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Abstract. We introduce the definition of p-limited completely continuous operators, $1 \leq p < \infty$. The question of whether a space of operators has the property that every p-limited subset is relative compact when the dual of the domain and the codomain have this property is studied using p-limited completely continuous evaluation operators.

Keywords: p-limited set; limited set; space of compact operators

Classification: 46B20, 46B25, 46B28

1. Introduction

A subset A of a Banach space X is called *limited* (or Dunford–Pettis (DP)) if every w^*-null (weakly null, respectively) sequence (x_n^*) in X^* tends to 0 uniformly on A; i.e.,

$$\sup_{x \in A} |x_n^*(x)| \to 0.$$

A subset A of a dual Banach space X^* is called an L-*set* in X^* if every weakly null sequence (x_n) in X converges uniformly on A.

A Banach space X has the *Gelfand–Phillips (GP) property* (or is a *Gelfand–Phillips space*) if every limited subset of X is relatively compact.

Spaces with the Gelfand–Phillips property include, among others, Schur spaces, spaces with w^*-sequential compact dual unit balls, separable spaces, reflexive spaces, spaces whose duals do not contain l_1, see [3], [11], [24, page 31].

A Banach space X has the *Dunford–Pettis relatively compact property* (DPrcP) if every DP subset of X is relatively compact.

It is known that $l_1 \not\hookrightarrow X$ if and only if X^* has the DPrcP if and only if any L-subset of X^* is relatively compact, see [12], [11].

An operator $T\colon X \to Y$ is called *limited completely continuous* (or lcc), see [23], (or *Dunford–Pettis completely continuous* (DPcc), see [25]) if T takes weakly null limited (DP, respectively) sequences in X to norm null ones in Y.

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A. K. Sinha and D. P. Karn in [19] extended the idea of limited sets to a p-level by introducing the following definition.

A subset A of a Banach space X is *p-limited*, $1 \leq p < \infty$, if for every weak* (weak) p-summable sequence (x_n^*) in X^*, there exists $(\alpha_n) \in l_p$ such that $|x_n^*(x)| \leq \alpha_n$ for all $x \in A$ and $n \in \mathbb{N}$.

We say that a Banach space X has the *p-limited relatively compact property*, p-limited rcP, if every p-limited set in X is relatively compact.

We say that an operator $T \colon X \to Y$ is called *p-limited completely continuous* (or p-limited cc) if T takes weakly null p-limited sequences in X to norm null ones in Y, $1 \leq p < \infty$.

Numerous papers have investigated whether spaces of operators inherit the Dunford–Pettis relative compact property or the Gelfand–Phillips property when the codomain and the dual of the domain possess the respective property; e.g., see [12], [14], [17], [10], [23], [4], and [25].

In [23], [14], and [25], limited completely continuous evaluation operators and Dunford–Pettis completely continuous evaluation operators were used to give sufficient conditions for the Gelfand–Phillips property and the DPrcP of some spaces of operators. In this paper p-limited completely continuous evaluation operators are used to study the p-limited relatively compact property in spaces of operators.

We show that if every operator $T \colon Y^* \to X^*$ is p-summing, Y has the p-limited rcP, and M is a closed subspace of $L(X, Y) = K(X, Y)$ such that the evaluation operator $\psi_{y^*} \colon M \to X^*$ is p-limited completely continuous for each $y^* \in Y^*$, then M has the p-limited rcP, $1 < p < \infty$. We prove that if X has the p-limited rcP, Y has the Schur property, and M is a closed subspace of $L_{w^*}(X^*, Y) = K_{w^*}(X^*, Y)$, then M has the p-limited rcP. We also prove that if $K_{w^*}(X^*, Y)$ has the p-limited rcP property, then at least one of the spaces X and Y does not contain l_2.

