notin F$. Then

$$\begin{aligned} \nabla(a \wedge b) &= \underline{\bigvee}\{F \in \mathrm{Fi}(A)\colon a \wedge b \notin F\} \\ &= \underline{\bigvee}\{F \in \mathrm{Fi}(A)\colon a \notin F \text{ or } b \notin F\} = \nabla a \veebar \nabla b. \end{aligned}$$

Hence, $\langle \mathbf{A}, \nabla \rangle$ is a qm-distributive nearlattice.

Let $\langle \mathbf{A}, \nabla \rangle$ be a qm-distributive nearlattice and $D \subseteq A$. Consider the set

$$\gamma(D) = \{a \in A\colon \nabla a \cap D = \emptyset\}$$

and the binary relation $R_{\nabla} \subseteq \mathrm{X}(A) \times \mathrm{X}(A)$ given by

$$(P, Q) \in R_{\nabla} \iff \gamma(P) \cap Q = \emptyset.$$

It is easy to check that $\subseteq^{-1} \circ R_{\nabla} \subseteq R_{\nabla}$.

Theorem 7 ([3])**.** *Let $\mathbf{A}$ be a distributive nearlattice and $\nabla\colon A \to \mathrm{Fi}(A)$ a map. Then the following conditions are equivalent:*

(1) *∇ is a quasi-modal operator on $\mathbf{A}$,*
(2) *∇ inverts the order and $\gamma(P) \in \mathrm{Fi}(A)$ for every $P \in \mathrm{X}(A)$.*

Proposition 8 ([3])**.** *Let $\langle \mathbf{A}, \nabla \rangle$ be a qm-distributive nearlattice. Let $a \in A$ and $P \in \mathrm{X}(A)$. Then $\nabla a \cap P \neq \emptyset$ if and only if there exists $Q \in \mathrm{X}(A)$ such that $\gamma(P) \cap Q = \emptyset$ and $a \in Q$.*

A possibility modal operator on a bounded distributive lattice $\mathbf{L} = \langle L, \vee, \wedge, 0, 1 \rangle$ is a map $\Diamond\colon L \to L$ such that $\Diamond 0 = 0$ and $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$ for every $a, b \in L$.

Definition 9. Let $\langle \mathbf{A}, \nabla \rangle$ be a qm-distributive nearlattice. For a subset $X \subseteq A$, we define

$$(\bullet) \qquad \Diamond_\nabla(X) = \mathrm{Fig}\Big(\bigcup\{\nabla x \colon x \in X\}\Big).$$

Remark 10. Note that $\Diamond_\nabla([a)) = \nabla a$ for every $a \in A$.

In the following result we show the connection between finite quasi-modal operators on a distributive nearlattice $\mathbf{A}$ and possibility modal operators on the bounded distributive lattice $\mathrm{Fi_f}(\mathbf{A})$.

Theorem 11 ([3])**.** *Let $\mathbf{A}$ be a distributive nearlattice.*

(1) *If $\nabla \colon A \to \mathrm{Fi}(A)$ is a finite quasi-modal operator on $\mathbf{A}$, then the map $\Diamond_\nabla \colon \mathrm{Fi_f}(A) \to \mathrm{Fi_f}(A)$ given by $(\bullet)$ is a possibility modal operator on $\mathrm{Fi_f}(\mathbf{A})$, i.e., $\Diamond_\nabla(\{1\}) = \{1\}$ and $\Diamond_\nabla(F \veebar G) = \Diamond_\nabla(F) \veebar \Diamond_\nabla(G)$ for every $F, G \in \mathrm{Fi_f}(A)$.*
(2) *If $\Diamond \colon \mathrm{Fi_f}(A) \to \mathrm{Fi_f}(A)$ is a possibility modal operator on $\mathrm{Fi_f}(\mathbf{A})$, then the map $\nabla_\Diamond \colon A \to \mathrm{Fi_f}(A)$ given by $\nabla_\Diamond a = \Diamond([a))$ is a finite quasi-modal operator on $\mathbf{A}$.*

If ∇ is a finite quasi-modal operator on $\mathbf{A}$, then $\nabla = \nabla_{\Diamond_\nabla}$. Analogously, if $\Diamond$ is a possibility modal operator on $\mathrm{Fi_f}(\mathbf{A})$, then $\Diamond = \Diamond_{\nabla_\Diamond}$. Moreover, there is a one to one correspondence between finite quasi-modal operators on $\mathbf{A}$ and possibility modal operators on $\mathrm{Fi_f}(\mathbf{A})$.

1.3 Qm-subnearlattices. Let $\mathbf{A}$ be a distributive nearlattice. We say that a structure $\mathbf{B} = \langle B, \vee, 1 \rangle$ is a *subnearlattice* of $\mathbf{A}$ if B is a subset of A, B is closed under the operation "$\vee$", $1 \in B$ and if $a, b \in B$ are such that if $a \wedge b$ exists in A then $a \wedge b \in B$. It follows that subnearlattices are equivalent to structures $\langle B, m, 1 \rangle$ such that B is a subset of A, $1 \in B$ and $m(a, b, c) \in B$ for every $a, b, c \in B$.

Now we introduce the notion of *qm-subnearlattice* in the class of quasi-modal distributive nearlattices. In what follows to distinguish about the algebra we are working on, we are going to use subscripts.

