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ON FEEBLY NIL-CLEAN RINGS

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Abstract. A ring R is feebly nil-clean if for any $a \in R$ there exist two orthogonal idempotents $e, f \in R$ and a nilpotent $w \in R$ such that $a = e - f + w$. Let R be a 2-primal feebly nil-clean ring. We prove that every matrix ring over R is feebly nil-clean. The result for rings of bounded index is also obtained. These provide many classes of rings over which every matrix is the sum of orthogonal idempotent and nilpotent matrices.

Keywords: orthogonal idempotent matrix; nilpotent matrix; matrix ring; feebly nil-clean ring

MSC 2020: 15A23, 15B33, 16U99

1. Introduction

Throughout, all rings are associative with an identity. It is of interest to write a matrix as the sum of certain elements such as idempotents, nilpotents, see [3], [10]. The motivation of this paper is to find a class of rings over which every square matrix is the sum of orthogonal idempotent and nilpotent matrices.

A ring is called *nil-clean* if every element can be written as the sum of an idempotent and a nilpotent. A ring R is weakly nil-clean provided that every element in R is the sum or difference of a nilpotent element and an idempotent. Such rings have been the object of much investigation over the last decade, as they are related to the well-studied clean rings of Nicholson. Though nil and weakly clean rings are popular, the conditions a bit restrictive (for example, there are even fields which are not weakly nil clean). The subjects of nil-clean and weakly nil-clean rings are interesting for so many mathematicians, see e.g., [4], [6], [10], [11] and [13]. In the current paper, we seek to remedy this by looking at an interesting generalization of nil and weakly nil cleanness, which they call feebly nil-clean. That is, a ring R is feebly nil-clean provided that every element in R is the sum of two orthogonal idempotents

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and a nilpotent. This new class enjoys many interesting properties and examples (for example, all tripotent rings are feebly nil-clean). We easily see that every feebly nil-clean ring is feebly clean. Here, a ring is feebly clean if for any $a \in R$ there exist two orthogonal idempotents $e, f \in R$ and a unit $u \in R$ such that $a = e - f + u$. Feebly clean rings were extensively studied in [2]. We shall investigate when a matrix ring over certain feebly nil-clean rings is feebly nil-clean, i.e., when every matrix over certain feebly nil-clean ring can be written as the sum of two orthogonal idempotents and a nilpotent matrices. A ring R is 2-primal if its prime radical coincides with the set of nilpotent elements of the ring. Examples of 2-primal rings include commutative rings and reduced rings. Let R be a 2-primal feebly nil-clean ring. We prove that $M_n(R)$ is feebly nil-clean for all $n \in \mathbb{N}$. A ring R is of bounded index if there is a positive integer n such that $a^n = 0$ for each nilpotent element a of R. We also prove that the matrix ring is feebly nil-clean for a feebly nil-clean ring of bounded index. These provide many classes of rings over which every matrix is the sum of orthogonal idempotent and nilpotent matrices.

We use $N(R)$ to denote the set of all nilpotent elements in R and $J(R)$ the Jacobson radical of R, and let $\mathbb{N}$ stand for the set of all natural numbers.

2. Structure theorems

The aim of this section is to investigate elementary properties of feebly nil-clean rings which will be used in the sequel. We begin with the following example.

Example 2.1. The class of feebly nil-clean rings contains many familiar examples.

(1) Every weakly nil-clean ring is feebly nil-clean, e.g., nil-clean rings, Boolean rings, weakly Boolean rings.
(2) Every strongly 2-nil-clean ring is feebly nil-clean, see [5].
(3) $\mathbb{Z}_3 \times \mathbb{Z}_3$ is feebly nil-clean, while it is not weakly nil-clean.
(4) A local ring R is feebly nil-clean if and only if $R/J(R) \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$, and $J(R)$ is nil.

We also provide some examples illustrating which ring-theoretic extensions of feebly nil-clean rings produce feebly nil-clean rings.

Example 2.2.

