Czechoslovak Mathematical Journal

Xia Xu; Yong Yang

Results related to Huppert's ρ-σ conjecture

Czechoslovak Mathematical Journal, Vol. 73 (2023), No. 4, 1273–1280

Persistent URL: http://dml.cz/dmlcz/151959

Czechoslovak Mathematical Journal, 73 (148) (2023), 1273–1280

RESULTS RELATED TO HUPPERT'S ϱ-σ CONJECTURE

Xia Xu, Yichang, Yong Yang, San Marcos

Received November 3, 2022. Published online November 2, 2023.

Abstract. We improve a few results related to Huppert's ϱ-σ conjecture. We also generalize a result about the covering number of character degrees to arbitrary finite groups.

Keywords: character degree; Huppert's conjecture

MSC 2020: 20C15

1. Introduction

Let $\pi(n)$ denote the set of prime divisors of a positive integer n. Let G be a finite group and let $\pi(G)$ denote the set of prime divisors of its order $|G|$. Let $\mathrm{Irr}(G)$ denote the set of irreducible characters of G. We set

$$\sigma(G) = \max\{|\pi(\chi(1))|\colon\ \chi \in \mathrm{Irr}(G)\}$$

and

$$\varrho(G) = \{p \text{ prime}\colon\ p \mid \chi(1) \text{ for some } \chi \in \mathrm{Irr}(G)\}.$$

Huppert's ϱ-σ conjecture states that $|\varrho(G)|$ can be bounded in terms of $\sigma(G)$ and if G is solvable, then $|\varrho(G)| \leqslant 2\sigma(G)$. It is a problem of central importance in character theory. Many analogues of Huppert's conjecture were proposed and studied. In fact, the conjugacy class version of Huppert's conjecture was proposed by Huppert himself. The element order version of Huppert's conjecture was first introduced by Shi in [10]. Let g be an element in G and let $\pi(o(g))$ denote the set of prime divisors

This project was supported by grants from the Simons Foundation (Nos. 499532 and 918096, to Yong Yang).

DOI: 10.21136/CMJ.2023.0513-22

of the order of g. We set

$$\sigma^e(G) = \alpha(G) = \max\{|\pi(o(g))|: g \in G\}$$

and

$$\varrho^e(G) = \{p \text{ prime}: p \mid o(g) \text{ for some } g \in G\}.$$

Shi asked in [10] if $|\varrho^e(G)|$ is bounded by a function of $\sigma^e(G)$ (we note that $\varrho^e(G) = \pi(G)$). Answering Shi's question, Zhang in [13] proved that if G is solvable, then $|\varrho^e(G)|$ is bounded by a quadratic function of $\alpha(G)$ and that for arbitrary G, $|\varrho^e(G)|$ is bounded by a super-exponential function of $\alpha(G)$. The result for solvable groups was improved later by Keller (see [3]) to a linear bound. Keller showed that for $C > 4$, it is true that $|\varrho^e(G)| \leqslant C\sigma^e(G)$ when $\sigma^e(G) > \frac{1}{2}\mathrm{e}^{6C/(C-4)}$. Later, the constant of a solvable group was determined by Yang in [11]. Using this result, we can improve the element order version of Huppert's conjecture for arbitray finite groups.

Character codegrees were first defined in [9]. Let the codegree of a character χ be $\operatorname{codeg}(\chi) = |G : \ker\chi|/\chi(1)$. Then, we set

$$\operatorname{codeg}(G) = \{\operatorname{codeg}(\chi): \chi \in \operatorname{Irr}(G)\},$$
$$\sigma(\operatorname{codeg}(G)) = \max\{|\pi(\operatorname{codeg}(\chi))|: \chi \in \operatorname{Irr}(G)\},$$
$$\varrho(\operatorname{codeg}(G)) = \{p \text{ prime}: p \mid \operatorname{codeg}(\chi) \text{ for some } \chi \in \operatorname{Irr}(G)\}.$$

The codegree version of Huppert's conjecture was studied in [7], [11], [12].

