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RELAXATION-TIME LIMITS OF GLOBAL SOLUTIONS IN FULL
QUANTUM HYDRODYNAMIC MODEL FOR SEMICONDUCTORS

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Abstract. This paper is concerned with the global well-posedness and relaxation-time limits for the solutions in the full quantum hydrodynamic model, which can be used to analyze the thermal and quantum influences on the transport of carriers in semiconductor devices. For the Cauchy problem in \mathbb{R}^3 , we prove the global existence, uniqueness and exponential decay estimate of smooth solutions, when the initial data are small perturbations of an equilibrium state. Moreover, we show that the solutions converge into that of the simplified quantum energy-transport model and the quantum drift-diffusion model for the moment relaxation limit, and the moment and energy relaxation limit, respectively.

Keywords: quantum hydrodynamic equation; quantum Euler-Poisson system; bipolar semiconductor model; relaxation-time limit

MSC 2020: 35Q40, 76Y05, 82D37, 35B40

1. INTRODUCTION

We consider the following full quantum hydrodynamical (FQHD) model for semiconductors (see [15], [6]):

$$(1.1) \quad \begin{cases} \varrho_t + \operatorname{div}(\varrho \mathbf{u}) = 0, \\ (\varrho \mathbf{u})_t + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho, \theta) = \hbar^2 \varrho \nabla \left(\frac{\Delta \sqrt{\varrho}}{\sqrt{\varrho}} \right) + \varrho \mathbf{V} - \frac{\varrho \mathbf{u}}{\tau_m}, \\ \varrho \theta_t + \varrho \mathbf{u} \cdot \nabla \theta + \frac{2p(\varrho, \theta)}{3} \operatorname{div} \mathbf{u} = \frac{2}{3} \operatorname{div}(\kappa \nabla \theta) + \frac{\hbar^2}{3} \operatorname{div}(\varrho \Delta \mathbf{u}) - \frac{2\beta}{3\tau_e} \varrho(\theta - \bar{\theta}), \\ \lambda \operatorname{div} \mathbf{V} = \varrho - \bar{\varrho}, \quad \operatorname{rot} \mathbf{V} = 0, \quad x \in \mathbb{R}^3, \quad t > 0, \end{cases}$$

where $\varrho(x, t)$ is the electron density, $\mathbf{u}(x, t) = (u_1, u_2, u_3)$ is the velocity, $\theta(x, t)$ is the temperature, $\mathbf{V}(x, t) = (V_1, V_2, V_3)$ is the electric field and $p(\varrho, \theta) = \varrho \theta$ is the

pressure. Also, $\hbar > 0$ is the (scaled) Planck constant, $\tau_m > 0$ is the momentum relaxation time, $\kappa > 0$ is the heat conductivity, $\tau_e > 0$ is the energy relaxation time, $\bar{\theta} > 0$ is the temperature of the semiconductor lattice in energy units, $\lambda > 0$ is the Debye length, $\bar{\varrho} > 0$ is the doping profile and $\beta > 0$.

If p is a function of ϱ only, then the corresponding barotropic model reads as

$$(1.2) \quad \begin{cases} \varrho_t + \operatorname{div}(\varrho \mathbf{u}) = 0, \\ (\varrho \mathbf{u})_t + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) = \hbar^2 \varrho \nabla \left(\frac{\Delta \sqrt{\varrho}}{\sqrt{\varrho}} \right) + \varrho \mathbf{V} - \frac{\varrho \mathbf{u}}{\tau_m}, \\ \lambda \operatorname{div} \mathbf{V} = \varrho - \bar{\varrho}, \quad \operatorname{rot} \mathbf{V} = 0, \end{cases}$$

which is called the unipolar isentropic (or isothermal) QHD model for semiconductors.

In the present paper, we investigate the relaxation-time limits ($\tau_m \rightarrow 0$ and/or $\tau_e \rightarrow 0$) for FQHD model (1.1). In the real simulations of semiconductor devices, the size of the device is rather small (in nano-size, for instance). This in turn makes the scaled parameters τ_m, τ_e rather smaller due to different situations under consideration. The typical values of the parameters for semiconductors are given in [24]. Therefore, one of the both mathematically and physically important problems is to justify the asymptotic approximation (or behavior) of the macroscopic observable of the quantum hydrodynamical model subject to the small parameters mentioned above.

First, we are interested in the global well-posedness of solutions for QHD model (1.2) or FQHD model (1.1). For last two decades, there have been many mathematical studies about the unipolar QHD model (1.2) in different settings. For the stationary system in 1-D case, Jüngel, Li [12], [13] proved the existence, uniqueness and exponential stability of the subsonic stationary solution to the Dirichlet-Neumann mixed boundary value problem. For the stationary system in multi-D case, Jüngel [11] proved the existence of solutions to the boundary value problem in bounded domains, and Dong [5] studied the existence and semi-classical limit of the boundary value problem with a mixed boundary condition. For the non-stationary system in 1-D case, the existence and semi-classical limit on the solutions were studied in [9], [10] for the Cauchy problem and in [27] for the initial boundary value problem of bounded interval. Also, for non-stationary system in multi-D case, Li, Marcati [21] proved the existence and exponential decay of the smooth solutions to the periodic boundary value problem in torus \mathbb{T}^3 , and Jüngel, Li, Matsumura [14] showed the existence and relaxation-time limit of the smooth solutions to the Cauchy problem in \mathbb{R}^3 . However, to the best of our knowledge, there seems not to be any result for the existence of solutions to non-isentropic unipolar

QHD model (1.1) except for [28], where they proved the global well-posedness and exponential decay for system (1.1) with $\tau_m = \tau_e$.

Next, we turn to the analysis of relaxation-time limits. To this end, let us introduce the diffusion scaling as

$$x \rightarrow x, \quad t \rightarrow \frac{t}{\tau_m}, \quad (\varrho^{\tau_m}, \mathbf{u}^{\tau_m}, \theta^{\tau_m}, \mathbf{V}^{\tau_m})(x, t) = \left(\varrho, \frac{\mathbf{u}}{\tau_m}, \theta, \mathbf{V} \right) \left(x, \frac{t}{\tau_m} \right).$$

Then (1.1) can be rewritten as

$$(1.3) \quad \begin{aligned} \varrho_t^{\tau_m} + \operatorname{div}(\varrho^{\tau_m} \mathbf{u}^{\tau_m}) &= 0, \\ \tau_m^2 (\varrho^{\tau_m} \mathbf{u}^{\tau_m})_t + \tau_m^2 \operatorname{div}(\varrho^{\tau_m} \mathbf{u}^{\tau_m} \otimes \mathbf{u}^{\tau_m}) + \nabla(\varrho^{\tau_m} \theta^{\tau_m}) \\ &= \hbar^2 \varrho^{\tau_m} \nabla \left(\frac{\Delta \sqrt{\varrho^{\tau_m}}}{\sqrt{\varrho^{\tau_m}}} \right) + \varrho^{\tau_m} \mathbf{V}^{\tau_m} - \varrho^{\tau_m} \mathbf{u}^{\tau_m}, \\ \tau_m \varrho^{\tau_m} \theta_t^{\tau_m} + \tau_m \varrho^{\tau_m} \mathbf{u}^{\tau_m} \cdot \nabla \theta^{\tau_m} + \tau_m \frac{2\varrho^{\tau_m} \theta^{\tau_m}}{3} \operatorname{div} \mathbf{u}^{\tau_m} \\ &= \frac{2}{3} \operatorname{div}(\kappa \nabla \theta^{\tau_m}) + \tau_m \frac{\hbar^2}{3} \operatorname{div}(\varrho^{\tau_m} \Delta \mathbf{u}^{\tau_m}) - \frac{2\beta}{3\tau_e} \varrho^{\tau_m} (\theta^{\tau_m} - \bar{\theta}), \\ \lambda \operatorname{div} \mathbf{V}^{\tau_m} &= \varrho^{\tau_m} - \bar{\varrho}, \quad \operatorname{rot} \mathbf{V}^{\tau_m} = 0. \end{aligned}$$

Formally, if we take only the moment relaxation-time limit $\tau_m \rightarrow 0$ for the fixed energy relaxation-time $\tau_e > 0$, then the simplified quantum energy-transport (QET) model

$$(1.4) \quad \begin{aligned} \varrho_t + \operatorname{div} \left[\varrho \mathbf{V} - \nabla(\varrho \theta) + \hbar^2 \varrho \nabla \left(\frac{\Delta \sqrt{\varrho}}{\sqrt{\varrho}} \right) \right] &= 0, \\ -\operatorname{div}(\kappa \nabla \theta) &= \frac{\beta \varrho}{\tau_e} (\bar{\theta} - \theta), \\ \lambda \operatorname{div} \mathbf{V} = \varrho - \bar{\varrho}, \quad \operatorname{rot} \mathbf{V} &= 0, \end{aligned}$$

is obtained, which is considered by [16], where they proved existence of global weak solutions to the periodic boundary value problem of system (1.4) in torus \mathbb{T}^d ($d \leq 3$). Recently, [19] showed the stability of the stationary solution of system (1.4) with $\hbar = 0$ on an interval.

On the other hand, if we take the moment and energy relaxation-time limit, that is, $\tau_m \rightarrow 0$ and $\tau_e \rightarrow 0$, then the quantum drift-diffusion (QDD) model

$$(1.5) \quad \varrho_t + \operatorname{div} \left[\varrho \mathbf{V} - \bar{\theta} \nabla \varrho + \hbar^2 \varrho \nabla \left(\frac{\Delta \sqrt{\varrho}}{\sqrt{\varrho}} \right) \right] = 0, \quad \lambda \operatorname{div} \mathbf{V} = \varrho - \bar{\varrho}, \quad \operatorname{rot} \mathbf{V} = 0,$$

is obtained, which is considered by many papers (see [17], [8], [18], [4], [2], [22], [1], [3], [7], [26], [29] and the references therein).

In fact, there are some mathematical results for the relaxation-time limit of the isentropic QHD model (1.2), which is reduced to system (1.5) as $\tau_m \rightarrow 0$ (see [14], [31], [23], [2], [1] and the references therein). However, to the best of our knowledge, there seems not to be any result for the relaxation-time limits of non-isentropic QHD model (1.1), which is the main goal of this paper.

Notation 1.1. *In this paper, $L^p(\mathbb{R}^3)$ and $W_p^k(\mathbb{R}^3)$ denote the usual Lebesgue and Sobolev spaces on \mathbb{R}^3 , with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$, respectively. When $p = 2$, we denote $W_p^k(\mathbb{R}^3)$ by $H^k(\mathbb{R}^3)$ with the norm $\|\cdot\|_{H^k}$ and $\|\cdot\|_{H^0} = \|\cdot\|$, which will be used to denote the usual L^2 -norm. We use the following notation:*

$$\mathcal{H}^k(\mathbb{R}^3) = \{w \in L^6(\mathbb{R}^3) \mid \nabla w \in H^{k-1}(\mathbb{R}^3)\}, \quad k \geq 1.$$

The notation $\|(A_1, A_2, \dots, A_l)\|_{H^k}$ means the summation of $\|A_i\|_{H^k}$ from $i = 1$ to $i = l$. For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we denote $D^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$ and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. For an integer m , the symbol ∇^m denotes the summation of all terms D^α with the multi-index α satisfying $|\alpha| = m$. We use C, c to denote the constants which are independent of x, t and may change from line to line. We also omit the spatial domain \mathbb{R}^3 in integrals for convenience. For simplicity, we use the notation $a \lesssim b$ which means $a \leq C_0 b$ for a universal constant C_0 depending only on $\hbar, \bar{\varrho}, \bar{\theta}, \beta$ and κ .

2. MAIN RESULTS AND PRELIMINARY

2.1. Main results. We consider the Cauchy problem of system (1.1) with the initial condition in \mathbb{R}^3 :

$$(2.1) \quad (\varrho, \mathbf{u}, \theta)(x, 0) = (\varrho_0, \mathbf{u}_0, \theta_0)(x) \rightarrow (\bar{\varrho}, 0, \bar{\theta}) \quad \text{as } |x| \rightarrow \infty.$$

From now on, we set the scaled Debye length to be one $\lambda = 1$ for simplicity.

First of all, we have the global existence and uniqueness theory of the Cauchy problem (1.1), (2.1).