2. Definitions and notation

Throughout this paper, X and Y will denote Banach spaces. The unit ball of X will be denoted by B_X, the closed linear span of a sequence (x_n) in X will be denoted by $[x_n]$, and X^* will denote the continuous linear dual of X. The space X embeds in Y (in symbols $X \hookrightarrow Y$) if X is isomorphic to a closed subspace of Y. An operator $T \colon X \to Y$ will be a continuous and linear function. The set of all operators, weakly compact operators, and compact operators from X to Y will be denoted by $L(X, Y)$, $W(X, Y)$, and $K(X, Y)$. The space of all $w^* - w$ continuous (or $w^* - w$ continuous compact) operators from X^* to Y will be denoted by $L_{w^*}(X^*, Y)$ ($K_{w^*}(X^*, Y)$, respectively). The projective tensor product of X and Y will be denoted by $X \otimes_\pi Y$.

An operator $T\colon X \to Y$ is called *completely continuous* (or *Dunford–Pettis*) if T maps weakly convergent sequences to norm convergent sequences.

A Banach space X has the *Dunford–Pettis property* (DPP) if every weakly compact operator $T\colon X \to Y$ is completely continuous for any Banach space Y. If X is a $C(K)$-space or an L_1-space, then X has the DPP. The reader can check [7] and [6] for results related to the DPP.

For $1 \le p < \infty$, p^* denotes the conjugate of p. If $p = 1$, c_0 plays the role of l_{p^*}. The unit vector basis of l_p will be denoted by (e_n).

Let $1 \le p < \infty$. The space of *weakly p-summable* (or p-summable) sequences in X is denoted by $l_p^w(X)$ ($l_p(X)$, respectively) endowed with its norm

$$\|(x_n)\|_p^w = \sup\left\{\left(\sum_{n=1}^{\infty} |\langle x^*, x_n\rangle|^p\right)^{1/p} : x^* \in B_{X^*}\right\}$$

$$\left(\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} \|x_n\|^p\right)^{1/p}, \text{ respectively}\right).$$

If $p < q$, then $l_p^w(X) \subseteq l_q^w(X)$. Further, the unit vector basis of l_{p^*} is weakly p-summable for all $1 < p < \infty$. The weakly 1-summable sequences are precisely the weakly unconditionally convergent series.

We recall the following isometries: $L(l_{p^*}, X) \simeq l_p^w(X)$ for $1 < p < \infty$; $L(c_0, X) \simeq l_p^w(X)$ for $p = 1$; $T \to (T(e_n))$, see [8, Proposition 2.2, page 36].

Let $1 \le p < \infty$. A sequence (x_n^*) in X^* is called *weak* p-summable* if $(\langle x_n^*, x\rangle) \in l_p$ for each $x \in X$. Let $l_p^{w^*}(X^*)$ denote the set of all weak* p-summable sequences in X^*. This is a Banach space with the norm

$$\|(x_n^*)\|_p^{w^*} = \sup\left\{\left(\sum_{n=1}^{\infty} |\langle x_n^*, x\rangle|^p\right)^{1/p} : x \in B_X\right\}.$$

The map $(x_i^*) \to L_{(x_i^*)}$, where $L_{(x_i^*)}(x) = (\langle x_i^*, x\rangle)$, identifies $l_p^{w^*}(X^*)$ and $L(X, l_p)$ isometrically for all $1 < p < \infty$. The spaces $l_p^{w^*}(X^*)$ and $l_p^w(X^*)$ are the same for $1 \le p < \infty$, see [13].

Let $1 \le p < \infty$. An operator $T\colon X \to Y$ is called *p-summing* (or *absolutely p-summing*) if $(Tx_n) \in l_p(Y)$ whenever $(x_n) \in l_p^w(X)$. The set of all p-summing operators from X to Y is denoted by $\Pi_p(X, Y)$.

3. Spaces of operators in which p-limited subsets are relatively compact

In the following we use p-limited completely continuous evaluation operators to give necessary and sufficient conditions for some spaces of operators to have the p-limited rcP.

If $H \subseteq L(X,Y)$, $x \in X$ and $y^* \in Y^*$, let $H(x) = \{T(x)\colon T \in H\}$ and $H^*(y^*) = \{T^*(y^*)\colon T \in H\}$.