The following can be viewed as another alternative of Huppert's ϱ-σ conjecture. Let G be a finite simple group and S be a subset of $\operatorname{Irr}(G)$. Then S is called a *covering set* of G if for every $p \in \pi(G)$ there is a character χ in S such that p divides $\chi(1)$. The covering number of G, denoted by $\operatorname{cn}(G)$, is defined as the minimal number of $\operatorname{Card}(S)$, where S is a covering set of G and $\operatorname{Card}(S)$ is the cardinality of the set S, see [4]. We know that for an arbitrary finite group G, there might exist $p \in \pi(G)$ that does not divide any irreducible character degree of G. Thus, in order to generalize the concept of covering number to arbitrary finite groups, we will consider the set $\varrho(G)$ instead of $\pi(G)$. In this note, we generalize the main result of [4] to arbitrary finite groups.

We will prove the following three results.

Theorem 1.1. *Let G be an arbitrary finite group. Then* $|\varrho(\operatorname{codeg}(G))| \leqslant \frac{43}{6}\sigma(\operatorname{codeg}(G))$.

Theorem 1.2. *Let G be a finite group. Then* $|\pi(G)| < 168\alpha(G)^4$.

Theorem 1.3. *Let G be a finite group, then the covering number* $\operatorname{cn}(G) \leqslant 6$.

2. Preliminaries

Lemma 2.1. *Let G be a finite solvable group. Then $|\varrho^e(G)| \leqslant \frac{17}{3}\alpha(G)$.*

Proof. See [11], Theorem 1.1. □

Lemma 2.2. *Let G be a finite group with trivial solvable radical, then $|\pi(G)| \leqslant \frac{3}{2}\sigma(\mathrm{codeg}(G))$.*

Proof. This is Theorem 1.3. of [7]. □

Lemma 2.3. *Suppose that $K/\mathrm{F}(K)$ is nilpotent and $C \trianglelefteq K$. Then there exists $\mu \in \mathrm{Irr}(C)$ such that $\pi(\mu(1)) = \pi(C/\mathrm{F}(K) \cap C)$.*

Proof. This is Proposition 17.3 of [5]. In fact, $\mu(1) = |C/\mathrm{F}(C)| = |(C/(\mathrm{F}(K) \cap C)|$. □

We set $\varrho_0(G) = \varrho(G) \setminus \{2,3\}$ and $\pi_0(G) = \pi(G) \setminus \{2,3\}$.

Lemma 2.4. *Suppose that M is a normal elementary abelian subgroup of the solvable group G. Assume that $M = \mathrm{C}_G(M)$ is a completely reducible G-module (possibly of mixed characteristic). Set $V = \mathrm{Irr}(M)$ and write $V = V_1 \oplus \ldots \oplus V_m$ for irreducible G-modules V_i. Write $V_i = Y_i^G$ for primitive modules Y_i for each i. Assume that $\mathrm{N}_G(Y_i)/\mathrm{C}_G(Y_i)$ is nilpotent by nilpotent for each i. If $M \leqslant N \trianglelefteq G$, there exist $\theta_1, \theta_2 \in \mathrm{Irr}(N)$ such that $\theta_1(1)\theta_2(1)$ is divisible by each prime in $\pi_0(N/M)$.*

Proof. First, we can write each V_i as a direct sum of the G-conjugates of Y_i, $i = 1, \ldots, m$. Thus, $V = X_1 \oplus \ldots \oplus X_n$ for subspaces X_i of V permuted by G (not necessarily transitively) with $\{Y_1, \ldots, Y_m\} \subseteq \{X_1, \ldots, X_n\}$. Furthermore, if $N_i = \mathrm{N}_G(X_i)$, $C_i = \mathrm{C}_G(X_i)$ and $F_i/C_i = \mathrm{F}(N_i/C_i)$, then X_i is a primitive, faithful N_i/C_i-module and N_i/F_i is nilpotent.