Remark 12. Note that if $\mathbf{A}$ is a distributive nearlattice and $\mathbf{B}$ is a subnearlattice of $\mathbf{A}$, then for each $Q \in \mathrm{X}(A)$ we have $Q \cap B \in \mathrm{X}(B) \cup \{\emptyset\}$.

Proposition 13. *Let $\mathbf{A}$ be a distributive nearlattice and $\mathbf{B}$ be a subnearlattice of $\mathbf{A}$. Then for each $P \in \mathrm{X}(B)$ there exists $Q \in \mathrm{X}(A)$ such that $P = Q \cap B$.*

Proof: Let $P \in \mathrm{X}(B)$. Then $B - P \in \mathrm{Fi}(B)$ and we consider the ideal $\mathrm{Idg}_A(P)$ generated by P in A. On the other hand, since $B - P$ is closed under existing meets, we take the filter $\mathrm{Fig}_A(B - P)$ generated by $B - P$ in A. Thus we have

$\mathrm{Idg}_A(P) \cap \mathrm{Fig}_A(B-P) = \emptyset$ and by Theorem 2 there exists $Q \in \mathrm{X}(A)$ such that $\mathrm{Idg}_A(P) \subseteq Q$ and $Q \cap \mathrm{Fig}_A(B-P) = \emptyset$. It follows that $P = Q \cap B$. □

Remark 14. It is easy to see that if $\mathbf{A}$ is a distributive nearlattice and $\mathbf{B}$ is a subnearlattice of $\mathbf{A}$, then $\mathrm{Fig}_B(X) = \mathrm{Fig}_A(X) \cap B$ for every $X \subseteq B$.

Definition 15. Let $\langle \mathbf{A}, \nabla_A \rangle, \langle \mathbf{B}, \nabla_B \rangle$ be two qm-distributive nearlattices. We say that the structure $\langle \mathbf{B}, \nabla_B \rangle$ is a *qm-subnearlattice* of $\langle \mathbf{A}, \nabla_A \rangle$ if $\mathbf{B}$ is a subnearlattice of $\mathbf{A}$, and for each $b \in B$ we have

$$\mathrm{Fig}_A(\nabla_B(b)) = \nabla_A(b).$$

Theorem 16. *Let $\langle \mathbf{A}, \nabla_A \rangle$ be a qm-distributive nearlattice and $\mathbf{B}$ be a subnearlattice of $\mathbf{A}$. Then the following conditions are equivalent:*

(1) *There exists a map $\nabla_B \colon B \to \mathrm{Fi}(B)$ such that $\langle \mathbf{B}, \nabla_B \rangle$ is a qm-subnearlattice of $\langle \mathbf{A}, \nabla_A \rangle$,*
(2) *For each $b \in B$ we have*

$$\mathrm{Fig}_A(\nabla_A(b) \cap B) = \nabla_A(b).$$

Proof: (1) ⇒ (2) Let $b \in B$. As ∇_B is a quasi-modal operator on $\mathbf{B}$, we have

$$\begin{aligned} \nabla_A(b) &= \mathrm{Fig}_A(\nabla_B(b)) \subseteq \mathrm{Fig}_A(\mathrm{Fig}_A(\nabla_B(b)) \cap B) \\ &= \mathrm{Fig}_A(\nabla_A(b) \cap B) \subseteq \nabla_A(b). \end{aligned}$$

Hence, $\mathrm{Fig}_A(\nabla_A(b) \cap B) = \nabla_A(b)$.

(2) ⇒ (1) We define the map $\nabla_B \colon B \to \mathrm{Fi}(B)$ given by $\nabla_B(b) = \nabla_A(b) \cap B$. It is easy to see that ∇_B is well defined and $\nabla_B(1) = \{1\}$. Let $a, b \in B$ be such that $a \wedge b$ exists in B. We prove the equality $\nabla_A(a \wedge b) = \mathrm{Fig}_A((\nabla_A(a) \cup \nabla_A(b)) \cap B)$. By Theorem 7, ∇ inverts the order and

$$\mathrm{Fig}_A((\nabla_A(a) \cup \nabla_A(b)) \cap B) \subseteq \nabla_A(a \wedge b).$$

We see the other inclusion. If we suppose the contrary, then there is $x \in \nabla_A(a \wedge b)$ such that $x \notin \mathrm{Fig}_A((\nabla_A(a) \cup \nabla_A(b)) \cap B)$. Then by Theorem 2 there exists $P \in \mathrm{X}(A)$ such that $x \in P$ and $P \cap \mathrm{Fig}_A((\nabla_A(a) \cup \nabla_A(b)) \cap B) = \emptyset$. So,

$$P \cap (\nabla_A(a) \cup \nabla_A(b)) \cap B = (P \cap B \cap \nabla_A(a)) \cup (P \cap B \cap \nabla_A(b)) = \emptyset.$$

Since $x \in \nabla_A(a \wedge b) = \mathrm{Fig}_A(\nabla_A(a \wedge b) \cap B)$, there exist $x_1, \ldots, x_n \in [\nabla_A(a \wedge b) \cap B)_A$ such that $x_1 \wedge \ldots \wedge x_n$ exists and $x = x_1 \wedge \ldots \wedge x_n$. So, there exist $y_1, \ldots, y_n \in \nabla_A(a \wedge b) \cap B$ such that $y_i \leq x_i$ for all $i \in \{1, \ldots, n\}$. It follows that $y_1, \ldots, y_n \in \nabla_A(a) \veebar \nabla_A(b) = \mathrm{Fig}_A(\nabla_A(a) \cup \nabla_A(b))$. Thus, there exist $z_1^a, \ldots, z_n^a \in \nabla_A(a)$ and $z_1^b, \ldots, z_n^b \in \nabla_A(b)$ such that $y_i = z_i^a \wedge z_i^b$ for all $i \in$

$\{1,\ldots,n\}$. As $x = x_1 \wedge \ldots \wedge x_n \in P$ and P is prime, there is $j \in \{1,\ldots,n\}$ such that $x_j \in P$. Then $y_j = z_j^a \wedge z_j^b \in P$ and again, since P is prime, we have $z_j^a \in P$ or $z_j^b \in P$.