Theorem 2.1 (Global existence). *Let $\bar{\varrho}, \bar{\theta}, \kappa$ and β be fixed positive constants. Assume that for the positive parameters τ_m, τ_e of system (1.1), it holds*

$$(2.2) \quad \tau_m \leq 1, \quad \tau_e \leq 1 \quad \text{and} \quad \frac{4\bar{\theta}^2}{\beta} \tau_m \tau_e \leq \hbar^2 < \frac{8\kappa\bar{\theta}}{\bar{\varrho}}.$$

Also, suppose that

$$(2.3) \quad (\sqrt{\varrho_0} - \sqrt{\bar{\varrho}}, \mathbf{u}_0, \theta_0 - \bar{\theta}) \in H^4(\mathbb{R}^3) \times \mathcal{H}^3(\mathbb{R}^3) \times H^2(\mathbb{R}^3).$$

Then there exists a positive constant $\eta_0 > 0$ depending on $\hbar, \beta, \kappa, \bar{\varrho}, \bar{\theta}$ but independent of τ_m, τ_e such that if

$$(2.4) \quad \begin{aligned} & \|\sqrt{\varrho_0} - \sqrt{\bar{\varrho}}\|_{H^2}^2 + \|\nabla \mathbf{u}_0\|^2 + \tau_m \|\theta_0 - \bar{\theta}\|_{H^2}^2 \\ & \quad + \tau_m (\|\nabla^3 \sqrt{\varrho_0}\|_{H^1}^2 + \|\nabla^2 \mathbf{u}_0\|_{H^1}^2) \leq \eta_0^2, \end{aligned}$$

the Cauchy problem (1.1), (2.1) admits a unique solution $(\varrho, \mathbf{u}, \theta, \mathbf{V})$ on $[0, \infty)$ satisfying

$$(2.5) \quad \begin{aligned} & \inf_{(x,t) \in \mathbb{R}^3 \times [0, \infty)} \varrho(x, t) > 0, \quad \inf_{(x,t) \in \mathbb{R}^3 \times [0, \infty)} \theta(x, t) > 0, \\ & \sqrt{\varrho} - \sqrt{\bar{\varrho}} \in C([0, \infty); H^4(\mathbb{R}^3)) \cap C^1([0, \infty); H^2(\mathbb{R}^3)), \\ & \mathbf{u} \in C([0, \infty); \mathcal{H}^3(\mathbb{R}^3)), \quad \mathbf{V} \in C([0, \infty); \mathcal{H}^5(\mathbb{R}^3)), \\ & \theta - \bar{\theta} \in C([0, \infty); H^2(\mathbb{R}^3)), \quad \nabla \theta \in L^2(0, \infty; H^2(\mathbb{R}^3)) \end{aligned}$$

and

$$(2.6) \quad \begin{aligned} & \|(\sqrt{\varrho} - \sqrt{\bar{\varrho}})(t)\|_{H^4}^2 + \|\mathbf{u}(t)\|_{\mathcal{H}^3}^2 + \|(\theta - \bar{\theta})(t)\|_{H^2}^2 + \|\mathbf{V}(t)\|_{\mathcal{H}^5}^2 \\ & \leq C(\|\sqrt{\varrho_0} - \sqrt{\bar{\varrho}}\|_{H^4}^2 + \|\mathbf{u}_0\|_{\mathcal{H}^3}^2 + \|\theta_0 - \bar{\theta}\|_{H^2}^2) e^{-ct} \end{aligned}$$

for any $t \in [0, \infty)$, where C, c are positive constants dependent on t and η_0 .

Remark 2.1. (1) In the real simulations of semiconductor devices, the size of the device is rather small (in nano-size, for instance). This in turn makes the scaled parameters \hbar, τ_m, τ_e rather smaller due to different situations under consideration. The typical values of the parameters for semiconductors are given in [24]. Therefore, one of the both mathematically and physically important problems is to justify the asymptotic approximation of the macroscopic observable of the quantum hydrodynamical model subject to the small parameters mentioned above. The third condition in (2.2) requires that the relaxation parameters τ_m and τ_e are rather smaller than the scaled Planck constant \hbar .

(2) Condition (2.2) holds for small τ_m and τ_e , which is not needed for isentropic QHD model (1.2). In fact, Jüngel-Li-Matsumura [14] proved the existence, exponential decay and relaxation-time limit for the Cauchy problem of isentropic QHD model (1.2) with $D(x) \neq \text{const.}$, when $(\sqrt{\varrho_0} - \sqrt{D}, \mathbf{u}_0) \in H^6 \times \mathcal{H}^5$. Also, we would like to emphasise there exists a recent result ([28]) similar to Theorem 2.1 in the case of the full QHD model (1.1) with $\tau_m = \tau_e$.

Next, we consider the relaxation-time limits for FQHD model (1.1). To this end, we consider indeed the initial value problem for the re-scaled system (1.3) together with the following initial data:

$$(2.7) \quad (\varrho^{\tau_m}, \mathbf{u}^{\tau_m}, \theta^{\tau_m})(x, 0) := (\varrho_0^{\tau_m}, \mathbf{u}_0^{\tau_m}, \theta_0^{\tau_m})(x) = \left(\varrho_0, \frac{\mathbf{u}_0}{\tau_m}, \theta_0 \right)(x), \quad x \in \mathbb{R}^3.$$

By Theorem 2.1, it is easy to verify that there is a unique global smooth solution $(\varrho^{\tau_m}, \mathbf{u}^{\tau_m}, \theta^{\tau_m}, \mathbf{V}^{\tau_m})$ of system (1.3), (2.7) satisfying (2.5). What is left is to establish the uniform estimates with respect to the parameters τ_m and/or τ_e in order to pass into the limits. We have the following theorem.

Theorem 2.2 (Momentum relaxation-time limit). *Under the assumptions of Theorem 2.1, let $(\varrho^{\tau_m}, \mathbf{u}^{\tau_m}, \theta^{\tau_m}, \mathbf{V}^{\tau_m})$ be the unique global solution of the system (1.3), (2.7) satisfying (2.5), (2.6). Then there exists a limit $(\varrho, \theta, \mathbf{V})$ as $\tau_m \rightarrow 0$ satisfying*

$$(2.8) \quad \begin{aligned} \sqrt{\varrho} - \sqrt{\bar{\varrho}} &\in L^\infty(0, \infty; H^2(\mathbb{R}^3)) \cap L^2(0, \infty; H^4(\mathbb{R}^3)), \quad \varrho_t \in L^2(0, \infty; L^2(\mathbb{R}^3)), \\ \theta - \bar{\theta} &\in L^2(0, \infty; H^3(\mathbb{R}^3)), \quad \mathbf{V} \in L^\infty(0, \infty; \mathcal{H}^3(\mathbb{R}^3)), \end{aligned}$$

where $(\varrho, \theta, \mathbf{V})$ is the strong solution of the Cauchy problem for QET model (1.4) with initial condition

$$(2.9) \quad \varrho(x, 0) = \varrho_0(x), \quad x \in \mathbb{R}^3.$$

Theorem 2.3 (Momentum and energy relaxation-time limit). *Under the assumptions of Theorem 2.1, let $(\varrho^{\tau_m, \tau_e}, \mathbf{u}^{\tau_m, \tau_e}, \theta^{\tau_m, \tau_e}, \mathbf{V}^{\tau_m, \tau_e})$ be the unique global solution of system (1.3), (2.7) satisfying (2.5). Then there exists a limit (ϱ, \mathbf{V}) as $\tau_m \rightarrow 0$ and $\tau_e \rightarrow 0$ such that*

$$(2.10) \quad \begin{aligned} \sqrt{\varrho} - \sqrt{\bar{\varrho}} &\in L^\infty(0, \infty; H^2(\mathbb{R}^3)) \cap L^2(0, \infty; H^4(\mathbb{R}^3)), \\ \varrho_t &\in L^2(0, \infty; L^2(\mathbb{R}^3)), \quad \mathbf{V} \in L^\infty(0, \infty; \mathcal{H}^3(\mathbb{R}^3)), \end{aligned}$$

and (ϱ, \mathbf{V}) is the strong solution of the Cauchy problem for QDD model (1.5) with initial condition (2.9).

Remark 2.2. (1) The same results as in Theorems 2.1–2.3 are also obtained for the periodic boundary value problem of the full QHD model (1.1) in torus $\mathbb{T}^d = (0, 1)^d$ ($d = 2, 3$) (see Li, Marcati [21] for the isentropic QHD model (1.2)).

(2) It is possible to generalize the results in Theorems 2.1–2.3 into the case of the more general state equations, that is, $p_\varrho(\varrho, \theta) > 0$ and $e_\theta(\varrho, \theta) > 0$, instead of $p = \varrho\theta$ and $e = 3\theta/2$, respectively, where e is the inertial energy. It will be left for the future work.

Highlight of this paper. The main purpose of this paper is to justify the asymptotic approximation of the macroscopic observable of the full QHD model subject to the small relaxation-time parameters τ_m and τ_e . The main steps in proving Theorems 2.1–2.3 are to derive the a priori estimates for $(\psi, \mathbf{u}, \zeta, \mathbf{V})$ independent

of the time T and the small relaxation-time parameters τ_m, τ_e . Here, we briefly review the main differences, compared with our previous work [28], where we obtained the same result as in Theorem 2.1 for fixed relaxation-time $\tau = \tau_m = \tau_e$. In fact, for the fixed relaxation-time $\tau_m > 0$, if we multiply the reformulated equation (3.6) on the momentum equation (1.1)₂ by ψ_t , then we have the uncontrolled term $\frac{1}{2}\bar{\theta} \int (1 + \psi) \nabla \zeta \cdot \nabla \psi_t dx$ (see (3.18)), while if we multiply the reformulated equation (3.8) on the energy equation (1.3)₃ by $(1 + \psi)^2 \zeta$, then we have the uncontrolled term $-(2\hbar^2 \tau_m / (3\bar{\theta})) \int (1 + \psi) \nabla \psi_t \cdot \nabla \zeta dx$ (see (3.20)). In [28], Lemma 3.1, we have taken the strategy to eliminate two uncontrolled terms by multiplying the second term by $3\bar{\theta}^2 / (4\hbar^2 \tau_m)$ for the fixed $\tau_m > 0$. However, in this paper, we cannot borrow the strategy because we study the relaxation-time limit $\tau_m \rightarrow 0$. To circumvent the difficulties, we first make a term $\frac{1}{2}(\tau_m - 1)\bar{\theta} \int (1 + \psi) \nabla \psi_t \cdot \nabla \zeta dx$ to survive by multiplying the second term by $3\bar{\theta}^2 / (4\hbar^2)$ (see (3.25)). Therefore, a term which can not be small was generated in lower order estimate (see (3.12)). Next, we obtain the high order estimate with all small controlled terms (see (3.33)). Then, combining the lower and higher estimates, we could get the desired estimate, independent of T , τ_m and τ_e , needed for the study of relaxation-time limits.

2.2. Auxiliary results. We need the following standard results.

Lemma 2.1 (Gagliardo-Nirenberg inequality, [25]). *Let l, s and k be any integers satisfying $0 \leq l, s < k$, and let $p, r, q \in [1, \infty]$ and $l/k \leq \theta \leq 1$ such that*

$$\frac{1}{p} - \frac{l}{3} = \left(\frac{1}{q} - \frac{k}{3}\right)\theta + \left(\frac{1}{r} - \frac{s}{3}\right)(1 - \theta).$$

Then, for any $u \in W_q^k(\mathbb{R}^3)$, there exists a positive constant $C > 0$ depending only on q, k, r and s such that

$$\|\nabla^l u\|_{L^p} \leq C \|\nabla^k u\|_{L^q}^\theta \|\nabla^s u\|_{L^r}^{1-\theta},$$

where $\|\nabla^l \cdot\|_{L^p} = \sum_{|\alpha|=l} \|D^\alpha \cdot\|_{L^p}$.

Lemma 2.2 ([14], Lemma 2.7). *Let $f \in H^s(\mathbb{R}^3)$, $s \geq 3/2$. There exists a unique solution $\mathbf{V} \in \mathcal{H}^{s+1}$ of the divergence equation*

$$\operatorname{div} \mathbf{V} = f, \quad \operatorname{rot} \mathbf{V} = 0, \quad \mathbf{V}(x) \rightarrow 0 \quad (|x| \rightarrow \infty)$$

satisfying

$$\|\mathbf{V}\|_{L^6} \leq C_s \|f\|, \quad \|\nabla \mathbf{V}\|_{H^s} \leq C_s \|f\|_{H^s}.$$

Last, we shall frequently use the following Moser-type calculus inequalities:

Lemma 2.3 ([20]). *For $f, g \in H^s(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $|\alpha| \leq s$, $s \geq \frac{3}{2}$, it holds that*

$$\|D^\alpha(fg)\| \leq C_s(\|f\|_{L^\infty} \|\nabla^s g\| + \|g\|_{L^\infty} \|\nabla^s f\|).$$

3. A PRIORI ESTIMATES

This section is devoted to derive the a priori estimates, which play a crucial role in the proof of our main results.

Since we are interested in not only the global existence theory but also the relaxation-time limits of smooth solutions with respect to small parameters, we deal with the Cauchy problem (1.3), (2.7) directly, because the scaled Cauchy problem (1.3), (2.7) is equivalent to the original Cauchy problem (1.1), (2.1) for smooth solutions.