Let $K = \bigcap\limits_i N_i \trianglelefteq G$ be the kernel of the permutation representation of G on $\{X_1, \ldots, X_n\}$. Since $\bigcap\limits_i C_i = M$, we have $\bigcap\limits_i F_i/M = \mathrm{F}(K/M) \trianglelefteq G/M$. Let $H = \bigcap\limits_i F_i$, so that $H/M = \mathrm{F}(K/M)$. Observe that K/H is nilpotent. Set $C = K \cap N$ and $F = H \cap N = C \cap H$. By Lemma 2.3, there exists $\theta \in \mathrm{Irr}(C/M)$ such that $\theta(1) = |C/F|$. Since $C \trianglelefteq N$, there exists $\theta_1 \in \mathrm{Irr}(N)$ such that $|C/F| \mid \theta_1(1)$. Consequently it suffices to show that there exists $\theta_2 \in \mathrm{Irr}(N)$ with $\theta_2(1)$ divisible by each prime in $\pi_0(N/C) \cup \pi_0(F/M)$. To do this, we need just to find some $\lambda \in V$ such that $\pi_0(N : \mathrm{C}_N(\lambda)) \supseteq \pi_0(N/C) \cup \pi_0(F/M)$.

We can choose $\Delta \subseteq \{X_1, \ldots, X_n\}$ such that $\mathrm{stab}_N(\Delta)/(N \cap K) = \mathrm{stab}_N(\Delta)/C$ is a $\{2,3\}$-group by [5], Corollary 5.7. Assess that Δ intersects each N-orbit non-trivially. Without losing generality, $\Delta = \{X_1, \ldots, X_l\}$ for some $l \in \{1, \ldots, n\}$.

Set $\lambda = \lambda_1 \dots \lambda_l \in V$ for nonprincipal $\lambda_i \in X_i$. Finally suppose that $Q \in \mathrm{Syl}_q(N)$ for a prime $q \geqslant 5$ and Q centralizes λ. Thus, $Q \leqslant \mathrm{stab}_N(\Delta)$. But $\mathrm{stab}_N(\Delta)/C$ is a $\{2,3\}$-group. Thus, $Q \leqslant C$. The intersection $F_i \cap C/C_i \cap C$ is isomorphic to a normal nilpotent subgroup of N_i/C_i and N_i/C_i acts irreducibly on X_i for each i. Thus, λ_i is not centralized by a nontrivial Sylow subgroup of $F_i \cap C/C_i \cap C$ for $i = 1, \dots, l$. Since $Q \cap F_i \in \mathrm{Syl}_q(F_i \cap C)$, we have that $q \nmid |F_i \cap C/C_i \cap C|$ for $i = 1, \dots, l$. Each F_j/C_j $(j = 1, \dots, n)$ is conjugate to some F_i/C_i with $i \in \{1, \dots, l\}$ by our choice of Δ. Hence, $q \nmid |F_j \cap C/C_j \cap C|$ for $j = 1, \dots, n$. Since $\bigcap\limits_i C_i = M$ and $\bigcap\limits_i (F_i \cap C) = F$, we have that $q \nmid |F/M|$. We have already seen above that $Q \leqslant C$ and so $q \nmid |N/C|$. Thus, $|N : \mathrm{C}_N(\lambda)|$ is divisible by each prime in $\pi_0(N/C) \cup \pi_0(F/M)$. □

Lemma 2.5. *Suppose that $M = \mathrm{C}_G(M)$ is a normal elementary abelian subgroup of a solvable group G and a completely reducible G-module (possibly of mixed characteristic). Assume that G splits over M, then there exists $\chi_1, \chi_2 \in \mathrm{Irr}(G)$ such that $\chi_1(1)\chi_2(1)$ is divisible by each prime in $\pi_0(G/M)$.*