We suppose $z_j^a \in P$. As $z_j^a \in \nabla_A(a) = \mathrm{Fig}_A(\nabla_A(a) \cap B)$, then there exist $w_1,\ldots,w_m \in [\,\nabla_A(a) \cap B)_A$ such that $w_1 \wedge \ldots \wedge w_m$ exists and $z_j^a = w_1 \wedge \ldots \wedge w_m$. So, there exist $\overline{w}_1,\ldots,\overline{w}_m \in \nabla_A(a) \cap B$ such that $\overline{w}_i \leq w_i$ for all $i \in \{1,\ldots,m\}$. Since P is prime and $z_j^a \in P$, there is $k \in \{1,\ldots,m\}$ such that $w_k \in P$. Thus, $\overline{w}_k \in P$. In summary, $\overline{w}_k \in P \cap B \cap \nabla_A(a)$ and

$$(P \cap B \cap \nabla_A(a)) \cup (P \cap B \cap \nabla_A(b)) \neq \emptyset,$$

which is a contradiction. If we suppose $z_j^b \in P$, the argument is analogous. Then $x \in \mathrm{Fig}_A((\nabla_A(a) \cup \nabla_A(b)) \cap B)$ and we have

$$\nabla_A(a \wedge b) = \mathrm{Fig}_A((\nabla_A(a) \cup \nabla_A(b)) \cap B).$$

Then, by Remarks 14, we get

$$\begin{aligned} \nabla_B(a \wedge b) &= \nabla_A(a \wedge b) \cap B = \mathrm{Fig}_A((\nabla_A(a) \cup \nabla_A(b)) \cap B) \cap B \\ &= \mathrm{Fig}_A(\nabla_B(a) \cup \nabla_B(b)) \cap B = \mathrm{Fig}_B((\nabla_B(a) \cup \nabla_B(b)) \\ &= \nabla_B(a) \veebar \nabla_B(b). \end{aligned}$$

Therefore, the pair $\langle \mathbf{B}, \nabla_B \rangle$ is a qm-subnearlattice of $\langle \mathbf{A}, \nabla_A \rangle$. $\square$

Remark 17. Following Example 5, let $\langle \mathbf{A}, \Box \rangle$ be a distributive nearlattice with a necessity modal operator and $\nabla_\Box$ the quasi-modal operator associated with $\Box$ given by $\nabla_\Box(a) = [\,\Box a)$. Let $\mathbf{B}$ be a subnearlattice of $\mathbf{A}$. If condition (2) of Theorem 16 is satisfied, then for each $b \in B$ we have $\mathrm{Fig}_A([\,\Box b) \cap B) = [\,\Box b)$. Thus, $\Box b \in \mathrm{Fig}_A([\,\Box b) \cap B)$ and there exist $x_1,\ldots,x_n \in [\,[\,\Box b) \cap B)_A$ such that $x_1 \wedge \ldots \wedge x_n$ exists and $\Box b = x_1 \wedge \ldots \wedge x_n$. So, there exist $y_1,\ldots,y_n \in [\,\Box b) \cap B$ such that $y_i \leq x_i$ for all $i \in \{1,\ldots,n\}$. It follows

$$\Box b \leq y_1 \wedge \ldots \wedge y_n \leq x_1 \wedge \ldots \wedge x_n = \Box b,$$

i.e., $\Box b = y_1 \wedge \ldots \wedge y_n$. On the other hand, since $\mathbf{B}$ is a subnearlattice of $\mathbf{A}$, $y_1 \wedge \cdots \wedge y_n \in B$ and $\Box b \in B$. Therefore, the qm-subnearlattices are a generalization of the $\Box$-subalgebras studied in [7].

2. Some extensions of qm-distributive nearlattices

Our aim is to introduce and study the classes of topological and monadic quasi-modal distributive nearlattices through the binary relation R_∇ and the operator

$\Diamond_\nabla$ given by ($\bullet$), and the connection that exists with the distributive lattice of its finitely generated filters. We begin this section by noting that Proposition 8 can be reformulated in a more general context. This result will be of great importance for the rest of the paper.

Proposition 18. *Let $\langle \mathbf{A}, \nabla \rangle$ be a qm-distributive nearlattice. Let $F \in \mathrm{Fi}(A)$ and $P \in \mathrm{X}(A)$. Then $\Diamond_\nabla(F) \cap P \neq \emptyset$ if and only if there exists $Q \in \mathrm{X}(A)$ such that $(P, Q) \in R_\nabla$ and $F \cap Q \neq \emptyset$.*

Proof: It follows from Proposition 8 and ($\bullet$). □

If $\langle \mathbf{A}, \nabla \rangle$ is a qm-distributive nearlattice and $X \subseteq A$, we define recursively

$$\Diamond_\nabla^0(X) = \mathrm{Fig}(X),$$

and

$$\Diamond_\nabla^n(X) = \Diamond_\nabla(\Diamond_\nabla^{n-1}(X))$$

for $n > 0$. It follows that $\Diamond_\nabla^1(X) = \Diamond_\nabla(X)$, which agrees with Definition 9. The next result is a generalization of Proposition 18.