Suppose that X and Y are Banach spaces and M is a closed subspace of $L(X,Y)$. If $x \in X$ and $y^* \in Y^*$, the evaluation operators $\varphi_x\colon M \to Y$ and $\psi_{y^*}\colon M \to X^*$ are defined by

$$\varphi_x(T) = T(x), \quad \psi_{y^*}(T) = T^*(y^*), \qquad T \in M.$$

Let $1 \le p < \infty$. A subset A of X^* is called a *p-L-set* in X^*, see [16], if for every weakly p-summable sequence (x_n) in X, there exists $(\alpha_n) \in l_p$ such that $|x^*(x_n)| \le \alpha_n$ for all $x^* \in A$ and $n \in \mathbb{N}$.

We will use the following facts:

(A) Let $1 \le p < \infty$. A subset A of X^* is a p-L-set in X^* if and only if it is a p-limited set in X^* [16, Corollary 3 (ii)].

(B) A sequence (x_n^*) in X^* is p-limited (i.e. the set of its terms is p-limited) if and only if $(x_n^*(x_n)) \in l_p$ for every $(x_n) \in l_p^w(X)$, by [16, Theorem 14].

(C) A sequence (x_n) in X is p-limited if and only if $(x_n^*(x_n)) \in l_p$ for every $(x_n^*) \in l_p^w(X^*)$, by [16, Corollary 15].

(D) The Banach space X has the p-limited rcP if and only if every weakly null p-limited sequence in X is norm null (since p-limited sets are relatively weakly compact, see [5, Proposition 2.1]).

(E) The unit vector basis (e_n) of c_0 is p-limited for all $p \ge 1$ and not relatively compact [5, page 717]. If X has the p-limited rcP, then $c_0 \not\hookrightarrow X$.

If $l_1 \not\hookrightarrow X$, then X^* has the p-limited rcP, $1 \le p < \infty$, see [16, Corollary 11]. Schur spaces have the p-limited rcP.

Lemma 1. *Let $1 \le p < \infty$. Suppose that $L(Y,X^*) = \Pi_p(Y,X^*)$. If (y_n) is weakly p-summable in Y and (x_n) is bounded in X, then $(x_n \otimes y_n)$ is weakly p-summable in $X \otimes_\pi Y$.*

Proof: Without loss of generality suppose $\|x_n\| \le 1$. Let $T \in (X \otimes_\pi Y)^* \simeq L(X,Y^*)$, see [9, page 230]. Then $T^*|_Y\colon Y \to X^*$ is p-summing and

$$\sum_n |\langle T, x_n \otimes y_n\rangle|^p \le \sum_n \|T^*(y_n)\|^p < \infty.$$

□

Theorem 2. *Let $1 < p < \infty$. Suppose that $L(Y, X^*) = \Pi_p(Y, X^*)$ and Y^* has the p-limited rcP. If M is a closed subspace of $L(X, Y^*) = K(X, Y^*)$ such that the evaluation operator $\psi_{y^{**}}\colon M \to X^*$ is p-limited completely continuous for each $y^{**} \in Y^{**}$, then M has the p-limited rcP.*

PROOF: Let $T\colon X \to Y^*$ be an operator. Since $T^*|_Y\colon Y \to X^*$ is p-summing, $T(B_X)$ is a p-limited set in Y^*, see [16, Theorem 7], thus relatively compact. Hence $L(X, Y^*) = K(X, Y^*)$.

Let (T_n) be a weakly null p-limited sequence in M such that $\|T_n\| = 1$ for each n. Let (x_n) be a sequence in B_X such that $\|T_n(x_n)\| > 1/2$ for each n.

Let $y^{**} \in Y^{**}$. The sequence $(\psi_{y^{**}}(T_n)) = (T_n^*(y^{**}))$ is norm null in X^*. Hence $\langle y^{**}, T_n(x_n)\rangle \le \|T_n^*(y^{**})\| \to 0$, and thus $(T_n(x_n))$ is weakly null.