P r o o f. By induction on $|M|$. Write $M = M_1 \oplus \dots \oplus M_n$ for $n \geqslant 1$ irreducible G-modules M_i. Set $V_i = \mathrm{Irr}(M_i)$ so that each V_i is an irreducible G-module and $V = V_1 \oplus \dots \oplus V_n$ is a faithful G/M-module by Proposition 12.1 of [5]. For each i, choose $H_i \leqslant G$ and X_i to be an irreducible primitive H_i-module with $X_i^G = V_i$. If $H_i/\mathrm{C}_{H_i}(X_i) \leqslant \Gamma(X_i)$ for each i, the result follows from Lemma 2.4. We assess without losing generality that $H_1/\mathrm{C}_{H_1}(X_1) \nleqslant \Gamma(X_1)$.

Let $K = \mathrm{C}_G(M_1) \trianglelefteq G$. Let H be a complement for M in G and let $J = NH$, where $N = M_2 \oplus \dots \oplus M_n$. Then $J \cap M = N$. Now $J \cap K = N(H \cap K)$ acts on N and $\mathrm{C}_{J\cap K}(N) = N$. By induction, there exist $\mu_1, \mu_2 \in \mathrm{Irr}(J \cap K)$ such that $\mu_1(1)\mu_2(1)$ is divisible by the primes in $\pi_0((J \cap K)/N) = \pi_0(K/M)$, as $(J \cap K)/N \cong K/M$. Now $J \cap K \trianglelefteq J$ and centralizes $M/N \cong M_1$. Thus, $J \cap K \trianglelefteq KJ = G$ and $K/N = M/N \times (J \cap K)/N$. By the choice of M_1, there exists $\lambda \in V_1$ such that $\pi_0(G/K) = \pi_0(G : I_G(\lambda))$. Set $\beta_1 = \lambda \cdot \mu_1 \in \mathrm{Irr}(K)$, $\beta_2 = \lambda \cdot \mu_2 \in \mathrm{Irr}(K)$. Now $I_G(\beta_1) \cup I_G(\beta_2) \subseteq I_G(\lambda)$. Thus, $\pi_0(G : I_G(\beta_1)) \cup \pi_0(G : I_G(\beta_2)) \supseteq \pi_0(G/K)$. Choose $\chi_i \in \mathrm{Irr}(G \mid \beta_i)$ $(i = 1, 2)$, then, as $K \trianglelefteq G$, we have $\pi(G/K) \cup \pi(\mu_1(1)) \cup \pi(\mu_2(1)) \subseteq \pi(\chi_1(1)) \cup \pi(\chi_2(1))$. Since $\mu_1(1)\mu_2(1)$ is divisible by each prime in $\pi_0(K/M)$, $\chi_1(1)\chi_2(1)$ is divisible by each prime in $\pi_0(G/M)$. □

We note that the statements of Lemmas 2.4 and 2.5 are stronger than Lemmas 17.4 and 17.5 of [5], but the proof is similar.

Lemma 2.6. *Let G be a finite solvable group, then there exists $\mu_1, \mu_2 \in \mathrm{Irr}(G/\Phi(G))$ such that $\pi_0(G/\mathrm{F}(G)) \subseteq \bigcup\limits_{i=1}^{2} \pi_0(\mu_i(1))$.*

P r o o f. Apply Lemma 2.5 with $G/\Phi(G)$ and $\mathrm{F}(G)/\Phi(G)$, respectively, in the role of G and M. Note that $\mathrm{F}(G/\Phi(G)) = \mathrm{F}(G)/\Phi(G)$ is a completely reducible and faithful $G/\mathrm{F}(G)$-module (possibly of mixed characteristic). Furthermore, $G/\Phi(G)$ splits over $\mathrm{F}(G)/\Phi(G)$. □

Lemma 2.7. *Let G be a finite group with trivial fitting subgroup, then the covering number $\mathrm{cn}(G) \leqslant 3$. Especially, if $\mathrm{PSL}_2(q)$ or J_1 is not involved in G, then $\mathrm{cn}(G) \leqslant 2$.*

P r o o f. This is Theorem 1.1 of [4]. □

3. Main results

We now prove the first main result.

