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SYMMETRIC AND REVERSIBLE PROPERTIES OF BI-AMALGAMATED RINGS

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Abstract. Let $f\colon A \to B$ and $g\colon A \to C$ be two ring homomorphisms and let K and K' be two ideals of B and C, respectively, such that $f^{-1}(K) = g^{-1}(K')$. We investigate unipotent, symmetric and reversible properties of the bi-amalgamation ring $A \bowtie^{f,g} (K, K')$ of A with (B, C) along (K, K') with respect to (f, g).

Keywords: amalgamated ring; unipotent; symmetric ring; reversible ring

MSC 2020: 16N40, 16U40, 16S99

1. Introduction and background

Throughout this paper all rings are associative with an identity. We denote by $Nil(R)$, $U(R)$, $C(R)$ and $Id(R)$, the set of nilpotent elements, unit elements, central elements and the set of idempotents of R, respectively.

Let $\alpha\colon A \to C$, $\beta\colon A \to C$ and $f\colon A \to B$ be ring homomorphisms. In [4], the authors studied amalgamated algebras within the frame of pullback $\alpha \times \beta$ such that $\alpha = \beta \circ f$, see [4], Propositions 4.2 and 4.4. Based on the amalgamated constructions in [7], the authors created the new constructions, called *bi-amalgamation*, which arise as pullbacks $\alpha \times \beta$ such that the diagram of ring homomorphims

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \alpha} \\ C & \xrightarrow{\beta} & D \end{array}$$

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is commutative with $\alpha \circ \pi_B(\alpha \times \beta) = \alpha \circ f(A)$, where π_B denotes the canonical projection of $B \times C$ over B. Namely, let $f\colon A \to B$ and $g\colon A \to C$ be two ring homomorphisms and let K and K' be two ideals of B and C, respectively, such that $f^{-1}(K) = g^{-1}(K')$. The bi-amalgamation of A with (B, C) along (K, K') with respect to (f, g) is a subring of $B \times C$ given by

$$A \bowtie^{f,g} (K, K') := \{(f(a) + k, g(a) + k')\colon\ a \in A,\ (k, k') \in K \times K'\}.$$

In [5], the authors have studied amalgamated rings with clean-type properties. In this motivation we study many ring theoretical properties of the bi-amalgamation ring $A \bowtie^{f,g} (K, K')$, in the case, where the rings are not assumed to be commutative. We give characterizations for the bi-amalgamation ring $A \bowtie^{f,g} (K, K')$ to be UU ring, symmetric and reversible ring.

This paper aims at studying the transfer of the notion of UU rings, symmetric rings, reversible rings to the bi-amalgamation of rings along ideals.

2. UU bi-amalgamated rings

Definition 2.1 ([6]). An element a in a ring A is called *von Neumann regular* if $a = aba$ for some $b \in A$. A ring A is called von Neumann regular if every element of A is a von Neumann regular.

Proposition 2.2. *If $A \bowtie^{f,g} (K, K')$ is a von Neumann regular ring, then $f(A)+K$ and $g(A) + K'$ are von Neumann regular rings.*

Proof. Let $a \in A$. We have $(f(a) + k, g(a) + k') \in A \bowtie^{f,g} (K, K')$. Since $A \bowtie^{f,g} (K, K')$ is a von Neumann regular ring, there exists $(f(b) + l, g(b) + l') \in A \bowtie^{f,g} (K, K')$ such that $(f(a)+k, g(a)+k') = (f(a)+k, g(a)+k')(f(b)+l, g(b)+l')\times (f(a)+k, g(a)+k') = ((f(a)+k)(f(b)+l)(f(a)+k), (g(a)+k')(g(b)+l')(g(a)+k'))$. So $f(a)+k = (f(a)+k)(f(b)+l)(f(a)+k)$ and $g(a)+k' = (g(a)+k')(g(b)+l')(g(a)+k')$. Hence, $f(A) + K$ and $g(A) + K'$ are von Neumann regular rings. $\square$

The following example shows that if $f(A)+K$ is a von Neumann regular ring, then the bi-amalgamation ring $A \bowtie^{f,g} (K, K')$ is not necessrily von Neumann regular.