We first reformulate the Cauchy problem (1.3), (2.7). For simplicity, we omit the index τ_m in the following argument. That is, we take

$$(\varrho^{\tau_m}, \mathbf{u}^{\tau_m}, \theta^{\tau_m}, \mathbf{V}^{\tau_m}) = (\varrho, \mathbf{u}, \theta, \mathbf{V}).$$

Then, setting

$$\psi = \frac{\sqrt{\bar{\varrho}} - \sqrt{\varrho}}{\sqrt{\bar{\varrho}}} \quad \text{and} \quad \zeta = \frac{\theta}{\bar{\theta}} - 1,$$

we obtain from (1.3) and (2.7) (see [28], Subsection 2.1) that

$$(3.1) \quad \begin{aligned} \psi_t + \mathbf{u} \cdot \nabla \psi + \frac{1 + \psi}{2} \operatorname{div} \mathbf{u} &= 0, \\ \tau_m^2 \mathbf{u}_t + \mathbf{u} + \frac{\tau_m^2}{2} \nabla(\mathbf{u}^2) - \tau_m^2 \operatorname{rot} \mathbf{u} \times \mathbf{u} + \frac{2\bar{\theta}(1 + \zeta)}{1 + \psi} \nabla \psi + \bar{\theta} \nabla \zeta &= \hbar^2 \nabla \left(\frac{\Delta \psi}{1 + \psi} \right) + \mathbf{V}, \\ \tau_m \zeta_t + \frac{2\beta}{3\tau_e} \zeta - \frac{2\kappa}{3\bar{\varrho}} \Delta \zeta + \frac{2\tau_m}{3} \operatorname{div} \mathbf{u} - \frac{\hbar^2 \tau_m}{3\bar{\theta}} \Delta(\operatorname{div} \mathbf{u}) &= F^{\tau_m}, \\ \operatorname{div} \mathbf{V} = \bar{\varrho}(\psi^2 + 2\psi), \quad \operatorname{rot} \mathbf{V} &= 0 \end{aligned}$$

and

$$(3.2) \quad (\psi, \tau_m \mathbf{u}, \zeta)(x, 0) = \left(\frac{\sqrt{\varrho_0(x)} - \sqrt{\bar{\varrho}}}{\sqrt{\bar{\varrho}}}, \mathbf{u}_0(x), \frac{\theta_0(x)}{\bar{\theta}} - 1 \right),$$

respectively, which is convenient to obtain the a priori estimates of $(\psi, \mathbf{u}, \zeta, \mathbf{V})$, where

$$(3.3) \quad F^{\tau_m} = \frac{\hbar^2 \tau_m}{3\varrho\bar{\theta}} \nabla[\bar{\varrho}(1 + \psi)^2] \cdot \Delta \mathbf{u} - \tau_m \mathbf{u} \cdot \nabla \zeta + \frac{2\kappa}{3} \left(\frac{1}{\bar{\varrho}(1 + \psi)^2} - \frac{1}{\bar{\varrho}} \right) \Delta \zeta - \frac{2\tau_m}{3} \zeta \operatorname{div} \mathbf{u}.$$

Define the solution space $X(0, T)$ for system (3.1), (3.2) by

$$\begin{aligned} X(0, T) := \{ & (\psi, \mathbf{u}, \zeta, \mathbf{V}) \mid \psi \in C([0, T]; H^4(\mathbb{R}^3)) \cap C^1([0, T]; H^2(\mathbb{R}^3)), \\ & \mathbf{u} \in C([0, T]; \mathcal{H}^3(\mathbb{R}^3)), \mathbf{V} \in C([0, T]; \mathcal{H}^5(\mathbb{R}^3)), \\ & \zeta \in C([0, T]; H^2(\mathbb{R}^3)), \nabla \zeta \in L^2(0, T; H^2(\mathbb{R}^3))\}, \end{aligned}$$

where $0 < T \leq \infty$. Also, we assume a priori that there exists a small positive constant $\eta_1 \leq 1$ depending on $\hbar, \beta, \kappa, \bar{\rho}, \bar{\theta}$ but independent of τ_m, τ_e such that

$$(3.4) \quad \sup_{0 \leq t \leq T} (\|\psi(t)\|_{H^2}^2 + \tau_m \|\nabla^3 \psi(t)\|_{H^1}^2 + \tau_m^2 \|(\psi_t, \nabla \mathbf{u})(t)\|^2 + \tau_m^3 \|\nabla(\psi_t, \nabla \mathbf{u})(t)\|_{H^1}^2 + \tau_m \|\zeta(t)\|_{H^2}^2) \leq \eta_1^2,$$

which is necessary to obtain the a priori estimates independent of T, τ_m and τ_e . By using (3.4) and Lemma 2.1, we can choose $\eta_1 > 0$ such that

$$(3.5) \quad 0 < \frac{1}{2} \leq 1 + \psi(x, t) \leq \frac{3}{2}.$$

Next, we reformulate some equations in (3.1) for the convenience in a priori estimates. To this end, we assume that $(\psi, \mathbf{u}, \zeta, \mathbf{V}) \in X(0, T)$ is the solution of Cauchy problem (3.1), (3.2) satisfying (3.5). Differentiating (3.1)₁ for t , we get

$$\psi_{tt} + \frac{1}{2(1+\psi)} \operatorname{div}((1+\psi)^2 \mathbf{u}_t) + \nabla \psi_t \cdot \mathbf{u} + \frac{\psi_t}{2} \operatorname{div} \mathbf{u} = 0,$$

and using (3.1)₂ yields (see [28], Subsection 2.1) that

$$(3.6) \quad \tau_m^2 \psi_{tt} + \psi_t + \bar{\rho} \psi + \frac{\hbar^2}{2} \Delta^2 \psi - \bar{\theta} \Delta \psi - \frac{\bar{\theta}}{2} (1+\psi) \Delta \zeta = f_1^{\tau_m},$$

where

$$(3.7) \quad \begin{aligned} f_1^{\tau_m} = & -\tau_m^2 \nabla \psi_t \cdot \mathbf{u} - \frac{\tau_m^2}{2} \psi_t \operatorname{div} \mathbf{u} - \frac{\bar{\theta}}{2} (3\psi^2 + \psi^3) - \nabla \psi \cdot \mathbf{V} \\ & + \frac{\tau_m^2}{2} (1+\psi) \operatorname{div}[(\mathbf{u} \cdot \nabla) \mathbf{u}] + \tau_m^2 \nabla \psi \cdot ((\mathbf{u} \cdot \nabla) \mathbf{u}) \\ & + \frac{\bar{\theta}}{1+\psi} \operatorname{div}[(1+\psi) \zeta \nabla \psi] + \frac{\hbar^2}{2} \frac{(\Delta \psi)^2}{1+\psi} + \bar{\theta} \frac{|\nabla \psi|^2}{1+\psi} + \bar{\theta} \nabla \psi \cdot \nabla \zeta. \end{aligned}$$

Also, using (3.1)₁, we rewrite (3.1)₃ as

$$(3.8) \quad \tau_m \zeta_t + \frac{2\beta}{3\tau_e} \zeta - \frac{2\kappa}{3\bar{\rho}} \Delta \zeta - \frac{4\tau_m}{3} \psi_t + \frac{2\hbar^2 \tau_m}{3\bar{\theta}} \Delta \left(\frac{\psi_t}{1+\psi} \right) = g_1^{\tau_m},$$

where

$$(3.9) \quad g_1^{\tau_m} = F^{\tau_m} - \frac{4\tau_m}{3(1+\psi)} (\psi \psi_t - \nabla \psi \cdot \mathbf{u}) - \frac{\hbar^2 \tau_m}{3\bar{\theta}} \Delta \left(\frac{\nabla \psi \cdot \mathbf{u}}{1+\psi} \right).$$

Last, taking rot to (3.1)₂ yields (see [28], Subsection 2.1) that

$$(3.10) \quad \tau_m^2 (\text{rot } \mathbf{u})_t + \text{rot } \mathbf{u} - \tau_m^2 \text{rot}(\text{rot } \mathbf{u} \times \mathbf{u}) + \text{rot} \left(\frac{2\bar{\theta}(1+\zeta)}{1+\psi} \nabla \psi + \bar{\theta} \nabla \zeta \right) = 0.$$

Noticing that

$$\begin{aligned} \text{rot}(\text{rot } \mathbf{u} \times \mathbf{u}) &= \text{rot } \mathbf{u} \text{div } \mathbf{u} + (\mathbf{u} \cdot \nabla) \text{rot } \mathbf{u} - (\text{rot } \mathbf{u} \cdot \nabla) \mathbf{u}, \\ \text{rot} \left(\frac{2\bar{\theta}(1+\zeta)}{1+\psi} \nabla \psi + \bar{\theta} \nabla \zeta \right) &= 2\bar{\theta} \nabla \left(\frac{1+\zeta}{1+\psi} \right) \times \nabla \psi = \frac{2\bar{\theta}}{1+\psi} \nabla \zeta \times \nabla \psi, \end{aligned}$$

we obtain from (3.10)

$$(3.11) \quad \begin{aligned} \tau_m^2 (\text{rot } \mathbf{u})_t + \text{rot } \mathbf{u} - \tau_m^2 \text{rot } \mathbf{u} \text{div } \mathbf{u} \\ - \tau_m^2 (\text{rot } \mathbf{u} \cdot \nabla) \mathbf{u} + \tau_m^2 (\mathbf{u} \cdot \nabla) \text{rot } \mathbf{u} = - \frac{2\bar{\theta}}{1+\psi} \nabla \zeta \times \nabla \psi. \end{aligned}$$

We first will prove the following lower order estimate:

Lemma 3.1. *Under assumption (3.4), it follows that*

$$(3.12) \quad \frac{d}{dt} E_0(t) + \Lambda_0(t) \leq 4\bar{\theta} \|\nabla^2 \zeta\| \|\psi_t\| + C\eta_1 (\|\nabla^2 \psi\|_{H^2}^2 + \|\nabla^2 \zeta\|^2 + \tau_m \|\nabla^2 \mathbf{u}\|^2),$$

where

$$(3.13) \quad \begin{aligned} E_0(t) &:= \frac{1}{2} \|\psi\|^2 + \tau_m^2 \int \psi \psi_t \, dx + \tau_m^2 \|\psi_t\|^2 + \bar{\varrho} \|\psi\|^2 \\ &\quad + \frac{\hbar^2}{2} \|\Delta \psi\|^2 + \bar{\theta} \|\nabla \psi\|^2 + \frac{3\bar{\theta}^2 \tau_m}{4\hbar^2} \|(1+\psi)\zeta\|^2 + \tau_m^2 \|\text{rot } \mathbf{u}\|^2, \\ \Lambda_0(t) &:= \left(1 - \frac{\tau_m^2}{2}\right) \|\psi_t\|^2 - \frac{\bar{\theta}^2 \tau_m}{\hbar^2} \int \psi_t \zeta \, dx + \frac{\beta \bar{\theta}^2}{2\hbar^2 \tau_e} \|\zeta\|^2 + \|\text{rot } \mathbf{u}\|^2 \\ &\quad + \frac{\bar{\theta}}{2} \|\nabla \psi\|^2 + \frac{\bar{\theta}}{4} \int \nabla \psi \cdot \nabla \zeta \, dx + \frac{\kappa \bar{\theta}^2}{2\hbar^2 \bar{\varrho}} \|\nabla \zeta\|^2 + \frac{\hbar^2}{4} \|\Delta \psi\|^2 + \frac{\bar{\varrho}}{2} \|\psi\|^2 \end{aligned}$$

and C is a positive constant depending only on \hbar , $\bar{\varrho}$, $\bar{\theta}$, β and κ .

Remark 3.1. By using (2.2) and (3.5), we obtain from (3.13)

$$(3.14) \quad E_0(t) \simeq \|\psi\|_{H^2}^2 + \tau_m^2 \|(\psi_t, \text{rot } \mathbf{u})\|^2 + \tau_m \|\zeta\|^2$$

and

$$(3.15) \quad \Lambda_0(t) \simeq \|\psi\|_{H^2}^2 + \|(\psi_t, \text{rot } \mathbf{u})\|^2 + \frac{1}{\tau_e} \|\zeta\|^2 + \|\nabla \zeta\|^2,$$

where $A \simeq B$ means $CA \leq B \leq A/C$ for a generic constant $C > 0$ depending only on \hbar , $\bar{\varrho}$, $\bar{\theta}$, β and κ .