Theorem 3.1. *Let G be a finite group. Then $|\varrho(\mathrm{codeg}(G))| \leqslant \frac{43}{6}\sigma(\mathrm{codeg}(G))$.*

P r o o f. We note that $\varrho(\mathrm{codeg}(G)) = \pi(G)$ by [9], Lemma 2.4.

Let S be the largest solvable normal subgroup of G. By the main result of [8], we know that $\sigma^e(S) \leqslant \sigma(\mathrm{codeg}(S))$. We also know that $\sigma(\mathrm{codeg}(S)) \leqslant \sigma(\mathrm{codeg}(G))$ by [11], Lemma 2.2 (1). Thus, we have $|\pi(S)| \leqslant \frac{17}{3}\sigma(\mathrm{codeg}(S)) \leqslant \frac{17}{3}\sigma(\mathrm{codeg}(G))$ by Lemma 2.1.

By Lemma 2.2, we know that $|\pi(G/S)| \leqslant \frac{3}{2}\sigma(\mathrm{codeg}(G/S))$.

Since $\sigma(\mathrm{codeg}(G/S)) \leqslant \sigma(\mathrm{codeg}(G))$ by [11], Lemma 2.2 (2), we have $|\pi(G/S)| \leqslant \frac{3}{2}\sigma(\mathrm{codeg}(G))$. Thus, we have

$$|\pi(G)| \leqslant |\pi(S)| + |\pi(G/S)| \leqslant \Big(\frac{17}{3} + \frac{3}{2}\Big)\sigma(\mathrm{codeg}(G)) = \frac{43}{6}\sigma(\mathrm{codeg}(G)).$$

□

We now are ready to prove the second main result.

Theorem 3.2. *Let G be a finite group. Then $|\pi(G)| < 168\alpha(G)^4$.*

P r o o f. First note that if $\alpha(G) = 1$ then $|\pi(G)| \leqslant 4$. Therefore we may assume that $\alpha(G) \geqslant 2$. By Lemma 2.1, we know that $|\pi(G)| \leqslant \frac{17}{3}\alpha(G)$ for every solvable group G.

Following the proof of Theorem A of [6] by Moretó, let G be a minimal counterexample and introduce the series

$$1 = S_0 \leqslant R_1 < S_1 < R_2 < S_2 < \ldots < R_m < S_m \leqslant R_{m+1} = G,$$

such that R_{i+1}/S_i is the largest normal solvable subgroup of G/S_i for every $i \geqslant 0$ and S_i/R_i is the socle of G/R_i for every $i \geqslant 1$. Moretó then constructed a solvable subgroup H of G in the following way. First, for each $i = 1, \dots, m$, it was proved that there is a prime divisor q_i of $|S_i/R_i|$ that is relatively prime to $|G/S_i||R_i|$. Then, put $H := Q_1 Q_2 \dots Q_m$, where Q_i ($i = 1, 2, \dots, m-1$) is a Sylow q_i-subgroup of R_{i+1} that is $Q_{i+1} \dots Q_m$-invariant, and Q_m is simply a Sylow q_m-subgroup of G. Note that the primes q_i are pairwise distinct and thus, $|\pi(H)| = m$. By Lemma 2.1, we have $m \leqslant \frac{17}{3}\alpha(H)$, and thus, $m \leqslant \frac{17}{3}\alpha(G)$.