Example 2.3. Let F be a field and

$$A = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : a \in F \right\}, \quad B = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : a, b \in F \right\} \quad \text{and} \quad C = B.$$

Let

$$K = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix} : a \in F \right\} \quad \text{and} \quad K' = \left\{ \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : b \in F \right\}$$

be two ideals of B and C, respectively. Consider $f\colon A \to B$ defined by

$$f\left(\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \right) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$$

and $g\colon A \to C$ defined by

$$g\left(\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \right) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}.$$

Clearly, $f(A)+K$ is a von Neumann regular ring but $g(A)+K'$ is not von Neumann regular, because for every

$$\begin{pmatrix} p & 0 & q \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} \in g(A) + K',$$

where $p, q \in F$ are nonzero,

$$\begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p & 0 & q \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, $A \bowtie^{f,g} (K, K') \cong (f(A)+K, g(A)+K')$ is not von Neumann regular.

Lemma 2.4. *Let $a \in U(A)$, $a \in Nil(A)$ and $e \in Id(A)$. Then:*

(1) $Id(A \bowtie^{f,g} (K,K')) = \{(f(e)+k, g(e)+k')\colon f(e)+k \in Id(f(A)+K),\ g(e)+k' \in Id(g(A)+K')\}$.

(2) $Nil(A \bowtie^{f,g} (K,K')) = \{(f(a)+k, g(a)+k')\colon f(a)+k \in Nil(f(A)+K),\ g(a)+k' \in Nil(g(A)+K')\}$.

(3) $U(A \bowtie^{f,g} (K,K')) = \{(f(a)+k, g(a)+k')\colon f(a)+k \in U(f(A)+K),\ g(a)+k' \in U(g(A)+K')\}$.

In [2], a ring R is called an NR ring if $Nil(R)$ is a subring of R.

Proposition 2.5. *If B and C are NR rings and $a \in Nil(A)$, then*

$$Nil(A \bowtie^{f,g} (K, K')) = \{(f(a) + k, g(a) + k') \colon k \in Nil(K), k' \in Nil(K')\}.$$

Proof. Let $k \in Nil(K)$ and $k' \in Nil(K')$. Since B is an NR ring and $a \in Nil(A)$, $f(a) + k \in Nil(f(A) + K)$ and also since C is NR and $a \in Nil(A)$, $g(a) + k' \in Nil(g(A) + K'))$. In light of Lemma 2.4, $(f(a) + k, g(a) + k') \in Nil(A \bowtie^{f,g} (K, K'))$. Next, we consider $(f(a) + k, g(a) + k') \in Nil(A \bowtie^{f,g} (K, K')$. Therefore, there exists a positive integer m such that $(f(a) + k, g(a) + k')^m = 0$. Thus, $(f(a) + k)^m = 0$ and $(g(a) + k')^m = 0$. We have $f(a) + k \in Nil(B)$ and $g(a) + k' \in Nil(C)$. Since $f(a) \in Nil(B)$ and B is NR, $k \in Nil(B)$ and also $g(a) \in Nil(C)$. Also, since C is NR, $k' \in Nil(C)$. Hence, the proof is complete. □

Theorem 2.6. *We have the following statements.*

(1) *If A, $f(A) + K$ and $g(A) + K'$ are NR rings, then $A \bowtie^{f,g} (K, K')$ is an NR ring.*

(2) *Let T be the set of all regular central elements of B such that $T \cap K \neq \emptyset$ and let T' be the set of all regular central elements of C such that $T' \cap K' \neq \emptyset$. If $A \bowtie^{f,g} (K, K')$ is an NR, then $f(A) + K$ and $g(A) + K'$ are NR.*

Proof. (1) Let $(f(a)+k_1, g(a)+k_1')$, $(f(b)+k_2, g(b)+k_2') \in Nil(A \bowtie^{f,g} (K, K'))$. In view of Lemma 2.4, $(f(a) + k_1), (f(b) + k_2) \in Nil(f(A) + K)$ and $(g(a) + k_1')$, $(g(b) + k_2') \in Nil(g(A) + K')$. Since $f(A) + K$ is NR, $(f(a) + k_1) - (f(b) + k_2)$, $(f(a)+k_1)(f(b)+k_2) \in Nil(f(A)+K)$ and $g(A)+K'$ is NR, $(g(a)+k_1')-(g(b)+k_2')$, $(g(a) + k_1')(g(b) + k_2') \in Nil(g(A) + K')$. Hence, $(f(a) + k_1, g(a) + k_1') - (f(b) + k_2, g(b) + k_2'), (f(a) + k_1, g(a) + k_1')(f(b) + k_2, g(b) + k_2') \in Nil(A \bowtie^{f,g} (K, K'))$.