Proof of Lemma 3.1. Multiplying (3.6) by ψ and integrating it over \mathbb{R}^3 , we obtain (cf., [28], (3.4)) that

$$(3.16) \quad \frac{1}{2} \frac{d}{dt} \left(\|\psi\|^2 + 2\tau_m^2 \int \psi \psi_t \, dx \right) - \tau_m^2 \|\psi_t\|^2 + \bar{\varrho} \|\psi\|^2 + \frac{\hbar^2}{2} \|\Delta\psi\|^2 \\ + \bar{\theta} \|\nabla\psi\|^2 + \frac{\bar{\theta}}{2} \int \nabla\zeta \cdot \nabla\psi \, dx = I_0^1,$$

where

$$(3.17) \quad I_0^1 = \tau_m^2 \int \psi_t \operatorname{div}(\psi \mathbf{u}) \, dx - \tau_m^2 \int \frac{\psi_t}{2} \operatorname{div} \mathbf{u} \psi \, dx - \frac{\tau_m^2}{2} \int ((\mathbf{u} \cdot \nabla) \mathbf{u}) \cdot \nabla\psi \, dx \\ + \int \frac{\bar{\theta}}{1+\psi} \operatorname{div}[(1+\psi)\zeta \nabla\psi] \psi \, dx - \int \left(\frac{\bar{\varrho}}{2} (3\psi^2 + \psi^3) + \nabla\psi \cdot \mathbf{V} \right) \psi \, dx \\ + \int \left(\frac{\hbar^2}{2} \frac{(\Delta\psi)^2}{1+\psi} + \bar{\theta} \frac{|\nabla\psi|^2}{1+\psi} + \bar{\theta} \nabla\psi \cdot \nabla\zeta \right) \psi \, dx.$$

Multiplying (3.6) by ψ_t and integrating it over \mathbb{R}^3 , we obtain (cf., [28], (3.6)) that

$$(3.18) \quad \frac{1}{2} \frac{d}{dt} \left(\tau_m^2 \|\psi_t\|^2 + \bar{\varrho} \|\psi\|^2 + \frac{\hbar^2}{2} \|\Delta\psi\|^2 + \bar{\theta} \|\nabla\psi\|^2 \right) + \|\psi_t\|^2 \\ + \frac{\bar{\theta}}{2} \int (1+\psi) \nabla\zeta \cdot \nabla\psi_t \, dx = I_0^2,$$

where

$$(3.19) \quad I_0^2 = \tau_m^2 \int \left[\frac{1+\psi}{2} \operatorname{div}[(\mathbf{u} \cdot \nabla) \mathbf{u}] + \nabla\psi \cdot ((\mathbf{u} \cdot \nabla) \mathbf{u}) \right] \psi_t \, dx \\ - \int \left(\frac{\bar{\varrho}}{2} (3\psi^2 + \psi^3) + \nabla\psi \cdot \mathbf{V} - \frac{\bar{\theta}}{1+\psi} \operatorname{div}[(1+\psi)\zeta \nabla\psi] \right) \psi_t \, dx \\ + \int \left(\frac{\hbar^2}{2} \frac{(\Delta\psi)^2}{1+\psi} + \bar{\theta} \frac{|\nabla\psi|^2}{1+\psi} + \frac{\bar{\theta}}{2} \nabla\psi \cdot \nabla\zeta \right) \psi_t \, dx.$$

Multiplying (3.8) by $(1+\psi)^2\zeta$ and integrating it over \mathbb{R}^3 , we obtain (cf., [28], (3.9)) that

$$(3.20) \quad \frac{\tau_m}{2} \frac{d}{dt} \|(1+\psi)\zeta\|^2 + \frac{2\beta}{3\tau_e} \|\zeta\|^2 + \frac{2\kappa}{3\bar{\varrho}} \|\nabla\zeta\|^2 \\ - \frac{4\tau_m}{3} \int \psi_t \zeta \, dx - \frac{2\hbar^2\tau_m}{3\bar{\theta}} \int (1+\psi) \nabla\psi_t \cdot \nabla\zeta \, dx = I_0^3,$$

where

$$(3.21) \quad \begin{aligned} I_0^3 = & \int (1 + \psi)^2 g_1^{\tau_m} \zeta \, dx - \frac{2\beta}{3\tau_e} \int \psi(2 + \psi) \zeta^2 \, dx - \frac{2\kappa}{3\bar{\varrho}} \int \psi(2 + \psi) |\nabla \zeta|^2 \, dx \\ & - \frac{4\kappa}{3\bar{\varrho}} \int (1 + \psi) \zeta \nabla \psi \cdot \nabla \zeta \, dx + \frac{4\tau_m}{3} \int \psi(2 + \psi) \psi_t \zeta \, dx + \tau_m \int (1 + \psi) \psi_t \zeta^2 \, dx \\ & - \frac{2\hbar^2 \tau_m}{3\theta} \int \psi_t [2 \operatorname{div}(\zeta \nabla \psi) + \nabla \psi \cdot \nabla \zeta + 2\zeta \frac{|\nabla \psi|^2}{1 + \psi}] \, dx. \end{aligned}$$

Multiplying (3.11) by $\operatorname{rot} \mathbf{u}$ and integrating it over \mathbb{R}^3 , we have

$$(3.22) \quad \frac{\tau_m^2}{2} \frac{d}{dt} \|\operatorname{rot} \mathbf{u}\|^2 + \|\operatorname{rot} \mathbf{u}\|^2 = I_0^4,$$

where

$$(3.23) \quad \begin{aligned} I_0^4 = & \frac{3\tau_m^2}{2} \int (\operatorname{rot} \mathbf{u} \cdot \operatorname{rot} \mathbf{u}) \operatorname{div} \mathbf{u} \, dx + \tau_m^2 \int (\operatorname{rot} \mathbf{u} \cdot \nabla) \mathbf{u} \cdot \operatorname{rot} \mathbf{u} \, dx \\ & - 2\bar{\theta} \int \frac{\nabla \zeta \times \nabla \psi}{1 + \psi} \cdot \operatorname{rot} \mathbf{u} \, dx. \end{aligned}$$

Here, we used the following formula:

$$(3.24) \quad \int (\mathbf{a} \cdot \nabla) \mathbf{b} \cdot \mathbf{c} \, dx = - \int (\mathbf{b} \cdot \mathbf{c}) \operatorname{div} \mathbf{a} \, dx - \int (\mathbf{a} \cdot \nabla) \mathbf{c} \cdot \mathbf{b} \, dx.$$

Thus, taking the procedure

$$(3.16) \times \frac{1}{2} + (3.18) + (3.20) \times \frac{3\bar{\theta}^2}{4\hbar^2} + (3.22),$$

we get

$$(3.25) \quad \frac{1}{2} \frac{d}{dt} E_0(t) + \Lambda_0(t) = \frac{1}{2} I_0^1 + I_0^2 + \frac{3\bar{\theta}^2}{4\hbar^2} I_0^3 + I_0^4 + (\tau_m - 1) \frac{\bar{\theta}}{2} \int (1 + \psi) \nabla \zeta \cdot \nabla \psi_t \, dx,$$

where $E_0(t)$ and $\Lambda_0(t)$ are defined by (3.13).

We estimate all the terms of the right-hand side of (3.25). First, for the term I_0^1 , using (3.5), Lemma 2.1, Lemma 2.2 and (3.4), we obtain (cf., [28], (3.18)) from (3.17) that

$$(3.26) \quad I_0^1 \lesssim \eta_1 (\|\psi_t\|^2 + \|\nabla \mathbf{u}\|^2 + \|\psi\|_{H^2}^2 + \|\zeta\|_{H^2}^2).$$

Using (3.5), Lemma 2.1, Lemma 2.2 and (3.4), we obtain (cf., [28], (3.20)) from (3.19) that

$$(3.27) \quad |I_0^2| \lesssim \eta_1 (\|\psi_t\|^2 + \|\nabla \mathbf{u}\|^2 + \|\psi\|_{H^2}^2 + \|\Delta \psi\|_{H^2}^2 + \|\zeta\|_{H^2}^2).$$

Noticing that $\tau_m \leq 1$, we obtain from (3.23) that

$$(3.28) \quad \begin{aligned} |I_0^4| &\lesssim \tau_m^2 \|\nabla \mathbf{u}\|_{L^6} \|\nabla \mathbf{u}\|_{L^3} \|\nabla \mathbf{u}\| + \|\nabla \zeta\|_{L^3} \|\nabla \psi\|_{L^6} \|\nabla \mathbf{u}\| \\ &\lesssim (\tau_m^{3/2} \|\nabla^2 \mathbf{u}\|) \tau_m^{1/2} \|\nabla^2 \mathbf{u}\|^{1/2} \|\nabla \mathbf{u}\|^{3/2} + \|\nabla^2 \psi\| \|\nabla \zeta\|^{1/2} \|\nabla^2 \zeta\|^{1/2} \|\nabla \mathbf{u}\| \\ &\lesssim \eta_1 (\tau_m^2 \|\nabla^2 \mathbf{u}\|^2 + \|\nabla \mathbf{u}\|^2 + \|\nabla \zeta\|_{H^1}^2). \end{aligned}$$

On the other hand, by the same lines as in [28], (3.23), we get

$$(3.29) \quad \int |(1 + \psi)^2 g_1 \zeta| \, dx \lesssim \eta_1 (\|\psi\|_{H^3}^2 + \|\zeta\|_{H^2}^2).$$

Thus, by (3.21) and (3.29), we have

$$(3.30) \quad \begin{aligned} I_0^3 &\lesssim \int |(1 + \psi)^2 g_1 \zeta| \, dx + \|\psi\|_{L^\infty} \left(\frac{1}{\tau_e} \|\zeta\|^2 + \|\nabla \zeta\|^2 \right) \\ &\quad + \|\nabla \psi\|_{L^6} \|\zeta\|_{L^3} \|\nabla \zeta\| + \tau_m \|\psi_t\| (\|\psi\|_{L^6} \|\zeta\|_{L^3} + \|\zeta\|_{L^3} \|\zeta\|_{L^6}) \\ &\quad + \tau_m \|\psi_t\| (\|\nabla \psi\|_{L^3} \|\nabla \zeta\|_{L^6} + \|\Delta \psi\| \|\zeta\|_{L^\infty} + \|\zeta\|_{L^\infty} \|\nabla \psi\|_{L^6} \|\nabla \psi\|_{L^3}) \\ &\lesssim \eta_1 (\|\psi\|_{H^3}^2 + \frac{1}{\tau_e} \|\zeta\|^2 + \|\nabla \zeta\|_{H^1}^2). \end{aligned}$$

Also, using (3.5), (3.4) and $\tau_m \leq 1$, we have

$$(3.31) \quad \begin{aligned} \frac{(\tau_m - 1)\bar{\theta}}{2} \int (1 + \psi) \nabla \zeta \cdot \nabla \psi_t \, dx &= \frac{(1 - \tau_m)\bar{\theta}}{2} \int \operatorname{div}[(1 + \psi) \nabla \zeta] \psi_t \, dx \\ &\leq \bar{\theta} (\|\nabla \psi\|_{L^3} \|\nabla \zeta\|_{L^6} + \frac{3}{2} \|\Delta \zeta\|) \|\psi_t\| \\ &\leq \bar{\theta} (C\eta_1 \|\nabla^2 \zeta\| + \frac{3}{2} \|\Delta \zeta\|) \|\psi_t\| \leq 2\bar{\theta} \|\psi_t\| \|\nabla^2 \zeta\|. \end{aligned}$$

Substituting (3.26), (3.27), (3.28), (3.30) and (3.31) into (3.25), and using $\tau_m \leq 1$ and $\tau_e \leq 1$, we have

$$(3.32) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} E_0(t) + \Lambda_0(t) - 2\bar{\theta} \|\psi_t\| \|\zeta\|_{H^2} \\ &\lesssim \eta_1 \left(\|\psi_t\|^2 + \|\nabla \mathbf{u}\|^2 + \|\psi\|_{H^4}^2 + \frac{1}{\tau_e} \|\zeta\|^2 + \|\nabla \zeta\|_{H^1}^2 + \tau_m^2 \|\nabla^2 \mathbf{u}\|^2 \right) \\ &\lesssim \eta_1 \left(\|(\psi_t, \operatorname{rot} \mathbf{u})\|^2 + \|\psi\|_{H^4}^2 + \frac{1}{\tau_e} \|\zeta\|^2 + \|\nabla \zeta\|_{H^1}^2 + \tau_m^2 \|\nabla^2 \mathbf{u}\|^2 \right) \\ &\stackrel{(3.15)}{\lesssim} \eta_1 (\Lambda_0(t) + \|\nabla^2 \psi\|_{H^2}^2 + \|\nabla^2 \zeta\|^2 + \tau_m^2 \|\nabla^2 \mathbf{u}\|^2), \end{aligned}$$

where we used $\|\nabla \mathbf{u}\|^2 = \|\operatorname{div} \mathbf{u}\|^2 + \|\operatorname{rot} \mathbf{u}\|^2$ and

$$\|\operatorname{div} \mathbf{u}\| \lesssim \|\psi_t\| + \eta_1 \|\nabla \psi\|$$

due to (3.1)₁.