Proposition 19. *Let $\langle \mathbf{A}, \nabla \rangle$ be a qm-distributive nearlattice. Let $F \in \mathrm{Fi}(A)$ and $P \in \mathrm{X}(A)$. Then for $n \geq 1$, $\Diamond_\nabla^n(F) \cap P \neq \emptyset$ if and only if there exists $Q \in \mathrm{X}(A)$ such that $(P, Q) \in R_\nabla^n$ and $F \cap Q \neq \emptyset$.*

Proof: The proof is by induction on n. The case $n = 1$ is Proposition 18. Assume that $\Diamond_\nabla^n(F) \cap P \neq \emptyset$ if and only if there exists $Q \in \mathrm{X}(A)$ such that $(P, Q) \in R_\nabla^n$ and $F \cap Q \neq \emptyset$. Suppose $\Diamond_\nabla^{n+1}(F) \cap P \neq \emptyset$. Then $\Diamond_\nabla(\Diamond_\nabla^n(F)) \cap P \neq \emptyset$ and by Proposition 18 there exists $R \in \mathrm{X}(A)$ such that $(P, R) \in R_\nabla$ and $\Diamond_\nabla^n(F) \cap R \neq \emptyset$. By inductive hypothesis there is $Q \in \mathrm{X}(A)$ such that $(R, Q) \in R_\nabla^n$ and $F \cap Q \neq \emptyset$. Hence $(P, Q) \in R_\nabla^{n+1}$ and $F \cap Q \neq \emptyset$.

Conversely, suppose there exists $Q \in \mathrm{X}(A)$ such that $(P, Q) \in R_\nabla^{n+1}$ and $F \cap Q \neq \emptyset$. Then there is $R \in \mathrm{X}(A)$ such that $(P, R) \in R_\nabla$ and $(R, Q) \in R_\nabla^n$. It follows by inductive hypothesis $\Diamond_\nabla^n(F) \cap R \neq \emptyset$. Then there is $y \in R$ such that $y \in \Diamond_\nabla^n(F)$. So, $\nabla y \subseteq \Diamond_\nabla^{n+1}(F)$. On the other hand, since $(P, R) \in R_\nabla$, we have $\gamma(P) \cap R = \emptyset$ and $y \notin \gamma(P)$, i.e., $\nabla y \cap P \neq \emptyset$. Therefore, $\Diamond_\nabla^{n+1}(F) \cap P \neq \emptyset$. □

Theorem 20. *Let $\langle \mathbf{A}, \nabla \rangle$ be a qm-distributive nearlattice. Then the following properties are satisfied:*

(1) *$F \subseteq \Diamond_\nabla(F)$ for every $F \in \mathrm{Fi}(A)$ if and only if R_∇ is reflexive.*
(2) *$\Diamond_\nabla^2(F) \subseteq \Diamond_\nabla(F)$ for every $F \in \mathrm{Fi}(A)$ if and only if R_∇ is transitive.*
(3) *$\Diamond_\nabla^n(F) \subseteq F$ for every $F \in \mathrm{Fi}(A)$ if and only if for all $P, Q \in \mathrm{X}(A)$, $(P, Q) \in R_\nabla^n$ implies $Q \subseteq P$.*
(4) *$\Diamond_\nabla^{n+1}(F) \subseteq F \veebar \Diamond_\nabla(F) \veebar \ldots \veebar \Diamond_\nabla^n(F)$ for every $F \in \mathrm{Fi}(A)$ if and only if for all $P, Q \in \mathrm{X}(A)$, if $(P, Q) \in R_\nabla^{n+1}$ and $F \cap Q \neq \emptyset$, then there exists $j \in \{0, \ldots, n\}$ such that $\Diamond_\nabla^j(F) \cap P \neq \emptyset$.*

Proof: (1) Let $F \in \mathrm{Fi}(A)$ and suppose $F \subseteq \Diamond_{\nabla}(F)$. Let $P \in \mathrm{X}(A)$ such that $\gamma(P) \cap P \neq \emptyset$. Then, since $\gamma(P) \in \mathrm{Fi}(A)$ by Theorem 7, we have by hypothesis $\Diamond_{\nabla}(\gamma(P)) \cap P \neq \emptyset$. So, by Proposition 18, there exists $Q \in \mathrm{X}(A)$ such that $(P,Q) \in R_{\nabla}$ and $\gamma(P) \cap Q \neq \emptyset$, which is a contradiction. Then $\gamma(P) \cap P = \emptyset$ and R_{∇} is reflexive. Reciprocally, suppose there is $F \in \mathrm{Fi}(A)$ such that $F \nsubseteq \Diamond_{\nabla}(F)$. Then there is $x \in F$ such that $x \notin \Diamond_{\nabla}(F)$. By Theorem 2 there exists $P \in \mathrm{X}(A)$ such that $x \in P$ and $\Diamond_{\nabla}(F) \cap P = \emptyset$. Since $(P,P) \in R_{\nabla}$ and $F \cap P \neq \emptyset$, by Proposition 18 it follows $\Diamond_{\nabla}(F) \cap P \neq \emptyset$ which is impossible.