(2) Let $(f(a) + k_1), f(b) + k_2) \in Nil(A \bowtie^{f,g} (K, K'))$. We shall prove that $(f(a)+k_1)-(f(b)+k_2) \in Nil(f(A)+K)$, $(f(a)+k_1)(f(b)+k_2) \in Nil(f(A)+K)$ and $(g(a)+k_1')-(g(b)+k_2'), (g(a)+k_1')(g(b)+k_2') \in Nil(g(A)+K')$. We have $(e_1(f(a)+k_1), e_2(g(a) + k_1')), (e_1(f(b) + k_2), e_2(g(b) + k_2')) \in Nil(A \bowtie^{f,g} (K, K'))$ for $e_1 \in T \cap K$ and $e_2 \in T' \cap K'$. Since $A \bowtie^{f,g} (K, K')$ is NR, $(e_1(f(a) + k_1), e_2(g(a) + k_1')) - (e_1(f(b)+k_2), e_2(g(b)+k_2'))$ and $(e_1(f(a)+k_1), e_2(g(a)+k_1'))(e_1(f(b)+k_2), e_2(g(b)+k_2'))$ are in $Nil(A \bowtie^{f,g} (K, K'))$. Then we have $(e_1((f(a) + k_1) - (f(b) + k_2)), e_2((g(a)+k_1')-(g(b)+k_2')))$ and $(e_1^2((f(a)+k_1)(f(b)+k_2)), e_2^2((g(a)+k_1')(g(b)+k_2')))$ are in $Nil(A \bowtie^{f,g} (K, K'))$. In light of Lemma 2.4, $e_1((f(a) + k_1) - (f(b) + k_2))$, $e_1^2((f(a) + k_1)(f(b) + k_2)) \in Nil(f(A) + K)$ and $e_2((g(a) + k_1') - (g(b) + k_2'))$, $e_2^2((g(a)+k_1')(g(b)+k_2')) \in Nil(g(A)+K')$. Therefore, there are positive integers n, m such that $(e_1((f(a) + k_1) - (f(b) + k_2)))^n = 0, (e_1^2((f(a) + k_1)(f(b) + k_2)))^n = 0$ and $(e_2((g(a)+k_1')-(g(b)+k_2')))^m = 0, (e_2^2((g(a)+k_1')(g(b)+k_2')))^m = 0$. Since e_1 is a regular central element, $((f(a)+k_1)-(f(b)+k_2))^n = 0$ and $((f(a)+k_1)(f(b)+k_2))^n = 0$

and e_2 is a regular central element, $((g(a)+k'_1)-(g(b)+k'_2))^m=0$ and $((g(a)+k'_1)(g(b)+k'_2))^m=0$. Therefore, $(f(a)+k_1)-(f(b)+k_2),(f(a)+k_1)(f(b)+k_2)\in Nil(f(A)+K)$ and $(g(a)+k'_1)-(g(b)+k'_2),(g(a)+k'_1)(g(b)+k'_2)\in Nil(g(A)+K')$. Hence, the proof is complete. □

Calugareanu in [1] introduced and studied UU rings (rings whose units are unipotent).

Proposition 2.7. *If A,$f(A)+K$ and $g(A)+K'$ are UU rings, then $A\bowtie^{f,g}(K,K'))$ is a UU ring.*

P r o o f. Let $(f(u)+k,g(u)+k')\in U(A\bowtie^{f,g}(K,K'))$, so by Lemma 2.4, $f(u)+k\in U(f(A)+K)$ and $g(u)+k'\in U(g(A)+K')$. Since A and $f(A)+K$ are UU rings, $f(u)+k-1\in Nil(f(A)+K)$ and also A and $g(A)+K'$ are UU rings, $g(u)+k'-1\in Nil(g(A)+K')$. So $(f(u)+k,g(u)+k')-(1,1)\in Nil(A\bowtie^{f,g}(K,K'))$. Thus, $A\bowtie^{f,g}(K,K'))$ is a UU ring. □