(1) Any quotient of a feebly nil-clean ring is feebly nil-clean.
(2) Any finite product of feebly nil-clean rings is feebly nil-clean. But $R = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \dots$ is an infinite product of feebly nil-clean rings, which is not feebly nil-clean. Here, the element $(0, 2, 2, 2, \dots) \in R$ cannot be written as the sum of two idempotents and a nilpotent.

(3) The triangular matrix ring $T_n(R)$ over a feebly nil-clean ring R is feebly nil-clean.
(4) The quotient ring $R[[x]]/(x^n)$ $(n \geqslant 1)$ of a feebly nil-clean ring R is feebly nil-clean.

Theorem 2.3. *Let I be a nil ideal of the ring R. Then R is feebly nil-clean if and only if the quotient ring R/I is feebly nil-clean.*

Proof. $\Rightarrow$ It is obtained from Example 2.2 (1).

$\Leftarrow$ Let $a \in R$, there exist two orthogonal idempotents $\overline{\alpha}$ and $\overline{\beta} \in R/I$ and a nilpotent $\overline{w} \in R/I$ such that $\overline{a} = \overline{\alpha} - \overline{\beta} + \overline{w}$. Since $\overline{\alpha}$ and $\overline{\beta}$ are orthogonal, $1 - \beta\alpha \in 1 + I \subseteq U(R)$. Since I is nil, there exists idempotents $e, g \in R$ such that $\overline{e} = \overline{\alpha}$ and $\overline{g} = \overline{\beta}$. Set $f = (1-e)(1-ge)^{-1}g(1-e)$. Then

$$f^2 = (1-e)(1-ge)^{-1}g(1-e)(1-ge)^{-1}g(1-e) = f$$

and $\overline{f} = \overline{\beta}$. This implies that $e, f \in R$ are orthogonal. Thus, we have orthogonal $e, f \in R$ and $w \in N(R)$ such that $a = e - f + w + r$ for some $r \in I$. Since $w \in N(R)$, we may assume that $w^k = 0$ for some $k \in \mathbb{N}$, this implies that $(w+r)^k \in I$ and so $w + r \in N(R)$. This completes the proof. $\square$

We use $P(R)$ to denote the prime radical of a ring R. That is, $P(R) = \bigcap\{P\colon\ P$ is a prime ideal of $R\}$. We have:

Corollary 2.4. *A ring R is feebly nil-clean if and only if the quotient ring $R/P(R)$ is feebly nil-clean.*

Proof. As $P(R)$ is a nil ideal of R, the result follows from Theorem 2.3. $\square$

Corollary 2.5. *Let R be a ring. Then the following are equivalent:*
(1) *R is feebly nil-clean.*
(2) *$T_n(R)$ is feebly nil-clean for all $n \in \mathbb{N}$.*
(3) *$T_n(R)$ is feebly nil-clean for some $n \in \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (2) Let

$$I = \left\{ \begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ 0 & 0 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \colon\ a_{ij} \in R \right\}.$$

Then I is an ideal of $T_n(R)$. Clearly, $T_n(R)/I \cong R \times R \times \ldots \times R$. In light of Example 2.2 and Theorem 2.3, we show that $T_n(R)$ is feebly nil-clean.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) Straightforward. $\square$

An element $a \in R$ is a tripotent if $a^3 = a$. A ring R is a tripotent ring if every element in R is a tripotent. For further use, we record the following characterizations of a 2-primal feebly nil-clean ring.

Theorem 2.6. *Let R be a ring. Then the following are equivalent:*
(1) *R is a 2-primal feebly nil-clean ring.*
(2) *For all $a \in R$, $a - a^3 \in P(R)$.*
(3) *For any $a \in R$, there exists a tripotent $e \in R$ such that $a - e \in P(R)$.*
(4) *$R/P(R)$ is tripotent.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R$. Then there exist two orthogonal idempotents $e, f \in R$ and a nilpotent $w \in R$ such that $a = e - f + w$. As R is 2-primal, $w \in P(R)$. Hence,

$$a^3 - a = (e-f)^3 + (e-f)^2 w + (e-f)w(e-f) + (e-f)w^2 \\ + w(e-f)^2 + w(e-f)w + w^2(e-f) + w^3 - (e-f) \in P(R),$$

as desired.