(2) Let $F \in \mathrm{Fi}(A)$ and suppose $\Diamond^2_{\nabla}(F) \subseteq \Diamond_{\nabla}(F)$. Let $P, Q, R \in \mathrm{X}(A)$ be such that $(P,Q), (Q,R) \in R_{\nabla}$. If $(P,R) \notin R_{\nabla}$, then $\gamma(P) \cap R \neq \emptyset$. As $(Q,R) \in R_{\nabla}$, by Proposition 18 we have $\Diamond_{\nabla}(\gamma(P)) \cap Q \neq \emptyset$. Since $(P,Q) \in R_{\nabla}$, again by Proposition 18 it follows $\Diamond^2_{\nabla}(\gamma(P)) \cap P \neq \emptyset$. Thus, by Proposition 19 there exists $T \in \mathrm{X}(A)$ such that $(P,T) \in R_{\nabla}$ and $\gamma(P) \cap T \neq \emptyset$ which is impossible. Conversely, suppose there is $F \in \mathrm{Fi}(A)$ such that $\Diamond^2_{\nabla}(F) \nsubseteq \Diamond_{\nabla}(F)$. So, there is $x \in \Diamond^2_{\nabla}(F)$ such that $x \notin \Diamond_{\nabla}(F)$. Then by Theorem 2 there exists $P \in \mathrm{X}(A)$ such that $x \in P$ and $\Diamond_{\nabla}(F) \cap P = \emptyset$. On the other hand, $\Diamond^2_{\nabla}(F) \cap P \neq \emptyset$ and by Proposition 19 there exists $Q \in \mathrm{X}(A)$ such that $(P,Q) \in R^2_{\nabla}$ and $F \cap Q \neq \emptyset$. By hypothesis R_{∇} is transitive and $R^2_{\nabla} \subseteq R_{\nabla}$, then $(P,Q) \in R_{\nabla}$ and $F \cap Q \neq \emptyset$. Thus by Proposition 18 it follows $\Diamond_{\nabla}(F) \cap P \neq \emptyset$, which is a contradiction. Therefore $\Diamond^2_{\nabla}(F) \subseteq \Diamond_{\nabla}(F)$ for every $F \in \mathrm{Fi}(A)$.

(3) Let $F \in \mathrm{Fi}(A)$ be such that $\Diamond^n_{\nabla}(F) \subseteq F$. Let $P, Q \in \mathrm{X}(A)$ be such that $(P,Q) \in R^n_{\nabla}$. If $a \in Q$, then $[a) \cap Q \neq \emptyset$ and by Proposition 19 we have $\Diamond^n_{\nabla}([a)) \cap P \neq \emptyset$. As $\Diamond^n_{\nabla}([a)) \subseteq [a)$, then $[a) \cap P \neq \emptyset$ and $a \in P$. So, $Q \subseteq P$. For the other implication, suppose there is $F \in \mathrm{Fi}(A)$ such that $\Diamond^n_{\nabla}(F) \nsubseteq F$, i.e., there is $x \in \Diamond^n_{\nabla}(F)$ such that $x \notin F$. Then by Theorem 2 there exists $P \in \mathrm{X}(A)$ such that $x \in P$ and $F \cap P = \emptyset$. Since $\Diamond^n_{\nabla}(F) \cap P \neq \emptyset$, by Proposition 19 there exists $Q \in \mathrm{X}(A)$ such that $(P,Q) \in R^n_{\nabla}$ and $F \cap Q \neq \emptyset$. Then by hypothesis $Q \subseteq P$ and $F \cap P \neq \emptyset$, which is impossible. Hence $\Diamond^n_{\nabla}(F) \subseteq F$ for every $F \in \mathrm{Fi}(A)$.

(4) Let $F \in \mathrm{Fi}(A)$ and suppose $\Diamond^{n+1}_{\nabla}(F) \subseteq F \veebar \Diamond_{\nabla}(F) \veebar \ldots \veebar \Diamond^n_{\nabla}(F)$. Let $P, Q \in \mathrm{X}(A)$ be such that $(P,Q) \in R^{n+1}_{\nabla}$ and $F \cap Q \neq \emptyset$. By Proposition 19, $\Diamond^{n+1}_{\nabla}(F) \cap P \neq \emptyset$ and by hypothesis,

$$(F \veebar \Diamond_{\nabla}(F) \veebar \ldots \veebar \Diamond^n_{\nabla}(F)) \cap P \neq \emptyset,$$

i.e., there is $x \in P$ such that $x \in F \veebar \Diamond_{\nabla}(F) \veebar \ldots \veebar \Diamond^n_{\nabla}(F)$. So, there exist $x_1, \ldots, x_n \in F \cup \Diamond_{\nabla}(F) \cup \ldots \cup \Diamond^n_{\nabla}(F)$ such that $x_1 \wedge \ldots \wedge x_n$ exists and $x = x_1 \wedge \ldots \wedge x_n$. As $x \in P$ and P is prime, there is $k \in \{1, \ldots, n\}$ such that $x_k \in P$ and there is $j \in \{0, \ldots, n\}$ such that $x_k \in \Diamond^j_{\nabla}(F)$. Then $\Diamond^j_{\nabla}(F) \cap P \neq \emptyset$. Reciprocally, suppose there is $F \in \mathrm{Fi}(A)$ such that $\Diamond^{n+1}_{\nabla}(F) \nsubseteq F \veebar$