By using the smallness of $\eta_1 > 0$, estimate (3.12) follows directly from (3.32). The proof of Lemma 3.1 is completed. \square

Next, we prove the higher order estimate:

Lemma 3.2. *Under assumption (3.4), it follows that*

$$(3.33) \quad \frac{d}{dt}E_\alpha(t) + \Lambda_\alpha(t) \leq C\eta_1 \left(\|\psi\|_{H^4}^2 + \|(\psi_t, \operatorname{rot} \mathbf{u})\|^2 \right. \\ \left. + \tau_m \|\nabla(\psi_t, \operatorname{rot} \mathbf{u})\|_{H^1}^2 + \frac{1}{\tau_e} \|\zeta\|_{H^2}^2 + \|\nabla\zeta\|_{H^2}^2 \right)$$

for $|\alpha| = 1$ or 2 and any $t \in [0, T]$, where

$$(3.34) \quad E_\alpha(t) := \frac{1}{2} \|D^\alpha \psi\|^2 + \tau_m^2 \int D^\alpha \psi D^\alpha \psi_t \, dx + \tau_m^3 \|D^\alpha \psi_t\|^2 + \tau_m \bar{\varrho} \|D^\alpha \psi\|^2 \\ + \frac{\hbar^2 \tau_m}{2} \|D^\alpha \Delta \psi\|^2 + \bar{\theta} \tau_m \|D^\alpha \nabla \psi\|^2 \\ + \frac{3\bar{\theta}^2 \tau_m}{4\hbar^2} \|(1 + \psi)D^\alpha \zeta\|^2 + \tau_m^3 \|\operatorname{rot} D^\alpha \mathbf{u}\|^2, \\ \Lambda_\alpha(t) := \left(\tau_m - \frac{\tau_m^2}{2} \right) \|D^\alpha \psi_t\|^2 - \frac{\bar{\theta}^2 \tau_m}{\hbar^2} \int D^\alpha \psi_t D^\alpha \zeta \, dx + \frac{\beta \bar{\theta}^2}{2\hbar^2 \tau_e} \|D^\alpha \zeta\|^2 \\ + \frac{\bar{\theta}}{2} \|\nabla D^\alpha \psi\|^2 + \frac{\bar{\theta}}{4} \int \nabla D^\alpha \psi \cdot \nabla D^\alpha \zeta \, dx + \frac{\kappa \bar{\theta}^2}{2\hbar^2 \bar{\varrho}} \|\nabla D^\alpha \zeta\|^2 \\ + \frac{\hbar^2}{4} \|\Delta D^\alpha \psi\|^2 + \frac{\bar{\varrho}}{2} \|D^\alpha \psi\|^2 + \tau_m \|\operatorname{rot} D^\alpha \mathbf{u}\|^2$$

and C is a positive constant depending only on $\hbar, \bar{\varrho}, \bar{\theta}, \beta$ and κ .

Remark 3.2. By using (2.2) and (3.5), we obtain from (3.34)

$$(3.35) \quad E_\alpha(t) \simeq \|D^\alpha \psi\|^2 + \tau_m \|\nabla D^\alpha \psi\|_{H^1}^2 + \tau_m^3 \|D^\alpha(\psi_t, \operatorname{rot} \mathbf{u})\|^2 + \tau_m \|D^\alpha \zeta\|^2$$

and

$$(3.36) \quad \Lambda_\alpha(t) \simeq \|D^\alpha \psi\|_{H^2}^2 + \tau_m \|D^\alpha(\psi_t, \operatorname{rot} \mathbf{u})\|^2 + \frac{1}{\tau_e} \|D^\alpha \zeta\|^2 + \|D^\alpha \nabla \zeta\|^2.$$

Proof of Lemma 3.2. Applying D^α to (3.6), multiplying the resulting equation by $D^\alpha \psi$ and using the Leibniz formula

$$(3.37) \quad D^\alpha(wv) = wD^\alpha v + \sum_{1 \leq |\beta| \leq |\alpha|} \mathbb{C}_{|\alpha|}^{|\beta|} D^\beta w D^{\alpha-\beta} v,$$

we obtain (cf., [28], (3.29)) that

$$(3.38) \quad \frac{1}{2} \frac{d}{dt} \left(\|D^\alpha \psi\|^2 + 2\tau_m^2 \int D^\alpha \psi D^\alpha \psi_t \, dx \right) - \tau_m^2 \|D^\alpha \psi_t\|^2 + \bar{\varrho} \|D^\alpha \psi\|^2 \\ + \frac{\hbar^2}{2} \|\Delta D^\alpha \psi\|^2 + \bar{\theta} \|\nabla D^\alpha \psi\|^2 + \frac{\bar{\theta}}{2} \int \nabla D^\alpha \zeta \cdot \nabla D^\alpha \psi \, dx = I_{1,\alpha},$$

where

$$(3.39) \quad I_{1,\alpha} = \int \left(D^\alpha f_1^{\tau_m} + \frac{\bar{\theta}}{2} \sum_{1 \leq |\beta| \leq |\alpha|} \mathbb{C}_{|\alpha|}^{|\beta|} D^\beta (1 + \psi) \Delta D^{\alpha-\beta} \zeta \right) D^\alpha \psi \, dx \\ - \frac{\bar{\theta}}{2} \int \psi \nabla D^\alpha \zeta \cdot \nabla D^\alpha \psi \, dx - \frac{\bar{\theta}}{2} \int D^\alpha \psi \nabla D^\alpha \zeta \cdot \nabla \psi \, dx.$$

Applying D^α to (3.6), multiplying the resulting equation by $D^\alpha \psi_t$ and using (3.37), we obtain (cf., [28], (3.31)) that

$$(3.40) \quad \frac{1}{2} \frac{d}{dt} \left(\tau_m^2 \|D^\alpha \psi_t\|^2 + \bar{\varrho} \|D^\alpha \psi\|^2 + \frac{\hbar^2}{2} \|\Delta D^\alpha \psi\|^2 + \bar{\theta} \|\nabla D^\alpha \psi\|^2 \right) \\ + \|D^\alpha \psi_t\|^2 + \frac{\bar{\theta}}{2} \int (1 + \psi) \nabla D^\alpha \zeta \cdot \nabla D^\alpha \psi_t \, dx = I_{2,\alpha},$$

where

$$(3.41) \quad I_{2,\alpha} = \int \left(D^\alpha f_1^{\tau_m} + \frac{\bar{\theta}}{2} \sum_{1 \leq |\beta| \leq |\alpha|} \mathbb{C}_{|\alpha|}^{|\beta|} D^\beta (1 + \psi) \Delta D^{\alpha-\beta} \zeta \right) D^\alpha \psi_t \, dx \\ - \frac{\bar{\theta}}{2} \int D_t^\alpha \psi \nabla D^\alpha \zeta \cdot \nabla \psi \, dx.$$

Applying D^α to (3.8) and using the Leibniz formula (3.37), we have

$$(3.42) \quad \tau_m (D^\alpha \zeta)_t + \frac{2\beta}{3\tau_e} D^\alpha \zeta - \frac{2\kappa}{3\bar{\varrho}} \Delta D^\alpha \zeta - \frac{4\tau_m}{3} D^\alpha \psi_t \\ + \frac{2\hbar^2 \tau_m}{3\bar{\theta}} \Delta \left(\frac{D^\alpha \psi_t}{1 + \psi} \right) = D^\alpha g_1^{\tau_m} - \frac{2\hbar^2 \tau_m}{3\bar{\theta}} g_{2,\alpha},$$

where

$$(3.43) \quad g_{2,\alpha} = \sum_{1 \leq |\beta| \leq |\alpha|} \mathbb{C}_{|\alpha|}^{|\beta|} \Delta \left[D^\beta \left(\frac{1}{1 + \psi} \right) D^{\alpha-\beta} \psi_t \right].$$

Multiplying (3.42) by $(1 + \psi)^2 D^\alpha \zeta$ yields (cf., [28], (3.36)) that

$$(3.44) \quad \frac{\tau_m}{2} \frac{d}{dt} \|(1 + \psi) D^\alpha \zeta\|^2 + \frac{2\beta}{3\tau_e} \|D^\alpha \zeta\|^2 + \frac{2\kappa}{3\bar{\varrho}} \|\nabla D^\alpha \zeta\|^2 \\ - \frac{4\tau_m}{3} \int D^\alpha \psi_t D^\alpha \zeta \, dx - \frac{2\hbar^2 \tau_m}{3\bar{\theta}} \int (1 + \psi) \nabla D^\alpha \psi_t \cdot \nabla D^\alpha \zeta \, dx = I_{3,\alpha},$$

where

$$\begin{aligned}
(3.45) \quad I_{3,\alpha} &= \int (1 + \psi)^2 \left(D^\alpha g_1^{\tau_m} - \frac{2\hbar^2 \tau_m}{3\theta} g_{2,\alpha} \right) D^\alpha \zeta \, dx \\
&+ \tau_m \int (1 + \psi) \psi_t (D^\alpha \zeta)^2 \, dx - \frac{2\beta}{3\tau_e} \int \psi(2 + \psi) (D^\alpha \zeta)^2 \, dx \\
&- \frac{2\kappa}{3\bar{\rho}} \int \psi(2 + \psi) |\nabla D^\alpha \zeta|^2 \, dx - \frac{4\kappa}{3\bar{\rho}} \int (1 + \psi) D^\alpha \zeta \nabla \psi \cdot \nabla D^\alpha \zeta \, dx \\
&+ \frac{4\tau_m}{3} \int \psi(2 + \psi) D^\alpha \psi_t D^\alpha \zeta \, dx \\
&- \frac{2\hbar^2 \tau_m}{3\theta} \int D^\alpha \psi_t \left[2 \operatorname{div}(D^\alpha \zeta \nabla \psi) + \nabla \psi \cdot \nabla D^\alpha \zeta + 2D^\alpha \zeta \frac{|\nabla \psi|^2}{1 + \psi} \right] dx.
\end{aligned}$$

Applying D^α to (3.11) and using

$$\begin{aligned}
D^\alpha(f \times g) &:= \partial_{x_i}(f \times g) = g \times D^\alpha f + f \times D^\alpha g \quad \text{for } |\alpha| = 1, \\
D^\alpha(f \times g) &:= \partial_{x_i} \partial_{x_j}(f \times g) = g \times D^\alpha f + f \times D^\alpha g \\
&\quad + \partial_{x_i} f \times \partial_{x_j} g + \partial_{x_j} f \times g \partial_{x_i} g \quad \text{for } |\alpha| = 2,
\end{aligned}$$

we have

$$\begin{aligned}
(3.46) \quad &\tau_m^2 (\operatorname{rot} D^\alpha \mathbf{u})_t + \operatorname{rot} D^\alpha \mathbf{u} - \tau_m^2 \operatorname{rot} D^\alpha \mathbf{u} \operatorname{div} \mathbf{u} - \tau_m^2 (\operatorname{rot} D^\alpha \mathbf{u} \cdot \nabla) \mathbf{u} \\
&+ \tau_m^2 (D^\alpha \mathbf{u} \cdot \nabla) \operatorname{rot} \mathbf{u} - \tau_m^2 \operatorname{rot} \mathbf{u} \operatorname{div} D^\alpha \mathbf{u} \\
&- \tau_m^2 (\operatorname{rot} \mathbf{u} \cdot \nabla) D^\alpha \mathbf{u} + \tau_m^2 (\mathbf{u} \cdot \nabla) \operatorname{rot} D^\alpha \mathbf{u} \\
&= -D^\alpha \left(\frac{2\bar{\theta}}{1 + \psi} \nabla \zeta \right) \times \nabla \psi - \frac{2\bar{\theta}}{1 + \psi} \nabla \zeta \times \nabla D^\alpha \psi - \tau_m^2 \chi_{1,\alpha} + \chi_{2,\alpha},
\end{aligned}$$

where $\chi_{k,\alpha}$ ($k = 1, 2$) is zero for $|\alpha| = 1$ and

$$\begin{aligned}
\chi_{1,\alpha} &= -(\operatorname{rot} \mathbf{u})_{x_i} (\operatorname{div} \mathbf{u})_{x_j} - ((\operatorname{rot} \mathbf{u})_{x_i} \cdot \nabla) (\mathbf{u})_{x_j} + ((\mathbf{u})_{x_i} \cdot \nabla) (\operatorname{rot} \mathbf{u})_{x_j} \\
&\quad - (\operatorname{rot} \mathbf{u})_{x_j} (\operatorname{div} \mathbf{u})_{x_i} - ((\operatorname{rot} \mathbf{u})_{x_j} \cdot \nabla) (\mathbf{u})_{x_i} + ((\mathbf{u})_{x_j} \cdot \nabla) (\operatorname{rot} \mathbf{u})_{x_i}, \\
\chi_{2,\alpha} &= - \left(\frac{2\bar{\theta}}{1 + \psi} \nabla \zeta \right)_{x_i} \times (\nabla \psi)_{x_j} - \left(\frac{2\bar{\theta}}{1 + \psi} \nabla \zeta \right)_{x_j} \times (\nabla \psi)_{x_i}
\end{aligned}$$

for $|\alpha| = 2$. Multiplying (3.46) by $\operatorname{rot} D^\alpha \mathbf{u}$, integrating it over \mathbb{R}^3 and using (3.24), we have

$$(3.47) \quad \frac{\tau_m^2}{2} \frac{d}{dt} \|\operatorname{rot} D^\alpha \mathbf{u}\|^2 + \|\operatorname{rot} D^\alpha \mathbf{u}\|^2 = I_{4,\alpha},$$

where

$$\begin{aligned}
(3.48) \quad I_{4,\alpha} &= \tau_m^2 \int \left[\frac{3}{2} \operatorname{rot} D^\alpha \mathbf{u} \operatorname{div} \mathbf{u} + (\operatorname{rot} D^\alpha \mathbf{u} \cdot \nabla) \mathbf{u} - (D^\alpha \mathbf{u} \cdot \nabla) \operatorname{rot} \mathbf{u} \right] \operatorname{rot} D^\alpha \mathbf{u} \, dx \\
&\quad + \tau_m^2 \int [\operatorname{rot} \mathbf{u} \operatorname{div} D^\alpha \mathbf{u} + (\operatorname{rot} \mathbf{u} \cdot \nabla) D^\alpha \mathbf{u}] \operatorname{rot} D^\alpha \mathbf{u} \, dx \\
&\quad - \int \left[D^\alpha \left(\frac{2\bar{\theta}}{1+\psi} \nabla \zeta \right) \times \nabla \psi + \left(\frac{2\bar{\theta}}{1+\psi} \nabla \zeta \right) \times \nabla D^\alpha \psi \right] \operatorname{rot} D^\alpha \mathbf{u} \, dx \\
&\quad - \tau_m^2 \int \chi_{1,\alpha} \operatorname{rot} D^\alpha \mathbf{u} \, dx + \int \chi_{2,\alpha} \operatorname{rot} D^\alpha \mathbf{u} \, dx \\
&= I_{4,\alpha}^1 + I_{4,\alpha}^2 - I_{4,\alpha}^3 - I_{4,\alpha}^4 + I_{4,\alpha}^5.
\end{aligned}$$

Thus, from the procedure

$$(3.38) \times \frac{1}{2} + (3.40) \times \tau_m + (3.44) \times \frac{3\bar{\theta}^2}{4\hbar^2} + (3.47) \times \tau_m,$$

we get

$$(3.49) \quad \frac{1}{2} \frac{d}{dt} E_\alpha(t) + \Lambda_\alpha(t) = \frac{1}{2} I_{1,\alpha} + \tau_m I_{2,\alpha} + \frac{3\bar{\theta}^2}{4\hbar^2} I_{3,\alpha} + \tau_m I_{4,\alpha},$$

where $E_\alpha(t)$ and $\Lambda_\alpha(t)$ are defined by (3.34).