Suppose $(y_n) \in l_p^w(Y)$ and (x_n) is a sequence in B_X. By Lemma 1, $(x_n \otimes y_n)$ is weakly p-summable in $X \otimes_\pi Y$. Since (T_n) is a p-limited sequence in $L(X, Y^*) \simeq (X \otimes_\pi Y)^*$, see [9, page 230]), $(\langle T_n, x_n \otimes y_n\rangle) = (\langle T_n(x_n), y_n\rangle) \in l_p$. Therefore $(T_n(x_n))_n$ is p-limited in Y^*, see [16, Corollary 3 (ii)], hence relatively compact. Thus $\|T_n(x_n)\| \to 0$, and we have a contradiction. $\square$

The Banach–Mazur distance $d(E, F)$ between two isomorphic Banach spaces E and F is defined by $\inf(\|T\|\|T^{-1}\|)$, where the infimum is taken over all isomorphisms T from E onto F. A Banach space E is called an $\mathcal{L}_\infty$-space (or $\mathcal{L}_1$-space), see [2], if there is a $\lambda \ge 1$ so that every finite dimensional subspace of E is contained in another subspace N with $d(N, l_\infty^n) \le \lambda$ ($d(N, l_1^n) \le \lambda$, respectively) for some integer n. Complemented subspaces of $C(K)$ spaces (or $L_1(\mu)$) spaces) are $\mathcal{L}_\infty$-spaces ($\mathcal{L}_1$-spaces, respectively), see [2, Proposition 1.26]. The dual of an $\mathcal{L}_1$-space (or $\mathcal{L}_\infty$-space) is an $\mathcal{L}_\infty$-space ($\mathcal{L}_1$- space, respectively), see [2, Proposition 1.27]. The $\mathcal{L}_\infty$-spaces, $\mathcal{L}_1$-spaces, and their duals have the DPP, see [2, Corollary 1.30].

Observation 1. (i) Let $1 \le p \le 2$. If X is an $\mathcal{L}_\infty$-space and Y is an $\mathcal{L}_p$-space, then every operator $T\colon X \to Y$ is 2-summing, see [8, Theorem 3.7, page 64].

(ii) If X and Y are $\mathcal{L}_\infty$-spaces, then $L(X, Y^*) = \Pi_p(X, Y^*)$, $2 \le p < \infty$. Indeed, by (i), every operator $T\colon X \to Y^*$ is 2-summing, and thus p-summing, $2 \le p < \infty$.

Observation 2 ([1])**.** If $T\colon Y \to X^*$ is an operator such that $T^*|_X$ is compact (or weakly compact), then T is compact (weakly compact, respectively).

Corollary 3. *Let $1 \le p < \infty$. Suppose that $L(Y, X^*) = \Pi_p(Y, X^*)$ and X^* and Y^* have the p-limited rcP. If M is a closed subspace of $L(X, Y^*) = K(X, Y^*)$, then M has the p-limited rcP. Further, $l_1 \overset{c}{\not\hookrightarrow} X \otimes_\pi Y$.*

Proof: Since X^* has the p-limited rcP, $\psi_{y^{**}}\colon M \to X^*$ is p-limited cc for each $y^{**} \in Y^{**}$. Apply Theorem 2. Since $L(X,Y^*)$ has the p-limited rcP, $c_0 \not\hookrightarrow L(X,Y^*) \simeq (X \otimes_\pi Y)^*$. Thus $l_1 \overset{c}{\not\hookrightarrow} X \otimes_\pi Y$ by a result of Bessaga–Pełczyński, [7, Theorem 10, page 48]. □

A topological space S is called *dispersed* (or *scattered*) if every nonempty closed subset of S has an isolated point. A compact Hausdorff space K is dispersed if and only if $l_1 \not\hookrightarrow C(K)$, see [21, Main theorem].