$\Diamond_\nabla(F) \veebar \ldots \veebar \Diamond_\nabla^n(F)$, i.e., there is $x \in \Diamond_\nabla^{n+1}(F)$ such that $x \notin F \veebar \Diamond_\nabla(F) \veebar \ldots \veebar \Diamond_\nabla^n(F)$. By Theorem 2 there exists $P \in \mathrm{X}(A)$ such that $x \in P$ and

$$(\star) \qquad (F \veebar \Diamond_\nabla(F) \veebar \ldots \veebar \Diamond_\nabla^n(F)) \cap P = \emptyset.$$

On the other hand, $\Diamond_\nabla^{n+1}(F) \cap P \neq \emptyset$ and by Proposition 19 there exists $Q \in \mathrm{X}(A)$ such that $(P, Q) \in R_\nabla^{n+1}$ and $F \cap Q \neq \emptyset$. Thus, by hypothesis, there exists $j \in \{0, \ldots, n\}$ such that $\Diamond_\nabla^j(F) \cap P \neq \emptyset$ which is impossible by $(\star)$. □

Definition 21. Let $\langle \mathbf{A}, \nabla \rangle$ be a qm-distributive nearlattice. We say that ∇ is *topological* if it satisfies the following conditions for each $a \in A$:

(R) $[a) \subseteq \nabla a$,
(T) $\Diamond_\nabla(\nabla a) \subseteq \nabla a$.

Moreover, we say that ∇ is *monadic* if it is topological and verifies the following additional condition for each $a, b \in A$:

(M) $\nabla a \cap \nabla b \subseteq \Diamond_\nabla([a) \cap \nabla b)$.

A pair $\langle \mathbf{A}, \nabla \rangle$ is a *topological qm-distributive nearlattice* if $\mathbf{A}$ is a distributive nearlattice and ∇ is a topological quasi-modal operator on $\mathbf{A}$. Analogously, a pair $\langle \mathbf{A}, \nabla \rangle$ is a *monadic qm-distributive nearlattice* if $\mathbf{A}$ is a distributive nearlattice and ∇ is a monadic quasi-modal operator on $\mathbf{A}$.

The topological and monadic qm-distributive nearlattices are generalizations of the **S4**-nearlattices and the **S5**-nearlattices, respectively, studied in [7].

Remark 22. If $\langle \mathbf{A}, \nabla \rangle$ is a topological qm-distributive nearlattice, then we have $\Diamond_\nabla(\nabla a) = \nabla a$ for every $a \in A$.

Now we are going to focus on the class of fqm-distributive nearlattices.

Theorem 23. *Let $\langle \mathbf{A}, \nabla \rangle$ be a fqm-distributive nearlattice. Then $\langle \mathbf{A}, \nabla \rangle$ is topological if and only if $F \subseteq \Diamond_\nabla(F)$ and $\Diamond_\nabla^2(F) \subseteq \Diamond_\nabla(F)$ for every $F \in \mathrm{Fi}_{\mathrm{f}}(A)$.*

Proof: Let $F \in \mathrm{Fi}_{\mathrm{f}}(A)$. Then there exist $a_1, \ldots, a_n \in A$ such that $F = [a_1) \veebar \ldots \veebar [a_n)$. By Remark 10 and condition (R) of Definition 21 we have

$$\begin{aligned} F &\subseteq \nabla a_1 \veebar \ldots \veebar \nabla a_n = \Diamond_\nabla([a_1)) \veebar \ldots \veebar \Diamond_\nabla([a_n)) \\ &= \Diamond_\nabla([a_1) \veebar \ldots \veebar [a_n)) = \Diamond_\nabla(F). \end{aligned}$$

On the other hand, $F = [a_1) \veebar \ldots \veebar [a_n)$ implies $\Diamond_\nabla(F) = \nabla a_1 \veebar \ldots \veebar \nabla a_n$. Thus, by the condition (T) of Definition 21,

$$\begin{aligned} \Diamond_\nabla^2(F) &= \Diamond_\nabla(\nabla a_1 \veebar \ldots \veebar \nabla a_n) = \Diamond_\nabla(\nabla a_1) \veebar \ldots \veebar \Diamond_\nabla(\nabla a_n) \\ &\subseteq \nabla a_1 \veebar \ldots \veebar \nabla a_n = \Diamond_\nabla(F) \end{aligned}$$

i.e., $\Diamond_\nabla^2(F) \subseteq \Diamond_\nabla(F)$. The converse is just restriction to principal filters. □

Corollary 24. *Let $\langle \mathbf{A}, \nabla \rangle$ be a topological fqm-distributive nearlattice. Then*

$$\Diamond_\nabla(F \cap \Diamond_\nabla(G)) \subseteq \Diamond_\nabla(F) \cap \Diamond_\nabla(G),$$

for every $F, G \in \mathrm{Fi_f}(A)$.