We estimate the terms of the right-hand side of (3.49). For the estimate on $I_{1,\alpha}$, using (3.39) and (3.7), we rewrite it as

$$(3.50) \quad I_{1,\alpha} = I_{1,\alpha}^1 + I_{1,\alpha}^2 + I_{1,\alpha}^3 + I_{1,\alpha}^4 + \frac{\bar{\theta}}{2} I_{1,\alpha}^5 - \frac{\bar{\theta}}{2} I_{1,\alpha}^6,$$

where

$$\begin{aligned}
I_{1,\alpha}^1 &= -(-1)^{|\alpha|} \int \left[\tau_m^2 \nabla \psi_t \cdot \mathbf{u} + \tau_m^2 \frac{\psi_t}{2} \operatorname{div} \mathbf{u} + \frac{\bar{\theta}}{2} (3\psi^2 + \psi^3) \right] D^\alpha D^\alpha \psi \, dx, \\
I_{1,\alpha}^2 &= -(-1)^{|\alpha|} \int \left[\nabla \psi \cdot \mathbf{V} - \frac{\bar{\theta}}{1+\psi} \operatorname{div} [(1+\psi)\zeta \nabla \psi] \right] D^\alpha D^\alpha \psi \, dx, \\
I_{1,\alpha}^3 &= (-1)^{|\alpha|} \tau_m^2 \int \left[\frac{1+\psi}{2} \operatorname{div} [(\mathbf{u} \cdot \nabla) \mathbf{u}] + \nabla \psi \cdot ((\mathbf{u} \cdot \nabla) \mathbf{u}) \right] D^\alpha D^\alpha \psi \, dx, \\
I_{1,\alpha}^4 &= (-1)^{|\alpha|} \int \left[\frac{\hbar^2}{2} \frac{(\Delta \psi)^2}{1+\psi} + \bar{\theta} \frac{|\nabla \psi|^2}{1+\psi} + \bar{\theta} \nabla \psi \cdot \nabla \zeta \right] D^\alpha D^\alpha \psi \, dx, \\
I_{1,\alpha}^5 &= \sum_{1 \leq |\beta| \leq |\alpha|} \mathbb{C}_{|\alpha|}^{|\beta|} \int D^\beta (1+\psi) \Delta D^{\alpha-\beta} \zeta D^\alpha \psi \, dx, \\
I_{1,\alpha}^6 &= \int \psi \nabla D^\alpha \zeta \cdot \nabla D^\alpha \psi \, dx + \int D^\alpha \psi \nabla D^\alpha \zeta \cdot \nabla \psi \, dx.
\end{aligned}$$

By using Hölder's inequality, (3.5), Lemma 2.1 and (3.4), we obtain (cf., [28], (3.45)) from (3.50) that

$$(3.51) \quad |I_{1,\alpha}| \lesssim \eta_1 (\|\psi\|_{H^4}^2 + \|\nabla \mathbf{u}\|^2 + \|\zeta\|_{H^3}^2 + \|\psi_t\|^2).$$

Next, we estimate $I_{2,\alpha}$. To this end, using (3.41) and (3.7), we rewrite $I_{2,\alpha}$ as

$$(3.52) \quad I_{2,\alpha} = I_{2,\alpha}^1 + I_{2,\alpha}^2 + I_{2,\alpha}^3 + I_{2,\alpha}^4 + \frac{\bar{\theta}}{2} I_{2,\alpha}^5,$$

where

$$\begin{aligned} I_{2,\alpha}^1 &= - \int D^\alpha \left[\tau_m^2 \nabla \psi_t \cdot \mathbf{u} + \tau_m^2 \frac{\psi_t}{2} \operatorname{div} \mathbf{u} + \frac{\bar{\theta}}{2} (3\psi^2 + \psi^3) \right] D^\alpha \psi_t \, dx, \\ I_{2,\alpha}^2 &= - \int D^\alpha \left[\nabla \psi \cdot \mathbf{V} - \frac{\bar{\theta}}{1+\psi} \operatorname{div}[(1+\psi)\zeta \nabla \psi] \right] D^\alpha \psi_t \, dx, \\ I_{2,\alpha}^3 &= \tau_m^2 \int D^\alpha \left[\frac{1+\psi}{2} \operatorname{div}[(\mathbf{u} \cdot \nabla) \mathbf{u}] + \nabla \psi \cdot ((\mathbf{u} \cdot \nabla) \mathbf{u}) \right] D^\alpha \psi_t \, dx, \\ I_{2,\alpha}^4 &= \int D^\alpha \left[\frac{\hbar^2}{2} \frac{(\Delta \psi)^2}{1+\psi} + \bar{\theta} \frac{|\nabla \psi|^2}{1+\psi} + \bar{\theta} \nabla \psi \cdot \nabla \zeta \right] D^\alpha \psi_t \, dx, \\ I_{2,\alpha}^5 &= \sum_{1 \leq |\beta| \leq |\alpha|} \mathbb{C}_{|\alpha|}^{|\beta|} \int D^\beta (1+\psi) \Delta D^{\alpha-\beta} \zeta D^\alpha \psi_t \, dx - \int D^\alpha \psi_t \nabla D^\alpha \zeta \cdot \nabla \psi \, dx. \end{aligned}$$

Then, by the same lines as in [28], (3.47)–(3.49), we obtain from (3.52) that

$$(3.53) \quad \tau_m |I_{2,\alpha}| \lesssim \eta_1 (\|\psi\|_{H^4}^2 + \tau_m \|\psi_t\|_{H^2}^2 + \tau_m \|\nabla \mathbf{u}\|_{H^2}^2 + \|\nabla \zeta\|_{H^2}^2).$$

Also, we estimate $I_{3,\alpha}$. Using (3.45), we rewrite it as

$$(3.54) \quad I_{3,\alpha} = I_{3,\alpha}^1 + \frac{2\hbar^2}{3\theta} I_{3,\alpha}^2 + I_{3,\alpha}^3 + I_{3,\alpha}^4 + I_{3,\alpha}^5 + I_{3,\alpha}^6 + I_{3,\alpha}^7 + I_{3,\alpha}^8,$$

where

$$\begin{aligned} I_{3,\alpha}^1 &= \int (1+\psi)^2 D^\alpha g_1^{\tau_m} D^\alpha \zeta \, dx, \quad I_{3,\alpha}^2 = -\tau_m \int (1+\psi)^2 g_{2,\alpha} D^\alpha \zeta \, dx, \\ I_{3,\alpha}^3 &= \tau_m \int (1+\psi) \psi_t (D^\alpha \zeta)^2 \, dx, \quad I_{3,\alpha}^4 = -\frac{2\beta}{3\tau_e} \int \psi(2+\psi) (D^\alpha \zeta)^2 \, dx, \\ I_{3,\alpha}^5 &= -\frac{2\kappa}{3\bar{\theta}} \int \psi(2+\psi) |\nabla D^\alpha \zeta|^2 \, dx, \quad I_{3,\alpha}^6 = -\frac{4\kappa}{3\bar{\theta}} \int (1+\psi) D^\alpha \zeta \nabla \psi \cdot \nabla D^\alpha \zeta \, dx, \\ I_{3,\alpha}^7 &= \frac{4\tau_m}{3} \int \psi(2+\psi) D^\alpha \psi_t D^\alpha \zeta \, dx, \\ I_{3,\alpha}^8 &= \frac{2\hbar^2 \tau_m}{3\theta} \int D^\alpha \psi_t \left[2 \operatorname{div}(D^\alpha \zeta \nabla \psi) + \nabla \psi \cdot \nabla D^\alpha \zeta + 2D^\alpha \zeta \frac{|\nabla \psi|^2}{1+\psi} \right] dx. \end{aligned}$$

Then, by the same lines as in [28], (3.51)–(3.55), we obtain from (3.54) that

$$(3.55) \quad |I_{3,\alpha}| \lesssim \eta_1 \left(\tau_m \|\psi_t\|_{H^2}^2 + \|\psi\|_{H^4}^2 + \tau_m \|\nabla \mathbf{u}\|_{H^2}^2 + \frac{1}{\tau_e} \|\nabla \zeta\|_{H^1}^2 + \|\nabla \zeta\|_{H^2}^2 \right).$$

Last, we estimate $I_{4,\alpha}$. By using (3.5), Lemma 2.1 and (3.4), we obtain (cf., [28], (3.56)) from (3.48) that

$$(3.56) \quad \tau_m |I_{4,\alpha}| \lesssim \eta_1 (\|\psi\|_{H^4}^2 + \tau_m \|\nabla \mathbf{u}\|_{H^2}^2 + \|\nabla \zeta\|_{H^2}^2).$$

Then, substituting (3.51), (3.53), (3.55) and (3.56) into (3.49), and using $\tau_m \leq 1$ and $\tau_e \leq 1$, we get

$$(3.57) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} E_\alpha(t) + \Lambda_\alpha(t) &\lesssim \eta_1 (\|\nabla \mathbf{u}\|^2 + \|\psi_t\|^2 + \tau_m \|\nabla^2 \mathbf{u}\|_{H^1}^2 + \tau_m \|\nabla \psi_t\|_{H^1}^2) \\ &\quad + \eta_1 \left(\|\psi\|_{H^4}^2 + \frac{1}{\tau_e} \|\zeta\|_{H^2}^2 + \|\nabla \zeta\|_{H^2}^2 \right). \end{aligned}$$

Noticing that

$$(3.58) \quad \|\nabla \mathbf{u}\|_{H^k}^2 = \|\operatorname{div} \mathbf{u}\|_{H^k}^2 + \|\operatorname{rot} \mathbf{u}\|_{H^k}^2 \quad \text{for } k = 0, 1, 2$$

due to $\Delta \mathbf{u} = \nabla \operatorname{div} \mathbf{u} - \operatorname{rot}(\operatorname{rot} \mathbf{u})$ and

$$(3.59) \quad \|\operatorname{div} \mathbf{u}\|_{H^k} \lesssim \|\psi_t\|_{H^k} + \eta_1 \|\nabla \psi\|_{H^k} \quad \text{for } k = 0, 1, 2$$

due to (3.1)₁, we obtain (3.33) from (3.57). The proof of Lemma 3.2 is completed. \square

4. PROOF OF MAIN RESULTS

In this section, we prove the main results, combining the estimates that we have derived in the previous section.

Using Yong's inequality, (3.15) (3.58) and (3.59), we obtain from (3.12) that

$$(4.1) \quad \begin{aligned} \frac{d}{dt} E_0(t) + C_1 \left(\|\psi\|_{H^2}^2 + \|(\psi_t, \operatorname{rot} \mathbf{u})\|^2 + \frac{1}{\tau_e} \|\zeta\|^2 + \|\nabla \zeta\|^2 \right) \\ \leq C_2 (\|\nabla^2 \psi\|_{H^2}^2 + \|\nabla^2 \zeta\|^2 + \tau_m \|\nabla(\psi_t, \operatorname{rot} \mathbf{u})\|^2), \end{aligned}$$

where C_1, C_2 are positive constants depending only on $\hbar, \bar{\varrho}, \bar{\theta}, \beta$ and κ .