Theorem 2.8. *Let $f\colon A\to B$ be a ring homomorphism and K a nil ideal of B. Let $g\colon A\to C$ be a ring homomorphism and K' a nil ideal of C. If $A\bowtie^{f,g}(K,K'))$ is a UU ring, then $f(A)+K$ and $g(A)+K'$ are UU rings.*

P r o o f. Let $f(u)+k\in U(f(A)+K)$. Since $k\in K$ and $K\subseteq Nil(B)$, so $f(u)+k-k\in U(f(A)+K)$. Hence, $f(u)\in U(f(A)+K)$ and that $u\in U(A)$ and also $g(u)+k'\in U(g(A)+K')$. Since $k'\in K'$ and $K'\subseteq Nil(C)$, so $g(u)+k'-k'\in U(g(A)+K')$. Hence, $g(u)\in U(g(A)+K')$ and $u\in U(A)$. Therefore, by Lemma 2.4, $f(u)+k,g(u)+k'\in U(A\bowtie^{f,g}(K,K'))$. As $A\bowtie^{f,g}(K,K'))$ is UU, $(f(u)+k,g(u)+k')-(1,1)\in Nil(A\bowtie^{f,g}(K,K'))$. So $(f(u)+k-1,g(u)+k'-1)\in Nil(A\bowtie^{f,g}(K,K'))$. Thus, by Lemma 2.4, $f(u)+k-1\in Nil(f(A)+K)$ and $g(u)+k'-1\in Nil(g(A)+K')$. Hence, $f(A)+K$ and $g(A)+K'$ are UU rings. □

Theorem 2.9. *If A, $f(A)+K$ and $g(A)+K'$ are UU, then so is $A\bowtie^{f,g}(K,K')$.*

P r o o f. Assume that A, $f(A)+K$ and $g(A)+K'$ are UU rings. Let $x\in A$, $k\in K$ and $k'\in K'$ such that $(f(x)+k,g(x)+k')$. Then $f(x)+k\in U(f(A)+K)$, $g(x)+k'\in U(g(A)+K')$. So (i) $f(x)+k=1+b'$ for some $b'\in Nil(f(x)+K)$ and (ii) $g(x)+k'=1+b''$ for some $b''\in Nil(g(x)+K')$. Since A is a UU ring for $x\in A$, we have $x=1+b$ for some $b\in Nil(A)$. It follows that $f(x)=1+f(b)$. Substituting $f(x)$ into (i), we get $1+f(b)+k=1+b'$. Hence, $f(b)+k=b'$. Also applying g on x, we get $g(x)=g(1+b)$. Substituing $g(x)$ into (ii), we get $1+g(b)+k'=1+b''$. Hence, $g(b)+k'=b''$. Consequently, $(f(x)+k,g(x)+k')=(1+b',1+b'')=(1,1)+(b,b'')=(1,1)+(f(b)+k,g(b)+k')$, which is the unipotent element of $A\bowtie^{f,g}(K,K')$. Hence, $A\bowtie^{f,g}(K,K')$ is UU, as desired. □

Proposition 2.10. *We have the following statements.*

(1) *Let $A \bowtie^{f,g} (K, K')$ be an NR ring and K' be a nil ideal of C. If $f(A) + K$ is a UU ring, then $A \bowtie^{f,g} (K, K')$ is a UU ring.*

(2) *Let $A \bowtie^{f,g} (K, K')$ be an NR ring and K be a nil ideal of B. If $g(A) + K'$ is a UU ring, then $A \bowtie^{f,g} (K, K')$ is a UU ring.*

P r o o f. (1) Since K' is a nil ideal of C, $0 \times K'$ is a nil ideal of $A \bowtie^{f,g} (K, K')$. In light of Proposition 4.1 of [7], we have the natural projection $p_A\colon A \bowtie^{f,g} (K, K') \to f(A) + K$ defind by $p_A(f(a) + k, g(a) + k') = f(a) + k$. Hence, the following natural isomorphism holds:

$$\frac{A \bowtie^{f,g} (K, K')}{0 \times K'} \cong f(A) + K.$$

In view of Proposition 2.2 of [1], $f(A) + K$ is a UU ring, and this implies that $A \bowtie^{f,g} (K, K')$ is a UU ring.