Proof: Let $F, G \in \mathrm{Fi_f}(A)$ and suppose there is $x \in \Diamond_\nabla(F \cap \Diamond_\nabla(G))$ such that $x \notin \Diamond_\nabla(F) \cap \Diamond_\nabla(G)$. By Theorem 2 there exists $P \in \mathrm{X}(A)$ such that $x \in P$ and $\Diamond_\nabla(F) \cap \Diamond_\nabla(G) \cap P = \emptyset$. Since $\Diamond_\nabla(F \cap \Diamond_\nabla(G)) \cap P \neq \emptyset$, by Proposition 18 there exists $Q \in \mathrm{X}(A)$ such that $(P, Q) \in R_\nabla$ and $F \cap \Diamond_\nabla(G) \cap Q \neq \emptyset$. Since $(P, Q) \in R_\nabla$ and $F \cap Q \neq \emptyset$, again by Proposition 18 we have $\Diamond_\nabla(F) \cap P \neq \emptyset$. On the other hand, as $(P, Q) \in R_\nabla$ and $\Diamond_\nabla(G) \cap Q \neq \emptyset$, by Proposition 18 it follows $\Diamond^2_\nabla(G) \cap P \neq \emptyset$. By hypothesis $\langle \mathbf{A}, \nabla \rangle$ is a topological fqm-distributive nearlattice, then by Theorem 23 we have $\Diamond^2_\nabla(G) \subseteq \Diamond_\nabla(G)$ and $\Diamond_\nabla(G) \cap P \neq \emptyset$. So, $\Diamond_\nabla(F) \cap \Diamond_\nabla(G) \cap P \neq \emptyset$, which is impossible. We conclude $\Diamond_\nabla(F \cap \Diamond_\nabla(G)) \subseteq \Diamond_\nabla(F) \cap \Diamond_\nabla(G)$. $\square$

Consider the relation $E_\nabla = R_\nabla \cap R^{-1}_\nabla$.

Lemma 25. *Let $\langle \mathbf{A}, \nabla \rangle$ be a topological fqm-distributive nearlattice. Then*

$$E_\nabla = \{(P, Q) \in \mathrm{X}(A) \times \mathrm{X}(A) \colon \gamma(P) = \gamma(Q)\}.$$

Proof: Let $P, Q \in \mathrm{X}(A)$ be such that $(P, Q) \in E_\nabla$. Then $(P, Q) \in R_\nabla$ and $(Q, P) \in R_\nabla$. If $a \notin \gamma(Q)$, then $\nabla a \cap Q \neq \emptyset$ and since $(P, Q) \in R_\nabla$ by Proposition 18 we have $\Diamond_\nabla(\nabla a) \cap P \neq \emptyset$. By Remark 22 it follows $\nabla a \cap P \neq \emptyset$ and $a \notin \gamma(P)$. Thus, $\gamma(P) \subseteq \gamma(Q)$. The other inclusion is similar and $\gamma(P) = \gamma(Q)$. The reciprocal is immediate because R_∇ is reflexive by Theorems 23 and 20. $\square$

Theorem 26. *Let $\langle \mathbf{A}, \nabla \rangle$ be a topological fqm-distributive nearlattice. If $R_\nabla \subseteq E_\nabla \circ \subseteq^{-1}$, then $\langle \mathbf{A}, \nabla \rangle$ is monadic.*

Proof: Let $a, b \in A$. Suppose there is $x \in \nabla a \cap \nabla b$ such that $x \notin \Diamond_\nabla([a) \cap \nabla b)$. Then by Theorem 2 there exists $P \in \mathrm{X}(A)$ such that $x \in P$ and $\Diamond_\nabla([a) \cap \nabla b) \cap P = \emptyset$. Then $x \in \nabla a \cap P$, i.e., $\nabla a \cap P \neq \emptyset$ and by Proposition 8 there is $Q \in \mathrm{X}(A)$ such that $(P, Q) \in R_\nabla$ and $a \in Q$. Thus, by hypothesis, $(P, Q) \in E_\nabla \circ \subseteq^{-1}$ and there exists $R \in \mathrm{X}(A)$ such that $(P, R) \in E_\nabla$ and $Q \subseteq R$. By Lemma 25 we have $\gamma(P) = \gamma(R)$ and $a \in R$. On the other hand, $x \in \nabla b \cap P$ implies $b \notin \gamma(P) = \gamma(R)$ and $\nabla b \cap R \neq \emptyset$. So, there is $y \in \nabla b$ such that $y \in R$. Then $[a \vee y) \subseteq [a) \cap \nabla b$ and $\Diamond_\nabla([a \vee y)) = \nabla(a \vee y) \subseteq \Diamond_\nabla([a) \cap \nabla b)$. As $a, y \in R$, $a \vee y \in R$ and $\nabla(a \vee y) \cap R \neq \emptyset$ by condition (R) of Definition 21. Then $\Diamond_\nabla([a) \cap \nabla b) \cap R \neq \emptyset$ and as $(P, R) \in R_\nabla$, by Proposition 18 we have $\Diamond^2_\nabla([a) \cap \nabla b) \cap P \neq \emptyset$. Then, by Theorem 23, $\Diamond_\nabla([a) \cap \nabla b) \cap P \neq \emptyset$ which is a contradiction. Thus, $\nabla a \cap \nabla b \subseteq \Diamond_\nabla([a) \cap \nabla b)$. $\square$

Theorem 27. *Let $\langle \mathbf{A}, \nabla \rangle$ be a topological fqm-distributive nearlattice. Then $\langle \mathbf{A}, \nabla \rangle$ is monadic if and only if*

$$\Diamond_\nabla(F) \cap \Diamond_\nabla(G) \subseteq \Diamond_\nabla(F \cap \Diamond_\nabla(G))$$

for every $F, G \in \mathrm{Fi_f}(A)$.