Also, using the smallness of $\eta_1 > 0$ and (3.36), we obtain from (3.33) that

$$(4.2) \quad \begin{aligned} \frac{d}{dt} E_1(t) + C_3 \left(\|\nabla \psi\|_{H^3}^2 + \tau_m \|\nabla(\psi_t, \text{rot } \mathbf{u})\|_{H^1}^2 + \frac{1}{\tau_e} \|\nabla \zeta\|_{H^1}^2 + \|\nabla^2 \zeta\|_{H^1}^2 \right) \\ \leq C_4 \eta_1 \left(\|\psi\|_{H^2}^2 + \|(\psi_t, \text{rot } \mathbf{u})\|^2 + \frac{1}{\tau_e} \|\zeta\|^2 + \|\nabla \zeta\|^2 \right), \end{aligned}$$

where $E_1(t) = \sum_{1 \leq |\alpha| \leq 2} E_\alpha(t)$ and

$$(4.3) \quad E_1(t) \simeq \|\nabla \psi\|_{H^1}^2 + \tau_m \|\nabla^2 \psi\|_{H^2}^2 + \tau_m^3 \|\nabla(\psi_t, \text{rot } \mathbf{u})\|_{H^1}^2 + \tau_m \|\nabla \zeta\|_{H^2}^2$$

due to (3.35). Multiplying (4.1) by $C_3/(2C_2)$, and adding the resulting inequality with (4.2), since $\eta_1 > 0$ is small, we deduce that there exists a constant C_5 such that

$$(4.4) \quad \begin{aligned} \frac{dE(t)}{dt} + C_5 \left(\|\psi\|_{H^4}^2 + \|(\psi_t, \text{rot } \mathbf{u})\|^2 \right) \\ + C_5 \left(\tau_m \|\nabla(\psi_t, \text{rot } \mathbf{u})\|_{H^1}^2 + \frac{1}{\tau_e} \|\zeta\|_{H^2}^2 + \|\nabla \zeta\|_{H^2}^2 \right) \leq 0, \end{aligned}$$

where $E(t) = \frac{1}{2} C_3 E_0(t)/C_2 + E_1(t)$.

Also, by (3.14) and (4.3), we have

$$(4.5) \quad E(t) \simeq \|\psi\|_{H^2}^2 + \tau_m \|\nabla^2 \psi\|_{H^2}^2 + \tau_m^2 \|(\psi_t, \text{rot } \mathbf{u})\|^2 + \tau_m^3 \|\nabla(\psi_t, \text{rot } \mathbf{u})\|_{H^1}^2 + \tau_m \|\zeta\|_{H^3}^2.$$

Therefore, by using (3.58) and (3.59), we obtain from (4.4) and (4.5) that

$$(4.6) \quad \frac{dE(t)}{dt} + C \left(\|\psi\|_{H^4}^2 + \|(\psi_t, \nabla \mathbf{u})\|^2 + \tau_m \|\nabla(\psi_t, \nabla \mathbf{u})\|_{H^1}^2 + \frac{1}{\tau_e} \|\zeta\|_{H^2}^2 + \|\nabla \zeta\|_{H^2}^2 \right) \leq 0$$

and

$$(4.7) \quad E(t) \simeq \|\psi\|_{H^2}^2 + \tau_m \|\nabla^3 \psi\|_{H^1}^2 + \tau_m^2 \|(\psi_t, \nabla \mathbf{u})\|^2 + \tau_m^3 \|\nabla(\psi_t, \nabla \mathbf{u})\|_{H^1}^2 + \tau_m \|\zeta\|_{H^2}^2,$$

respectively.

4.1. The existence, uniqueness and exponential decay of global solutions.

In this subsection, we will prove Theorem 2.1. We first state the existence of the local solutions to Cauchy problem (3.1), (3.2). We omit the proof, because it is similar as in [28], Theorem 4.1, where we used the linearization of the non-linear system and the contraction mapping principle.

Lemma 4.1. *Let $\bar{\varrho}, \bar{\theta}, \tau_m, \hbar$ and τ_e be fixed. Assume that (2.3) and*

$$\exists m_1, m_2 > 0 \quad \forall x \in \mathbb{R}^3, \quad \frac{m_1}{2} \leq \varrho_0(x), \quad \theta_0(x) \leq 2m_2.$$

Then there exists a positive constant $t_0 > 0$, depending on $\hbar, \tau_m, \tau_e, \kappa, \bar{\varrho}, \bar{\theta}, \beta$ and $\|(\sqrt{\varrho_0} - \sqrt{\bar{\varrho}}, \mathbf{u}_0, \theta_0 - \bar{\theta})\|_{H^4 \times \mathcal{H}^3 \times H^2}$, such that the Cauchy problem (3.1), (3.2) admits a solution $(\varrho, \mathbf{u}, \theta, \mathbf{V})$ on $[0, t_0]$ satisfying

$$\begin{aligned} m_1 &\leq \varrho(x, t), \quad \theta(x, t) \leq m_2, \\ \sqrt{\varrho} - \sqrt{\bar{\varrho}} &\in C([0, t_0]; H^4(\mathbb{R}^3)) \cap C^1([0, t_0]; H^2(\mathbb{R}^3)), \\ \mathbf{u} &\in C([0, t_0]; \mathcal{H}^3(\mathbb{R}^3)), \quad \mathbf{V} \in C([0, t_0]; \mathcal{H}^5(\mathbb{R}^3)), \\ \theta - \bar{\theta} &\in C([0, t_0]; H^2(\mathbb{R}^3)), \quad \nabla \theta \in L^2([0, t_0]; H^2(\mathbb{R}^3)). \end{aligned}$$

By (4.6) and (4.7), there exists a positive constant $c_0 > 0$, depending on $\hbar, \tau_m, \tau_e, \kappa, \bar{\varrho}, \bar{\theta}, \beta$ satisfying

$$(4.8) \quad E(t) \leq E(0)e^{-c_0 t}$$

for fixed τ_m and τ_e . Therefore, by (4.7), (4.8) and (3.2), we get

$$\begin{aligned} (4.9) \quad &\|\psi(t)\|_{H^2}^2 + \tau_m \|\nabla^3 \psi(t)\|_{H^1}^2 + \tau_m^2 \|(\psi_t, \nabla \mathbf{u})(t)\|^2 \\ &\quad + \tau_m^3 \|\nabla(\psi_t, \nabla \mathbf{u})(t)\|_{H^1}^2 + \tau_m \|\zeta(t)\|_{H^2}^2 \\ &\lesssim E(t) \leq E(0) \\ &\lesssim \|\psi(0)\|_{H^2}^2 + \tau_m \|\nabla^3 \psi(0)\|_{H^1}^2 + \|\nabla \mathbf{u}(0)\|^2 \\ &\quad + \tau_m \|\nabla \mathbf{u}^2(0)\|_{H^1}^2 + \tau_m \|\zeta(0)\|_{H^2}^2 \\ &\lesssim \|\sqrt{\varrho_0} - \sqrt{\bar{\varrho}}\|_{H^2}^2 + \tau_m \|\nabla^3 \sqrt{\varrho_0}\|_{H^1}^2 + \|\nabla \mathbf{u}_0\|^2 \\ &\quad + \tau_m \|\nabla^2 \mathbf{u}_0\|_{H^1}^2 + \tau_m \|\theta_0 - \bar{\theta}\|_{H^2}^2. \end{aligned}$$

Using Lemma 4.1, (2.4) and (4.9), by a standard continuity argument, we can close the a priori estimate (4.6). Also, by using (4.7) and Lemma 2.2, we obtain the exponential decay (2.6) from (4.8). Now, the existence of the global solutions to Cauchy problem (3.1), (3.2) can be proved by extending the local solutions in time (see Subsection 4.2 in [28]). Moreover, the proof of the uniqueness is standard, relying on the estimates for the difference of two solutions (see Subsection 4.3 in [28]). So, we omit the details for brevity.

4.2. The relaxation-time limits. We first prove Theorem 2.2. Let $(\varrho^{\tau_m}, \mathbf{u}^{\tau_m}, \theta^{\tau_m}, \mathbf{V}^{\tau_m})$ be the global solution derived in Theorem 2.1 and we set

$$\psi^{\tau_m} = \frac{\sqrt{\varrho^{\tau_m}} - \sqrt{\bar{\varrho}}}{\sqrt{\bar{\varrho}}}, \quad \zeta^{\tau_m} = \frac{\theta^{\tau_m} - \bar{\theta}}{\bar{\theta}}.$$

Then, by (4.6), (4.7) and Lemma 2.2, $(\psi^{\tau_m}, \mathbf{u}^{\tau_m}, \zeta^{\tau_m}, \mathbf{V}^{\tau_m})$ is the unique solution to system (3.1), (3.2) satisfying

$$\begin{aligned}
(4.10) \quad & \|\psi^{\tau_m}(t)\|_{H^2}^2 + \|\nabla \mathbf{V}^{\tau_m}(t)\|_{H^2}^2 + \tau_m \|\nabla^3 \psi^{\tau_m}(t)\|_{H^1}^2 + \tau_m \|\nabla^4 \mathbf{V}^{\tau_m}(t)\|_{H^1}^2 \\
& + \tau_m^2 \|(\psi_t^{\tau_m}, \nabla \mathbf{u}^{\tau_m})(t)\|^2 + \tau_m^3 \|\nabla(\psi_t^{\tau_m}, \nabla \mathbf{u}^{\tau_m})(t)\|_{H^1}^2 + \tau_m \|\zeta^{\tau_m}(t)\|_{H^2}^2 \\
& + \int_0^t \|\psi^{\tau_m}(s)\|_{H^4}^2 ds + \int_0^t \|(\psi_t^{\tau_m}, \nabla \mathbf{u}^{\tau_m})(s)\|^2 ds + \frac{1}{\tau_e} \int_0^t \|\zeta^{\tau_m}(s)\|_{H^2}^2 ds \\
& + \tau_m \int_0^t \|(\psi_t^{\tau_m}, \nabla \mathbf{u}^{\tau_m})(s)\|_{H^1}^2 ds + \int_0^t \|\nabla \zeta^{\tau_m}(s)\|_{H^2}^2 ds \\
& \lesssim \|\psi(0)\|_{H^2}^2 + \tau_m \|\nabla^3 \psi(0)\|_{H^1}^2 + \|\nabla \mathbf{u}(0)\|^2 + \tau_m \|\nabla \mathbf{u}^2(0)\|_{H^1}^2 + \tau_m \|\zeta(0)\|_{H^2}^2
\end{aligned}$$

for any $t > 0$. The right-hand sides in (4.10) are independent of τ_m due to (2.4). Thus, these uniform estimates and the arguments on compact set in the space $L^p(0, T; B)$ (see [30]) imply the existence of a subsequence denoted also by $(\psi^{\tau_m}, \mathbf{u}^{\tau_m}, \zeta^{\tau_m}, \mathbf{V}^{\tau_m})$ such that

$$\begin{aligned}
(4.11) \quad & \psi^{\tau_m} \longrightarrow \psi \quad \text{*weakly in } L^\infty(0, \infty; H^2(\mathbb{R}^3)) \cap L^2(0, \infty; H^4(\mathbb{R}^3)), \\
& \psi_t^{\tau_m} \longrightarrow \psi_t \quad \text{weakly in } L^2(0, \infty; L^2(\mathbb{R}^3)), \\
& \psi^{\tau_m} \longrightarrow \psi \quad \text{strongly in } L^2(0, \infty; H_{\text{loc}}^3(\mathbb{R}^3)), \\
& \mathbf{u}^{\tau_m} \longrightarrow \mathbf{u} \quad \text{weakly in } L^2(0, \infty; \mathcal{H}^1(\mathbb{R}^3)), \\
& \zeta^{\tau_m} \longrightarrow \zeta \quad \text{weakly in } L^2(0, \infty; H^3(\mathbb{R}^3)), \\
& \mathbf{V}^{\tau_m} \longrightarrow \mathbf{V} \quad \text{*weakly in } L^\infty(0, \infty; \mathcal{H}^3(\mathbb{R}^3))
\end{aligned}$$

as $\tau_m \rightarrow 0$. Also, the above converging results allow the limit solutions $(\psi, \mathbf{u}, \zeta, \mathbf{V})$ from the FQHD model (3.1), (3.2) to the system

$$\begin{aligned}
(4.12) \quad & [(1 + \psi)^2]_t + \text{div}[(1 + \psi)^2 \mathbf{u}] = 0, \quad x \in \mathbb{R}^3, \quad 0 < t \leq T, \\
& \mathbf{u} + \frac{2\bar{\theta}(1 + \zeta)}{1 + \psi} \nabla \psi + \bar{\theta} \nabla \zeta = \hbar^2 \nabla \left(\frac{\Delta \psi}{1 + \psi} \right) + \mathbf{V}, \quad -\frac{\kappa}{\bar{\theta}(1 + \psi)^2} \Delta \zeta = -\frac{\beta}{\tau_e} \zeta, \\
& \text{div } \mathbf{V} = \bar{\varrho}(\psi^2 + 2\psi), \quad \text{rot } \mathbf{V} = 0, \quad \psi(x, 0) = \frac{\sqrt{\varrho_0(x)} - \sqrt{\bar{\varrho}}}{\sqrt{\bar{\varrho}}}.
\end{aligned}$$

Indeed, noticing that

$$F^{\tau_m} \rightarrow \frac{2\kappa}{3} \left(\frac{1}{\bar{\theta}(1 + \psi)^2} - \frac{1}{\bar{\theta}} \right) \Delta \zeta \quad \text{weakly in } L^2(0, \infty; L^2(\mathbb{R}^3))$$

due to (3.3), and using (4.11), we obtain (4.12)₃ from (3.1)₃. It is easy to check the other equations in (4.12).