(2) Since K is a nil ideal of B, $K \times 0$ is a nil ideal of $A \bowtie^{f,g} (K, K')$. So by Proposition 4.1 of [7], we have the natural projection $p_B\colon A \bowtie^{f,g} (K, K') \to g(A) + K'$ defined by $p_B(f(a) + k, g(a) + k') = g(a) + k'$. Hence, the following natural isomorphism holds:

$$\frac{A \bowtie^{f,g} (K, K')}{K \times 0} \cong g(A) + K'.$$

In view of Proposition 2.2 of [1], $g(A) + K'$ is a UU ring, and this implies that $A \bowtie^{f,g} (K, K')$ is a UU ring. □

3. Symmetric and reversible properties of bi-amalgamated rings along ideals

Symmetric rings are defined in [10]. A ring is called *symmetric* if $abc = 0$ implies $acb = 0$ for $a, b, c \in R$. In [8], this concept is extended to the central symmetric ring, that is, if $abc = 0$ implies acb is central in R for $a, b, c \in R$. In this section we study necessary and sufficient condition for $A \bowtie^{f,g} (K, K')$ to be symmetric.

Proposition 3.1. *Symmetric property of rings is preserved under isomorphims and subrings.*

P r o o f. The proof is straightforward. □

Theorem 3.2. *If A, $f(A) + K$ and $g(A) + K'$ are symmetric, then so is $A \bowtie^{f,g} (K, K')$.*

P r o o f. Assume A, $f(A)+K$ and $g(A)+K'$ are symmetric. Let $(f(a)+x, g(a)+x'), (f(b)+y, g(b)+y'), (f(c)+z, g(c)+z') \in A \bowtie^{f,g} (K, K')$ with $(f(a)+x, g(a)+x')(f(b)+y, g(b)+y')(f(c)+z, g(c)+z') = 0$. Then $(f(a)+x)(f(b)+y) \times (f(c)+z) = 0$ and $(g(a)+x')(g(b)+y')(g(c)+z') = 0$. By assumption, $(f(a)+x) \times (f(c)+z)(f(b)+y) = 0$ and $(g(a)+x')(g(c)+z')(g(b)+y') = 0$. Hence, $(f(a)+x, g(a)+x')(f(c)+z, g(c)+z')(f(b)+y, g(b)+y') = 0$. So $A \bowtie^{f,g} (K, K')$ is symmetric.
□

Theorem 3.3. *Assume that A is symmetric and $K \cap S \neq \emptyset$, where S is the set of regular central elements of B and $K' \cap S' \neq \emptyset$, where S' is the set of regular central elements of C. Then $A \bowtie^{f,g} (K, K')$ is symmetric if and only if $f(A)+K$ and $g(A)+K'$ are symmetric.*

P r o o f. It is enough to show that if $A \bowtie^{f,g} (K, K')$ is a symmetric ring, then $f(A)+K$ and $g(A)+K'$ are symmetric rings. For if $s \in S \cap K$ and $f(a)+x$, $f(b)+y, f(c)+z \in f(A)+K$ such that $(f(a)+x)(f(b)+y)(f(c)+z) = 0$, then $(s(f(a)+x), 0)(s(f(b)+y), 0)(s(f(c)+z), 0) = 0$. By hypothesis, $(s(f(a)+x), 0) \times (s(f(b)+y), 0)(s(f(c)+z), 0) = 0$. Hence, $s^3((f(a)+x)(f(b)+y)(f(c)+z)) = 0$. By regularity of s, we have $((f(a)+x)(f(b)+y)(f(c)+z)) = 0$. Also, if $s' \in S' \cap K'$ and $g(a)+x', g(b)+y', g(c)+z' \in g(A)+K'$ such that $(g(a)+x')(g(b)+y')(g(c)+z') = 0$, then $(0, s'(g(a)+x'))(0, s'(g(b)+y'))(0, s'(g(c)+z')) = 0$. By hypothesis, we have $(0, s'(g(a)+x')(0, s'(g(b)+y')(0, s'(g(c)+z')) = 0$. Hence, $s'^3((g(a)+x')(g(b)+y') \times (g(c)+z')) = 0$. By regularity of s', we have $(g(a)+x')(g(b)+y')(g(c)+z') = 0$. □

In [12], Ouyang and Chen discussed weak symmetric rings and they proved that all symmetric rings are weak symmetric.