(2) $\Rightarrow$ (3) Let $x \in N(R)$. Then $x^3 - x \in P(R)$. Hence, $x^2 - 1 \in U(R)$, hence, $x = (x^3 - x)(x^2 - 1)^{-1} \in P(R)$. This shows that $N(R) \subseteq P(R)$. Let $a \in R$. In view of [5], Theorems 2.3 and 2.8, there exists $e^3 = e \in R$ such that $a - e \in N(R) \subseteq P(R)$, as required.

(3) $\Rightarrow$ (4) This is obvious.

(4) $\Rightarrow$ (1) Let $S := R/P(R)$ be tripotent, and let $a \in R$. In view of [8], Theorem 1, there exists commutating idempotents $e, f \in S$ such that $a = e - f$. Hence, $e = e(1-f) - f(1-e)$ is the difference of two orthogonal idempotents. Hence, $R/P(R)$ is feebly nil-clean. As is well known, $P(R)$ is nil. Therefore, we complete the proof by Theorem 2.3. $\square$

A ring R is a right (left) quasi-duo ring if every maximal right (left) ideal of R is an ideal. For instance, local rings, duo rings and weakly right (left) duo rings are all right (left) quasi-duo rings. Every abelian exchange ring is a right (left) duo ring, see [14].

Corollary 2.7. *Every 2-primal feebly nil-clean is right (left) quasi-duo.*

Proof. By virtue of Theorem 2.6, $R/P(R)$ is tripotent. Since $R/J(R) \cong (R/P(R))/(J(R)/P(R))$, we see that $R/J(R)$ is commutative. Let M be a right (left) maximal ideal of R. Then $M/J(R)$ is an ideal of $R/J(R)$. Let $x \in M, r \in R$. Then $\overline{rx} \in M/J(R)$, and then $rx \in M + J(R) \subseteq M$. This shows that M is an ideal of R. Therefore, R is right (left) quasi-duo. $\square$

3. Feebly nil-clean matrix rings

In [7], Corollary 1, Han and Nicholson proved that every matrix ring of a clean ring (i.e., every element is the sum of an idempotent and a unit) is clean. By using a similar route, we easily see that every matrix over a feebly nil-clean ring is the sum of idempotent and an invertible matrices. The purpose of this section is to investigate certain feebly nil-clean rings over which every matrix is feebly nil-clean. We have:

Lemma 3.1. *$M_n(\mathbb{Z}_3)$ is feebly nil-clean.*

Proof. Let $A \in M_n(\mathbb{Z}_3)$. In view of [1], Theorem 2, we can find $E^3 = E \in M_n(\mathbb{Z}_3)$ and a nilpotent $W \in M_n(\mathbb{Z}_3)$ such that $A = E+W$. Set $E_1 = -E-E^2$ and $E_2 = E - E^2$. Since $2 = -1$ in $\mathbb{Z}_3$, we directly check that

$$E = E_1 - E_2, \quad E_1^2 = E_1, \quad E_2^2 = E_2.$$

Further, we see that $E_1E_2 = E_2E_1 = 0$. Therefore, $A = E_1 - E_2 + W$, hence the result. $\square$

Lemma 3.2. *Let R be tripotent. Then $M_n(R)$ is feebly nil-clean for all $n \in \mathbb{N}$.*

Proof. Let $A \in M_n(R)$, and let S be the subring of R generated by the entries of A. That is, S is formed by finite sums of monomials of the form: $a_1a_2\dots a_m$, where $a_1, \dots, a_m$ are the entries of A. Since R is a commutative ring in which $6 = 0$, S is a finite ring in which $x = x^3$ for all $x \in S$. By virtue of [5], Lemma 4.1, S is isomorphic to a finite direct product of $\mathbb{Z}_2$ and/or $\mathbb{Z}_3$. In terms of Lemmas 2.4 and Lemma 2.2 (2), $M_n(S)$ is feebly nil-clean. As $A \in M_n(S)$, A is the sum of two idempotent and a nilpotent matrices over S, as desired. $\square$