Corollary 4. (i) *Let $2 \le p < \infty$. Suppose X and Y are $\mathcal{L}_\infty$-spaces, $l_1 \not\hookrightarrow X$, and Y^* has the p-limited rcP. Then $L(X,Y^*) = K(X,Y^*)$ has the p-limited rcP.*

(ii) *Let $2 \le p < \infty$. Suppose $X = C(K_1)$, K_1 is dispersed, Y is an $\mathcal{L}_\infty$-space, and Y^* has the p-limited rcP. Then $L(X,Y^*) = K(X,Y^*)$ has the p-limited rcP.*

Proof: (i) By Observation 1, $L(Y,X^*) = \Pi_p(Y,X^*)$, $2 \le p < \infty$. Suppose $l_1 \not\hookrightarrow X$. Since X has the DPP and $l_1 \not\hookrightarrow X$, X^* has the Schur property [6, Theorem 3]. Apply Corollary 3. □

Theorem 5. *Let $1 < p < \infty$. Suppose that $L(Y^*,X^*) = \Pi_p(Y^*,X^*)$ and Y has the p-limited rcP. If M is a closed subspace of $L(X,Y) = K(X,Y)$ such that the evaluation operator $\psi_{y^*}\colon M \to X^*$ is p-limited cc for each $y^* \in Y^*$, then M has the p-limited rcP.*

Proof: Let $T\colon X \to Y$ be an operator. Since $T^*\colon Y^* \to X^*$ is p-summing, $T(B_X)$ is a p-limited set in Y, see [5, Theorem 3.1], and thus relatively compact. Hence T is compact. Thus $L(X,Y) = K(X,Y)$.

Let (T_n) be a weakly null p-limited sequence in M such that $\|T_n\| = 1$. Let (x_n) be a sequence in B_X such that $\|T_n(x_n)\| > 1/2$. Then $(\psi_{y^*}(T_n)) = (T_n^*(y^*))$ is norm null for each $y^* \in Y^*$ and $\langle T_n(x_n), y^*\rangle \le \|T_n^*(y^*)\| \to 0$. Therefore $(T_n(x_n))$ is weakly null.

Suppose (y_n^*) is a weakly p-summable sequence in Y^*. By Lemma 1, $(x_n \otimes y_n^*)$ is weakly p-summable in $X \otimes_\pi Y^*$. Now $L(X,Y)$ embeds isometrically in $L(X,Y^{**}) \simeq (X \otimes_\pi Y^*)^*$ and (T_n) is a p-limited sequence in $L(X,Y^{**})$. Then $(\langle T_n, x_n \otimes y_n^*\rangle) = (\langle T_n(x_n), y_n^*\rangle) \in l_p$. Therefore $(T_n(x_n))$ is p-limited in Y, and thus relatively compact. Thus $\|T_n(x_n)\| \to 0$, and we have a contradiction. □

Corollary 6. *Let $1 < p < \infty$. Suppose that X^* and Y have the p-limited rcP and $L(Y^*,X^*) = \Pi_p(Y^*,X^*)$. If M is a closed subspace of $L(X,Y) = K(X,Y)$, then M has the p-limited rcP.*

Proof: Since X^* has the p-limited rcP, $\psi_{y^*}\colon M \to X^*$ is p-limited cc for each $y^* \in Y^*$. Apply Theorem 5. □

Corollary 7. *Let $2 \leq p < \infty$. Suppose that X is an $\mathcal{L}_\infty$-space, $l_1 \not\hookrightarrow X$, Y is an $\mathcal{L}_1$-space, and Y has the p-limited rcP. Then $L(X,Y) = K(X,Y)$ has the p-limited rcP.*

Proof: By Observation 1, $L(Y^*, X^*) = \Pi_p(Y^*, X^*)$, $2 \leq p < \infty$. Since X has the DPP and $l_1 \not\hookrightarrow X$, X^* has the Schur property. Apply Theorem 5. $\square$

The operator $T\colon X \to Y$ is p-limited cc if and only if T takes p-limited sets to relatively compact sets (since p-limited sets are relatively weakly compact, see [5, Proposition 2.1]).