Definition 3.4. A ring is called *weak symmetric* if $abc \in Nil(R)$ implies $acb \in Nil(R)$ for $a, b, c \in R$.

Now we study necessary and sufficient conditions for $A \bowtie^{f,g} (K, K')$ to be weak symmetric.

Proposition 3.5. *Weak symmetric property of rings is preserved under isomorphims and subrings.*

P r o o f. The proof is straightforward. □

Theorem 3.6. *If A, $f(A)+K$ and $g(A)+K'$ are weak symmetric, then so is $A \bowtie^{f,g} (K, K')$.*

Proof. Assume A, $f(A)+K$ and $g(A)+K'$ are weak symmetric. Let $(f(a)+x, g(a)+x'), (f(b)+y, g(b)+y'), (f(c)+z, g(c)+z') \in Nil(A \bowtie^{f,g} (K,K'))$. Then $(f(a)+x)(f(b)+y)(f(c)+z) \in f(A)+K$ and $(g(a)+x')(g(b)+y')(g(c)+z') \in g(A)+K'$ are nilpotent. By assumption, $(f(a)+x)(f(c)+z)(f(b)+y)$ and $(g(a)+x') \times (g(c)+z')(g(b)+y')$ are nilpotent respectively in $f(A)+K$ and $g(A)+K'$. Hence, $(f(a)+x, g(a)+x')(f(c)+z, g(c)+z')(f(b)+y, g(b)+y')$ is nilpotent in $A \bowtie^{f,g} (K,K')$ and so $A \bowtie^{f,g} (K,K')$ is weak symmetric. □

Theorem 3.7. *Assume that A is weak symmetric and $K \cap S \neq \emptyset$, where S is the set of regular central elements of B and $K' \cap S' \neq \emptyset$, where S' is the set of regular central elements of C. Then $A \bowtie^{f,g} (K,K')$ is weak symmetric if and only if $f(A)+K$ and $g(A)+K'$ are weak symmetric rings.*

Proof. It is enough to show that if $A \bowtie^{f,g} (K,K')$ is weak symmetric, then $f(A)+K$ and $g(A)+K'$ are weak symmetric rings. For if $s \in S \cap K$ and $f(a)+x$, $f(b)+y, f(c)+z \in f(A)+K$ such that $(f(a)+x)(f(b)+y)(f(c)+z)$ is nilpotent, then $(f(0)+s(f(a)+x),0)(f(0)+s(f(b)+y),0)(f(0)+s(f(c)+z),0)$ is nilpotent in $A \bowtie^{f,g} (K,K')$. By hypothesis, $(f(0)+s(f(a)+x),0)(f(0)+s(f(b)+y),0)(f(0)+s(f(c)+z),0)$ is nilpotent in $A \bowtie^{f,g} (K,K')$. Hence, $s^3((f(a)+x)(f(b)+y)(f(c)+z))$ is nilpotent and $((f(a)+x)(f(b)+y)(f(c)+z))$ is nilpotent since s is central regular. Hence, $f(A)+K$ is weak symmetric. Also if $s' \in S' \cap K'$ and $g(a)+x'$, $g(b)+y', g(c)+z' \in g(A)+K'$ such that $(g(a)+x')(g(b)+y')(g(c)+z')$ is nilpotent, then $(0, g(0)+s'(g(a)+x'))(0, g(0)+s'(g(b)+y'))(0, g(0)+s'(g(c)+z'))$ is nilpotent in $A \bowtie^{f,g} (K,K')$. By hypothesis, $(0, g(0)+s'(g(a)+x'))(0, g(0)+s'(g(b)+y'))(0, g(0)+s'(g(c)+z')$ is nilpotent in $A \bowtie^{f,g} (K,K')$. Hence, $s'^3((g(a)+x')(g(b)+y')(g(c)+z'))$ is nilpotent and $((g(a)+x')(g(b)+y')(g(c)+z'))$ is nilpotent since s' is central regular. Hence, $g(A)+K'$ is weak symmetric. □

Definition 3.8. A ring R is called reversible if for any $a, b \in R, ab = 0$ implies $ba = 0$.