We now have

$$\begin{aligned} |\pi(G)| &\leqslant m \cdot \max_{1\leqslant i\leqslant m} |\pi(S_i/S_{i-1})| + |\pi(G/S_m)| \\ &\leqslant m \cdot \max_{1\leqslant i\leqslant m} \{|\pi(S_i/R_i)| + |\pi(R_i/S_{i-1})|\} + |\pi(G/S_m)| \\ &\leqslant \frac{17}{3}\alpha(G)\Big(28\alpha(G)^3 + \frac{17}{3}\alpha(G)\Big) + \frac{17}{3}\alpha(G) < 168\alpha(G)^4, \end{aligned}$$

where the third inequality follows from $\varrho(G) < 28\alpha(G)^3$ by [2], Theorem 3.2 and the fact that $|\pi(G)| \leqslant \frac{17}{3}\alpha(G)$ for every solvable group G, and the last inequality comes from $\alpha(G) \geqslant 2$. $\square$

Remark 3.1. After the paper has been submitted for publication, we noticed that the coefficient in Lemma 2.1 has been improved to 5 by [1], Theorem 1. In view of this, the coefficient in Theorem 3.1 can be improved to $\frac{13}{2}$ and the coefficient in Theorem 3.2 can be improved to 141.

We now prove the third main result.

Theorem 3.3. *Let G be a finite solvable group, then there exist $\chi_1, \chi_2, \chi_3 \in \operatorname{Irr}(G)$ such that $\varrho_0(G) \subseteq \bigcup_{i=1}^{3} \pi_0(\chi_i(1))$.*

Proof. By the Ito-Michler theorem a prime p does not divide the degree of any irreducible character of a group G if and only if G has a normal abelian Sylow p-subgroup. Thus, $p \in \varrho(G)$ if and only if $p \mid |G/\mathrm{F}(G)|$ or $\mathrm{F}(G)$ has a nonabelian Sylow p-subgroup. Let π be the set of primes r for which $\mathrm{O}_r(G)$ is nonabelian and $r \nmid |G/\mathrm{O}_p(G)|$. Then there exists $\mu \in \operatorname{Irr}(G)$ such that $r \mid \mu(1)$ for all $r \in \pi$. By Lemma 2.6, there exists $\chi_1, \chi_2 \in \operatorname{Irr}(G/\Phi(G))$ such that $\pi_0(G/\mathrm{F}(G)) \subseteq \bigcup_{i=1}^{2} \pi_0(\chi_i(1))$. Thus, we have

$$\varrho_0(G) = \pi_0(G/\mathrm{F}(G)) \cup \pi \subseteq \bigcup_{i=1}^{2} \pi_0(\chi_i(1)) \cup \pi_0(\mu(1)).$$

This completes the proof. $\square$

Theorem 3.4. *Let G be a finite group, then the covering number $\operatorname{cn}(G) \leqslant 6$.*

P r o o f. Let G be a solvable group, we have $\varrho_0(G) \subseteq \bigcup_{i=1}^{3} \pi_0(\chi_i(1))$ by Theorem 3.3, where $\chi_1, \chi_2, \chi_3 \in \operatorname{Irr}(G)$. If 2 or 3 belongs to $\varrho(G)$, we can always find an irreducible character of G to cover it. Thus $\operatorname{cn}(G) \leqslant 5$.

Let G be a nonsolvable group. Let S be the largest solvable normal subgroup of G. By Theorem 3.3, there exists $\beta_1, \beta_2, \beta_3 \in \operatorname{Irr}(S)$ such that $\varrho_0(S) \subseteq \bigcup_{i=1}^{3} \pi_0(\beta_i(1))$. By Lemma 2.7, there exists $\mu_1, \mu_2, \mu_3 \in \operatorname{Irr}(G/S)$ such that $\pi(G/S) \subseteq \bigcup_{i=1}^{3} \pi(\mu_i(1))$.