Theorem 8. *Let $1 < p < \infty$. Suppose that X^* has the p-limited rcP, $L(X, Y^{**}) = \Pi_p(X, Y^{**})$, and M is a closed subspace of $L(X,Y) = K(X,Y)$ such that the evaluation operator $\varphi_x\colon M \to Y$ is p-limited cc for each $x \in X$. Then M has the p-limited rcP.*

Proof: Let $T\colon X \to Y$ be an operator. Since $T^{**}|_X\colon X \to Y^{**}$ is p-summing, $T^*(B_{Y^*})$ is a p-limited set in X^*, see [16, Theorem 7], and thus relatively compact. Therefore T^*, and thus T, is compact.

Let H be a p-limited subset of M. We show that H is relatively compact. By [20, Theorems 2.1, 2.2], [14, Theorem 2.2], it is enough to show that (a) $H^*(B_{Y^*})$ is relatively compact and (b) $H(x)$ is relatively compact for each $x \in X$.

Let (T_n) be a sequence in H. For each $x \in X$, $\varphi_x\colon M \to Y$ is p-limited cc, hence $(T_n(x)) = (\varphi_x(T_n))$ is relatively compact.

Let (y_n^*) be a sequence in B_{Y^*} and let (x_n) be a weakly p-summable sequence in X. Then $(x_n \otimes y_n^*)$ is weakly p-summable in $X \otimes_\pi Y^*$ by [15, Lemma 9]. Now $L(X,Y)$ embeds isometrically in $L(X, Y^{**})$ and (T_n) is a p-limited sequence in $L(X, Y^{**}) \simeq (X \otimes_\pi Y^*)^*$. Hence $(\langle T_n, x_n \otimes y_n^* \rangle) = (\langle T_n^*(y_n^*), x_n \rangle) \in l_p$. Therefore $(T_n^*(y_n^*))$ is p-limited in X^*, and thus relatively compact.

Then H is relatively compact. $\square$

Corollary 9. *Let $1 < p < \infty$. Suppose that X^* and Y have the p-limited rcP, and $L(X, Y^{**}) = \Pi_p(X, Y^{**})$. Then any closed subspace M of $L(X,Y) = K(X,Y)$ has the p-limited rcP.*

Proof: Since Y has the p-limited rcP, $\varphi_x\colon M \to Y$ is p-limited cc for each $x \in X$. Apply Theorem 8. $\square$

We recall the following well-known isometries, see [22, page 60]:
1) $L_{w^*}(X^*, Y) \simeq L_{w^*}(Y^*, X)$, $K_{w^*}(X^*, Y) \simeq K_{w^*}(Y^*, X)$ $(T \to T^*)$
2) $W(X,Y) \simeq L_{w^*}(X^{**}, Y)$ and $K(X,Y) \simeq K_{w^*}(X^{**}, Y)$ $(T \to T^{**})$.

Theorem 10. *Let $1 < p < \infty$. Let X and Y be Banach spaces and M be a closed subspace of $L_{w^*}(X^*, Y)$ such that the evaluation operator $\psi_{y^*}\colon M \to X$*

is p-limited cc for each $y^* \in Y^*$. If M does not have the p-limited rcP, then there is a separable subspace Y_0 of Y and an operator $A\colon Y_0 \to c_0$ which is not completely continuous.

Proof: Suppose M does not have the p-limited rcP. Let (T_n) be a weakly null p-limited sequence in M such that $\|T_n\| \not\to 0$. By passing to a subsequence, suppose that for some $\varepsilon > 0$, $\|T_n\| > \varepsilon$ for each n. Let (x_n^*) be a sequence in B_{X^*} so that $\|T_n(x_n^*)\| > \varepsilon$ for each n.