In [3], Cohn studied reversible rings. Futher studies are done in [9], [11]. We start with an example to illustrate the definition.

Example 3.9. Let $A = \mathbb{Z}_2$ and $B = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ and $C = \begin{pmatrix} \mathbb{Z}_2 & 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 & 0 \\ 0 & 0 & \mathbb{Z}_2 \end{pmatrix}$ be rings and $K = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$ an ideal of B and $K' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbb{Z}_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ an ideal of C. Let $f\colon A \to B$ be defined by $f(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, where $a \in \mathbb{Z}_2$ and $g\colon A \to C$ be defined by $g(a) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$, where $a \in \mathbb{Z}_2$. Then

$$f(A)+K = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$$

and

$$g(A)+K' = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Hence, $f(A)+K$ and $g(A)+K'$ are reversible rings. Also,

$$\begin{aligned} A \bowtie^{f,g} (K,K') = \Bigg\{ & \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right), \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right), \\ & \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right), \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right), \\ & \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right), \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right), \\ & \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right), \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \Bigg\}, \end{aligned}$$

is a reversible ring.

Now we investigate the conditions on the reversibility of the rings of the form $A \bowtie^{f,g} (K,K')$.

Theorem 3.10. *Let A, B and C be rings, $f\colon A \to B$ and $g\colon A \to C$ be two ring homomorphisms, K and K' be the two ideals of B and C, respectively. Then the following hold.*

(1) *If A, $f(A)+K$ and $g(A)+K'$ are reversible, then $A \bowtie^{f,g} (K,K')$ is reversible.*

(2) (a) *If B is reversible, then $A \bowtie^{f,g} (K,K')$ is reversible.*

(b) *If C is reversible, then $A \bowtie^{f,g} (K,K')$ is reversible.*

(c) *If $f(A)+K$ is reversible and $K'=0$, then $A \bowtie^{f,g} (K,K')$ is reversible.*

(d) *If $g(A)+K'$ is reversible and $K=0$, then $A \bowtie^{f,g} (K,K')$ is reversible.*

Proof. (1) Let $(f(a)+x, g(a)+x'), (f(b)+y, g(b)+y') \in A \bowtie^{f,g} (K,K')$ with $(f(a)+x, g(a)+x')(f(b)+y, g(b)+y') = 0$. Then $(f(a)+x)(f(b)+y) = 0$ and $(g(a)+x')(g(b)+y') = 0$. By hypothesis, $(f(b)+y)(f(a)+x) = 0$ and $(g(b)+y')(g(a)+x') = 0$. It follows that $A \bowtie^{f,g} (K,K')$ is reversible.

(2) (a) Note that $f(A)+K$ is reversible as a subring of the reversible ring B. It follows that $f(A)+K$ is isomorphic to $A \bowtie^{f,g} (K,K')$ by the homomorphism α defined by $\alpha(f(a)+k, g(a)+k') = f(a)+k$, where $(f(a)+k, g(a)+k') \in A \bowtie^{f,g} (K,K')$.

(b) It is clear that $g(A)+K'$ is reversible as a subring of the reversible ring C. It follows that $g'(A)+K$ is isomorphic to $A \bowtie^{f,g} (K, K')$ by the homomorphism β defined by $\beta(f(a)+k, g(a)+k') = g(a)+k'$, where $(f(a)+k, g(a)+k') \in A \bowtie^{f,g} (K, K')$.

(c) By Proposition 4.1 of [7], we have the canonical isomorphism

$$\frac{A \bowtie^{f,g} (K, K')}{0 \times K'} \cong f(A) + K$$

since $K' = 0$. Hence, $A \bowtie^{f,g} (K, K')$ is reversible by the property that reversibility is preserved under isomorphism.

(d) By Proposition 4.1 of [7], we have the canonical isomorphism

$$\frac{A \bowtie^{f,g} (K, K')}{K \times 0} \cong g(A) + K'$$

since $K = 0$. Hence, $A \bowtie^{f,g} (K, K')$ is reversible by the property that reversibility is preserved under isomorphism. □

In [13], a ring A is called *weakly reversible* if for all $a, b, r \in A$ such that $ab = 0$, $Abra$ is a nil left ideal of A (equivalently, $braA$ is a nil ideal of A).