Proof: Let $F, G \in \mathrm{Fi_f}(A)$. Then there exist $a_1, \ldots, a_n,\ b_1, \ldots, b_m \in A$ such that $F = [\,a_1) \veebar \ldots \veebar [\,a_n)$ and $G = [\,b_1) \veebar \ldots \veebar [\,b_m)$. Thus, by Remark 10 we have $\Diamond_\nabla(F) = \nabla a_1 \veebar \ldots \veebar \nabla a_n$ and $\Diamond_\nabla(G) = \nabla b_1 \veebar \ldots \veebar \nabla b_m$. Then since $\mathrm{Fi_f}(\mathbf{A})$ is a distributive lattice and by condition (M) of Definition 21, it follows

$$\begin{aligned}
\Diamond_\nabla(F) \cap \Diamond_\nabla(G) &= (\nabla a_1 \veebar \ldots \veebar \nabla a_n) \cap (\nabla b_1 \veebar \ldots \veebar \nabla b_k) \\
&= \underline{\bigvee} \{\nabla a_i \cap \nabla b_j \colon\ 1 \leq i \leq n,\ 1 \leq j \leq m\} \\
&\subseteq \underline{\bigvee} \{\Diamond_\nabla([a_i) \cap \nabla b_j) \colon\ 1 \leq i \leq n,\ 1 \leq j \leq m\}.
\end{aligned}$$

Note that $[\,a_i) \cap \nabla b_j$ is a principal filter for all $i \in \{1, \ldots, n\}$ and for all $j \in \{1, \ldots, m\}$. Indeed, since $\nabla b_j \in \mathrm{Fi_f}(A)$, then there exist $c_1, \ldots, c_k \in A$ such that $\nabla b_j = [\,c_1) \veebar \ldots \veebar [\,c_k)$. So,

$$\begin{aligned}
[\,a_i) \cap \nabla b_j &= [\,a_i) \cap ([\,c_1) \veebar \ldots \veebar [\,c_k)) = [\,c_1 \vee a_i) \veebar \ldots \veebar [\,c_k \vee a_i) \\
&= [\,(c_1 \vee a_i) \wedge \ldots \wedge (c_k \vee a_i)) = [\,m^{k-1}(c_1, \ldots, c_k, a_i))
\end{aligned}$$

and $[\,a_i) \cap \nabla b_j = [m^{k-1}(c_1, \ldots, c_k, a_i))$. Then for each $i \in \{1, \ldots, n\}$ and each $j \in \{1, \ldots, m\}$, let $d_{ij} \in A$ be such that $[\,a_i) \cap \nabla b_j = [\,d_{ij})$. Thus,

$$\begin{aligned}
\Diamond_\nabla(F) \cap \Diamond_\nabla(G) &\subseteq \underline{\bigvee} \{\Diamond_\nabla([\,a_i) \cap \nabla b_j) \colon\ 1 \leq i \leq n,\ 1 \leq j \leq m\} \\
&= \underline{\bigvee} \{\Diamond_\nabla([\,d_{ij})) \colon\ 1 \leq i \leq n,\ 1 \leq j \leq m\} \\
&= \Diamond_\nabla\Big(\underline{\bigvee} \{[\,d_{ij}) \colon\ 1 \leq i \leq n,\ 1 \leq j \leq m\}\Big) \\
&= \Diamond_\nabla\Big(\underline{\bigvee} \{[\,a_i) \cap \nabla b_j \colon\ 1 \leq i \leq n,\ 1 \leq j \leq m\}\Big) \\
&= \Diamond_\nabla(([\,a_1) \veebar \ldots \veebar [\,a_n)) \cap (\nabla b_1 \veebar \ldots \veebar \nabla b_m)) \\
&= \Diamond_\nabla(F \cap \Diamond_\nabla(G)).
\end{aligned}$$

Reciprocally, if $a, b \in A$, then by Remark 10 and by hypothesis we have

$$\nabla a \cap \nabla b = \Diamond_\nabla([\,a)) \cap \Diamond_\nabla([\,b)) \subseteq \Diamond_\nabla\big([\,a) \cap \Diamond_\nabla([\,b))\big) = \Diamond_\nabla([\,a) \cap \nabla b).$$

Therefore, $\langle \mathbf{A}, \nabla \rangle$ is a monadic fqm-distributive nearlattice. $\square$

The notion of quantifier on a Boolean algebra was introduced by P. R. Halmos in [18] and later by R. Cignoli in the class of bounded distributive lattices, see [11]. Recall that a *quantifier* on a bounded distributive lattice $\mathbf{L} = \langle L, \vee, \wedge, 0, 1\rangle$ is a unary operator $\Delta\colon L \to L$ that verifies the following conditions for every $a, b \in A$:

- $\Delta\, 0 = 0$,
- $a \wedge \Delta\, a = a$,
- $\Delta\,(a \wedge \Delta\, b) = \Delta\, a \wedge \Delta\, b$,
- $\Delta\,(a \vee b) = \Delta\, a \vee \Delta\, b$.

We have the main result of this paper which is a consequence of Theorem 11, Corollary 24 and Theorem 27.

Theorem 28. *Let* $\mathbf{A}$ *be a distributive nearlattice. Then there is a one to one correspondence between monadic finite quasi-modal operators on* $\mathbf{A}$ *and quantifiers on* $\mathrm{Fi_f}(\mathbf{A})$.

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