If we set $\varrho = (1 + \psi)^2 \bar{\varrho}$ and $\theta = (1 + \zeta) \bar{\theta}$, then system (4.12) is equivalent to the Cauchy problem (1.4), (2.9). Moreover, we obtain from (4.11) that

$$\begin{aligned} \psi &\in L^\infty(0, \infty; H^2(\mathbb{R}^3)) \cap L^2(0, \infty; H^4(\mathbb{R}^3)), & \psi_t &\in L^2(0, \infty; L^2(\mathbb{R}^3)), \\ \zeta &\in L^2(0, \infty; H^3(\mathbb{R}^3)), & \mathbf{V} &\in L^\infty(0, \infty; \mathcal{H}^3(\mathbb{R}^3)), \end{aligned}$$

which implies (2.8). The proof of Theorem 2.2 is completed.

Next, we prove Theorem 2.3. Let $(\varrho^{\tau_m, \tau_e}, \mathbf{u}^{\tau_m, \tau_e}, \theta^{\tau_m, \tau_e}, \mathbf{V}^{\tau_m, \tau_e})$ be the global solution derived in Theorem 2.1 and we set

$$\psi^{\tau_m, \tau_e} = \frac{\sqrt{\varrho^{\tau_m, \tau_e}} - \sqrt{\bar{\varrho}}}{\sqrt{\bar{\varrho}}}, \quad \zeta^{\tau_m, \tau_e} = \frac{\theta^{\tau_m, \tau_e} - \bar{\theta}}{\bar{\theta}}.$$

Then, $(\psi^{\tau_m, \tau_e}, \mathbf{u}^{\tau_m, \tau_e}, \zeta^{\tau_m, \tau_e}, \mathbf{V}^{\tau_m, \tau_e})$ is also the unique solution to system (3.1), (3.2) satisfying (4.10). Thus, there exists a subsequence denoted also by $(\psi^{\tau_m, \tau_e}, \mathbf{u}^{\tau_m, \tau_e}, \zeta^{\tau_m, \tau_e}, \mathbf{V}^{\tau_m, \tau_e})$ such that

$$(4.13) \quad \begin{aligned} \psi^{\tau_m, \tau_e} &\longrightarrow \psi && \text{*weakly in } L^\infty(0, \infty; H^2(\mathbb{R}^3)) \cap L^2(0, \infty; H^4(\mathbb{R}^3)), \\ \psi_t^{\tau_m, \tau_e} &\longrightarrow \psi_t && \text{weakly in } L^2(0, \infty; L^2(\mathbb{R}^3)), \\ \psi^{\tau_m, \tau_e} &\longrightarrow \psi && \text{strongly in } L^2(0, \infty; H_{\text{loc}}^3(\mathbb{R}^3)), \\ \mathbf{u}^{\tau_m, \tau_e} &\longrightarrow \mathbf{u} && \text{weakly in } L^2(0, \infty; \mathcal{H}^1(\mathbb{R}^3)), \\ \zeta^{\tau_m, \tau_e} &\longrightarrow 0 && \text{strongly in } L^2(0, \infty; H^2(\mathbb{R}^3)), \\ \mathbf{V}^{\tau_m, \tau_e} &\longrightarrow \mathbf{V} && \text{*weakly in } L^\infty(0, \infty; \mathcal{H}^3(\mathbb{R}^3)) \end{aligned}$$

as $\tau_m \rightarrow 0$ and $\tau_e \rightarrow 0$. Also, the above converging results allow the limit solutions $(\psi, \mathbf{u}, \mathbf{V})$ from the FQHD model (3.1), (3.2) to the system

$$\begin{aligned} [(1 + \psi)^2]_t + \operatorname{div}[(1 + \psi)^2 \mathbf{u}] &= 0, & x &\in \mathbb{R}^3, & 0 < t \leq T, \\ \mathbf{u} + \frac{2\bar{\theta}(1 + \zeta)}{1 + \psi} \nabla \psi &= \hbar^2 \nabla \left(\frac{\Delta \psi}{1 + \psi} \right) + \mathbf{V}, \\ \operatorname{div} \mathbf{V} &= \bar{\varrho}(\psi^2 + 2\psi), & \operatorname{rot} \mathbf{V} &= 0, \\ \psi(x, 0) &= \frac{\sqrt{\varrho_0(x)} - \sqrt{\bar{\varrho}}}{\sqrt{\bar{\varrho}}}, \end{aligned}$$

which is equivalent to the Cauchy problem (1.5), (2.9) for QDD model, if we set $\varrho = (1 + \psi)^2 \bar{\varrho}$ and $\theta = (1 + \zeta) \bar{\theta}$. Moreover, we obtain from (4.13) that

$$\begin{aligned} \psi &\in L^\infty(0, \infty; H^2(\mathbb{R}^3)) \cap L^2(0, \infty; H^4(\mathbb{R}^3)), \\ \psi_t &\in L^2(0, \infty; L^2(\mathbb{R}^3)), & \mathbf{V} &\in L^\infty(0, \infty; \mathcal{H}^3(\mathbb{R}^3)), \end{aligned}$$

which implies (2.10). The proof of Theorem 2.3 is completed. \square

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References

- [1] *X. Chen*: The isentropic quantum drift-diffusion model in two or three dimensions. *Z. Angew. Math. Phys.* *60* (2009), 416–437. [zbl](#) [MR](#) [doi](#)
- [2] *X. Chen, L. Chen*: Initial time layer problem for quantum drift-diffusion model. *J. Math. Anal. Appl.* *343* (2008), 64–80. [zbl](#) [MR](#) [doi](#)
- [3] *X. Chen, L. Chen, H. Jian*: Existence, semiclassical limit and long-time behavior of weak solution to quantum drift-diffusion model. *Nonlinear Anal., Real World Appl.* *10* (2009), 1321–1342. [zbl](#) [MR](#) [doi](#)
- [4] *L. Chen, Q. Ju*: Existence of weak solution and semiclassical limit for quantum drift-diffusion model. *Z. Angew. Math. Phys.* *58* (2007), 1–15. [zbl](#) [MR](#) [doi](#)
- [5] *J. Dong*: Mixed boundary-value problems for quantum hydrodynamic models with semiconductors in thermal equilibrium. *Electron. J. Differ. Equ.* *2005* (2005), Article ID 123, 8 pages. [zbl](#) [MR](#)
- [6] *C. L. Gardner*: The quantum hydrodynamic model for semiconductor devices. *SIAM J. Appl. Math.* *54* (1994), 409–427. [zbl](#) [MR](#) [doi](#)
- [7] *U. Gianazza, G. Savaré, G. Toscani*: The Wasserstein gradient flow of the Fisher information and the quantum drift-diffusion equation. *Arch. Ration. Mech. Anal.* *194* (2009), 133–220. [zbl](#) [MR](#) [doi](#)
- [8] *M. P. Gualdani, A. Jüngel, G. Toscani*: A nonlinear fourth-order parabolic equation with nonhomogeneous boundary conditions. *SIAM. J. Math. Anal.* *37* (2006), 1761–1779. [zbl](#) [MR](#) [doi](#)
- [9] *F. Huang, H.-L. Li, A. Matsumura*: Existence and stability of steady-state of one-dimensional quantum hydrodynamic system for semiconductors. *J. Differ. Equations* *225* (2006), 1–25. [zbl](#) [MR](#) [doi](#)
- [10] *Y. Jia, H. Li*: Large-time behavior of solutions of quantum hydrodynamic model for semiconductors. *Acta Math. Sci., Ser. B, Engl. Ed.* *26* (2006), 163–178. [zbl](#) [MR](#) [doi](#)
- [11] *A. Jüngel*: A steady-state quantum Euler-Poisson system for potential flows. *Commun. Math. Phys.* *194* (1998), 463–479. [zbl](#) [MR](#) [doi](#)
- [12] *A. Jüngel, H. Li*: Quantum Euler-Poisson systems: Existence of stationary states. *Arch. Math., Brno* *40* (2004), 435–456. [zbl](#) [MR](#)
- [13] *A. Jüngel, H. Li*: Quantum Euler-Poisson systems: Global existence and exponential decay. *Q. Appl. Math.* *62* (2004), 569–600. [zbl](#) [MR](#) [doi](#)
- [14] *A. Jüngel, H.-L. Li, A. Matsumura*: The relaxation-time limit in the quantum hydrodynamic equations for semiconductors. *J. Differ. Equations* *225* (2006), 440–464. [zbl](#) [MR](#) [doi](#)
- [15] *A. Jüngel, D. Matthes, J. P. Milišić*: Derivation of new quantum hydrodynamic equations using entropy minimization. *SIAM J. Appl. Math.* *67* (2006), 46–68. [zbl](#) [MR](#) [doi](#)
- [16] *A. Jüngel, J. P. Milišić*: A simplified quantum energy-transport model for semiconductors. *Nonlinear Anal., Real World Appl.* *12* (2011), 1033–1046. [zbl](#) [MR](#) [doi](#)
- [17] *A. Jüngel, R. Pinnau*: A positivity-preserving numerical scheme for a nonlinear fourth order parabolic system. *SIAM J. Numer. Anal.* *39* (2001), 385–406. [zbl](#) [MR](#) [doi](#)
- [18] *A. Jüngel, I. Violet*: The quasineutral limit in the quantum drift-diffusion equations. *Asymptotic Anal.* *53* (2007), 139–157. [zbl](#) [MR](#)
- [19] *Y.-H. Kim, S. Ra, S.-C. Kim*: Asymptotic behavior of strong solutions of a simplified energy-transport model with general conductivity. *Nonlinear Anal., Real World Appl.* *59* (2021), Article ID 103261, 18 pages. [zbl](#) [MR](#) [doi](#)

- [20] *S. Klainerman, A. Majda*: Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. *Commun. Pure Appl. Math.* *34* (1981), 481–524. [zbl](#) [MR](#) [doi](#)
- [21] *H. Li, P. Marcati*: Existence and asymptotic behavior of multi-dimensional quantum hydrodynamic model for semiconductors. *Commun. Math. Phys.* *245* (2004), 215–247. [zbl](#) [MR](#) [doi](#)
- [22] *H. Li, G. Zhang, M. Zhang, C. Hao*: Long-time self-similar asymptotic of the macroscopic quantum models. *J. Math. Phys.* *49* (2008), Article ID 073503, 14 pages. [zbl](#) [MR](#) [doi](#)
- [23] *J. Mao, F. Zhou, Y. Li*: Some limit analysis in a one-dimensional stationary quantum hydrodynamic model for semiconductors. *J. Math. Anal. Appl.* *364* (2010), 186–194. [zbl](#) [MR](#) [doi](#)
- [24] *P. A. Markowich, C. A. Ringhofer, C. Schmeiser*: *Semiconductor Equations*. Springer, Vienna, 1990. [zbl](#) [MR](#) [doi](#)
- [25] *L. Nirenberg*: On elliptic partial differential equations. *Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser.* *13* (1959), 115–162. [zbl](#) [MR](#)
- [26] *S. Nishibata, N. Shigeta, M. Suzuki*: Asymptotic behaviors and classical limits of solutions to a quantum drift-diffusion model for semiconductors. *Math. Models Methods Appl. Sci.* *20* (2010), 909–936. [zbl](#) [MR](#) [doi](#)
- [27] *S. Nishibata, M. Suzuki*: Initial boundary value problems for a quantum hydrodynamic model of semiconductors: Asymptotic behaviors and classical limits. *J. Differ. Equations* *244* (2008), 836–874. [zbl](#) [MR](#) [doi](#)
- [28] *S. Ra, H. Hong*: The existence, uniqueness and exponential decay of global solutions in the full quantum hydrodynamic equations for semiconductors. *Z. Angew. Math. Phys.* *72* (2021), Article ID 107, 32 pages. [zbl](#) [MR](#) [doi](#)
- [29] *J. Ri, S. Ra*: Solution to a multi-dimensional isentropic quantum drift-diffusion model for bipolar semiconductors. *Electron. J. Differ. Equ.* *2018* (2018), Article ID 200, 19 pages. [zbl](#) [MR](#)
- [30] *J. Simon*: Compact sets in the space $L^p(0, T; B)$. *Ann. Mat. Pura Appl. (4)* *146* (1986), Article ID 146, 32 pages. [doi](#)
- [31] *G. Zhang, H.-L. Li, K. Zhang*: Semiclassical and relaxation limits of bipolar quantum hydrodynamic model for semiconductors. *J. Differ. Equations* *245* (2008), 1433–1453. [zbl](#) [MR](#) [doi](#)

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