Theorem 3.11. *Let A, B and C be rings. Let $f\colon A \to B$ and $g\colon A \to C$ be two ring homomorphisms, and K and K' be two ideals of B and C, respectively. Then the following hold.*

(1) *If A, $f(A)+K$ and $g(A)+K'$ are weakly reversible, then $A \bowtie^{f,g} (K, K')$ is weakly reversible.*

(2) *Assume that A is weakly reversible and f, g are injective. If $A \bowtie^{f,g} (K, K')$ is weakly reversible and $f(A) \cap K = 0$ and $g(A) \cap K' = 0$, then $f(A)+K$ and $g(A)+K'$ are weakly reversible.*

Proof. (1) Let $(f(a)+k_1, g(a)+k'_1), (f(b)+k_2, g(b)+k'_2) \in A \bowtie^{f,g} (K, K')$ with $(f(a)+k_1, g(a)+k'_1)(f(b)+k_2, g(b)+k'_2) = 0$. Since A is weakly reversible, we have $(f(a)+k_1)(f(b)+k_2) = 0$ and $(g(a)+k'_1)(g(b)+k'_2) = 0$. We have $Abca$ is a nil ideal of A, $f(A)+K(f(b)+k_2)(f(c)+k_3)(f(a)+k_1)$ is a nil ideal of $f(A)+K$ for all $(f(c)+k_3) \in f(A)+K$ and $g(A)+K'(g(b)+k'_2)(g(c)+k'_3)(g(a)+k'_1)$ is a nil ideal of $g(A)+K'$ for all $(g(c)+k'_3) \in g(A)+K'$ since A, $f(A)+K$ and $g(A)+K'$ are weakly reversible. Thus, there exist $m, n, p \in \mathbb{N}$ such that $(rbca)^m = 0$ for all $c \in A$, $[(f(r)+s)(f(b)+k_2)(f(c)+k_3)(f(a)+k_1)]^n = 0$ and $[(g(r)+s') \times (g(b)+k'_2)(g(c)+k'_3)(g(a)+k'_1)]^p = 0$. For $k = \max\{m, n, p\}$, $[(f(r)+s, g(r)+s') \times (f(b)+k_2, g(b)+k'_2)(f(c)+k_3, g(c)+k'_3)(f(a)+k_1, g(a)+k'_1)]^k = 0$. We have $A \bowtie^{f,g} (K, K')(f(b)+k_2, g(b)+k'_2)(f(c)+k_3, g(c)+k'_3)(f(a)+k_1, g(a)+k'_1)$ is a nil ideal of $A \bowtie^{f,g} (K, K')$ for all $(f(c)+k_3, g(c)+k'_3) \in A \bowtie^{f,g} (K, K')$.

(2) We have $(f(a)+0, g(a)+0)(f(b)+0, g(b)+0) = 0 \in A \bowtie^{f,g} (K, K')$ since A is weakly reversible and f, g are injective. By hypothesis, $A \bowtie^{f,g} (K, K')(f(b)+0, g(b)+0)(f(c)+z, g(c)+z')(f(a)+0, g(a)+0)$ is a nil ideal of $A \bowtie^{f,g} (K, K')$ for all $(f(c)+z, g(c)+z') \in A \bowtie^{f,g} (K, K')$. Let $(f(a)+k_1)(f(b)+k_2) = 0$ and $(g(a)+k_1')(g(b)+k_2') = 0$. Then $(f(a)+k_1, g(a)+k_1') = 0$. By hypothesis, $A \bowtie^{f,g} (K, K')(f(b)+k_2, g(b)+k_2')(f(c)+k_3, g(c)+k_3')(f(a)+k_1, g(a)+k_1')$ is a nil ideal of $A \bowtie^{f,g} (K, K')$ for all $(f(c)+k_3, g(c)+k_3') \in A \bowtie^{f,g} (K, K')$. Then $f(A)+K(f(b)+k_2)(f(c)+k_3)(f(a)+k_1)$ is a nil ideal of $f(A)+K$ for all $(f(c)+k_3) \in f(A)+K$ and $g(A)+K'(g(b)+k_2')(g(c)+k_3')(g(a)+k_1')$ is a nil ideal of $g(A)+K'$ for all $(g(c)+k_3') \in g(A)+K'$. □

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