Let $y^* \in Y^*$. Since $\psi_{y^*} : M \to X$ is p-limited cc, $\|\psi_{y^*}(T_n)\| = \|T_n^*(y^*)\| \to 0$. Then $\langle y^*, T_n(x_n^*)\rangle = \langle T_n^*(y^*), x_n^*\rangle \le \|T_n^*(y^*)\| \to 0$. Therefore $(y_n) := (T_n(x_n^*))$ is weakly null in Y. By the Bessaga–Pełczyński selection principle [7, page 43], we may (and do) assume that (y_n) is a seminormalized weakly null basic sequence in Y. Let $Y_0 = [y_n]$ be the closed linear span of (y_n) and let (y_n^*) be the sequence of coefficient functionals associated with (y_n). Define $A\colon Y_0 \to c_0$ by $A(y) = (y_k^*(y))$, $y \in Y_0$. Note that $\|A(y_n)\| \ge 1$ for each n. Then A is a bounded linear operator defined on a separable space, and A is not completely continuous. $\square$

Corollary 11. (i) *Let $1 < p < \infty$. Suppose that X has the p-limited rcP and M is a closed subspace of $L_{w^*}(X^*, Y)$. If M does not have the p-limited rcP, then there is a separable subspace Y_0 of Y and an operator $A\colon Y_0 \to c_0$ which is not completely continuous.*

(ii) *Let $1 < p < \infty$. Suppose that X has the p-limited rcP and Y has the Schur property. If M is a closed subspace of $L_{w^*}(X^*, Y) = K_{w^*}(X^*, Y)$, then M has the p-limited rcP.*

(iii) *Let $1 < p < \infty$. Suppose that X has the Schur property and Y has the p-limited rcP. If M is a closed subspace of $L_{w^*}(X^*, Y) = K_{w^*}(X^*, Y)$, then M has the p-limited rcP.*

Proof: (i) Since X has the p-limited rcP, $\psi_{y^*} : M \to X$ is p-limited cc. Apply Theorem 10.

(ii) Suppose that X has the p-limited rcP and Y has the Schur property. Let $T \in L_{w^*}(X^*, Y)$. Since T is weakly compact and Y has the Schur property, T is compact. Suppose that M does not have the p-limited rcP. By Theorem 10, there is a non-completely continuous operator defined on a closed linear subspace Y_0 of Y. This is a contradiction since Y has the Schur property.

(iii) By the previous argument and the isometries 1), M has the p-limited rcP. $\square$

The space l_2 has the p-limited rcP, since $l_1 \not\hookrightarrow l_2$, see [16, Corollary 11]. The space $K_{w^*}(l_2, l_2)$ does not have the p-limited rcP, since $c_0 \hookrightarrow K_{w^*}(l_2, l_2)$ by [18, Theorem 20].

Theorem 12. *Let $1 \leq p < \infty$. Suppose that $K_{w^*}(X^*, Y)$ has the p-limited rcP. Then X and Y have the p-limited rcP property and either $l_2 \not\hookrightarrow X$ or $l_2 \not\hookrightarrow Y$. If moreover Y is a dual space Z^*, the condition $l_2 \not\hookrightarrow Y$ implies $l_1 \not\hookrightarrow Z$.*

Proof: Suppose that $K_{w^*}(X^*, Y)$ has the p-limited rcP. Then X and Y have the p-limited rcP, since the p-limited rcP is inherited by closed subspaces. Suppose $l_2 \hookrightarrow X$ and $l_2 \hookrightarrow Y$. Then $c_0 \hookrightarrow K_{w^*}(X^*, Y)$ by [18, Theorem 20]. This contradiction proves the first assertion.

Now suppose $Y = Z^*$ and $l_1 \hookrightarrow Z$. Then $L_1 \hookrightarrow Z^*$, see [7, page 212]. Also, the Rademacher functions span l_2 inside of L_1, hence $l_2 \hookrightarrow Z^*$. □

Corollary 13. *Let $1 \leq p < \infty$. Suppose that $K(X, Y)$ has the p-limited rcP. Then X^* and Y have the p-limited rcP and either $l_1 \not\hookrightarrow X$ or $l_2 \not\hookrightarrow Y$. If moreover Y is a dual space Z^*, the condition $l_2 \not\hookrightarrow Y$ implies $l_1 \not\hookrightarrow Z$.*

Proof: Apply Theorem 12 and the isometries 2). □

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