It is easy to see that $\varrho(G) = \pi(G/S) \cup \varrho(S)$. Since the order of a nonabelian simple group is divisible by 2, we have $2 \in \pi(G/S)$. If $3 \mid |G/S|$, then $3 \in \pi(G/S)$. We have

$$\varrho(G) = \pi(G/S) \cup \varrho(S) \subseteq \bigcup_{i=1}^{3} \pi(\mu_i(1)) \cup \bigcup_{i=1}^{3} \pi_0(\beta_i(1)).$$

If $3 \nmid |G/S|$, then the only non-abelian simple groups involved in G/S will be $Sz(q)$. In this case, we have $\operatorname{cn}(G/S) \leqslant 2$ by Lemma 2.7. In other words, there exists $\theta_1, \theta_2 \in \operatorname{Irr}(G/S)$ such that $\pi(G/S) \subseteq \bigcup_{i=1}^{2} \pi(\theta_i(1))$. If $3 \in \varrho(S)$, then there exists $\tau \in \operatorname{Irr}(G)$ such that $3 \mid \tau(1)$. We have

$$\varrho(G) = \pi(G/S) \cup \varrho(S) \subseteq \bigcup_{i=1}^{2} \pi(\theta_i(1)) \cup \bigcup_{i=1}^{3} \pi_0(\beta_i(1)) \cup \pi(\tau(1)).$$

Therefore, $\operatorname{cn}(G) \leqslant 6$ for all finite groups. □

Acknowledgement. The authors are grateful to the referee for the valuable suggestions which improved the manuscript.

References

[1] *C. Bellotti, T. M. Keller, T. S. Trudgian*: New bounds for numbers of primes in element orders of finite groups. Available at `https://arxiv.org/abs/2211.05837` (2022), 6 pages. doi
[2] *N. N. Hung, Y. Yang*: On the prime divisors of element orders. Math. Nachr. *294* (2021), 1905–1911. zbl MR doi
[3] *T. M. Keller*: A linear bound for $\varrho(n)$. J. Algebra *178* (1995), 643–652. zbl MR doi
[4] *Y. Liu*: On covering number of groups with trivial Fitting subgroup. Acta Math. Sin., Engl. Ser. *38* (2022), 1277–1284. zbl MR doi
[5] *O. Manz, T. R. Wolf*: Representations of Solvable Groups. London Mathematical Society Lecture Notes Series 185. Cambridge University Press, Cambridge, 1993. zbl MR doi
[6] *A. Moretó*: On the number of different prime divisors of element orders. Proc. Am. Math. Soc. *134* (2006), 617–619. zbl MR doi
[7] *A. Moretó*: Huppert's conjecture for character codegrees. Math. Nachr. *294* (2021), 2232–2236. zbl MR doi

[8] *G. Qian*: A note on element orders and character codegrees. Arch. Math. *97* (2011), 99–103. zbl MR doi
[9] *G. Qian, Y. Wang, H. Wei*: Co-degrees of irreducible characters in finite groups. J. Algebra *312* (2007), 946–955. zbl MR doi
[10] *W. Shi*: Characterization of simple groups using orders and related topics. Adv. Math., Beijing *20* (1991), 135–141. zbl MR
[11] *Y. Yang*: On analogues of Huppert's conjecture. Bull. Aust. Math. Soc. *104* (2021), 272–277. zbl MR doi
[12] *Y. Yang, G. Qian*: The analog of Huppert's conjecture on character codegrees. J. Algebra *478* (2017), 215–219. zbl MR doi
[13] *J. Zhang*: Arithmetical conditions on element orders and group structure. Proc. Am. Math. Soc. *123* (1995), 39–44. zbl MR doi

Author's address: Xia Xu, College of Science, China Three Gorges University, Yichang, Hubei 443002, P. R. China, e-mail: `xuxia1128@hotmail.com`; Yong Yang (corresponding author), Department of Mathematics, Texas State University, 601 University Drive, San Marcos, TX 78666, e-mail: `yang@txstate.edu`.