Theorem 3.3. *Let R be a 2-primal feebly nil-clean ring, then $M_n(R)$ is feebly nil-clean for all $n \in \mathbb{N}$.*

Proof. Since R is feebly nil-clean, it follows by Lemma 2.6 that $R/P(R)$ is tripotent. In virtue of Lemma 3.2, $M_n(R/P(R))$ is feebly nil-clean. Further, $M_n(P(R)) = P(M_n(R))$ is nil. As $M_n(R/P(R)) \cong M_n(R)/M_n(P(R))$, it follows by Theorem 2.3 that $M_n(R)$ is feebly nil-clean. This completes the proof. $\square$

Corollary 3.4. *Let R be a commutative weakly nil-clean ring. Then $M_n(R)$ is feebly nil-clean for all $n \in \mathbb{N}$.*

Proof. As every commutative weakly nil-clean ring is a 2-primal feebly nil-clean ring, we obtain the result by Theorem 3.3. $\square$

Corollary 3.5. *Let R be a commutative ring in which every element is the sum of two idempotents and a nilpotent. Then $M_n(R)$ is feebly nil-clean for all $n \in \mathbb{N}$.*

P r o o f. Let $a \in R$. Then we have two idempotents $e, f \in R$ and a nilpotent $w \in R$ such that $1 - a = e + f + w$; hence, $a = (1 - e)(1 - f) - ef + w$. Clearly, $(1 - e)(1 - f)$ and ef are orthogonal idempotents. Hence, R is feebly nil-clean. Since every commutative ring is a 2-primal ring, we complete the proof in terms of Theorem 3.3. □

Example 3.6. Let $m = 2^k 3^l$ $(k, l \in \mathbb{N})$. Then $M_n(\mathbb{Z}_m)$ is feebly nil-clean for all $n \in \mathbb{N}$.

P r o o f. Clearly, $\mathbb{Z}_m$ is a commutative weakly nil-clean ring, hence, the result follows by Corollary 3.4. □

For instance, every square matrix over $\mathbb{Z}_6$ is the sum of two orthogonal idempotent and a nilpotent matrices.

Example 3.7. Let

$$\mathbb{Z}_{3^m}[\alpha] = \Big\{a + b\alpha \colon a, b \in \mathbb{Z}_{3^m},\ \alpha = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i},\ \mathrm{i}^2 = -1\Big\} \quad (m \in \mathbb{N}).$$

Then every square matrix over $\mathbb{Z}_{3^m}[\alpha]$ is the sum of two orthogonal idempotent and a nilpotent matrices.

P r o o f. Construct a map $\varphi \colon \mathbb{Z}_{3^m}[\alpha] \to \mathbb{Z}_3$ given by $\overline{a} + \overline{b}\alpha \mapsto \overline{a} + \overline{b}$. If $\overline{a} + \overline{b}\alpha = \overline{0}$, then $a, b \in 3^m\mathbb{Z} \subseteq 3\mathbb{Z}$. Hence, $\overline{a} + \overline{b} = \overline{0}$ in $\mathbb{Z}_3$. Thus, φ is well defined. Further, φ is an epimorphism of rings. Therefore, $\mathbb{Z}_{3^m}[\alpha]/\mathrm{Ker}(\varphi) \cong \mathbb{Z}_3$. Clearly, $x^2 + x + 1 = (x - \alpha)(x - \alpha^2)$. Hence, $3 = (1 - \alpha)(1 - \alpha^2)$. If $\overline{a} + \overline{b}\alpha \in \mathrm{Ker}(\varphi)$, then $a + b \equiv 0 \pmod 3$. Thus, $\overline{a} + \overline{b}\alpha = \overline{a + b} - \overline{b}(\overline{1} - \alpha) \in (1 - \alpha)$. On the other hand, $1 - \alpha \in \mathrm{Ker}(\varphi)$. Therefore, $\mathrm{Ker}(\varphi) = (1 - \alpha)$. As $\mathrm{Ker}(\varphi)$ is a maximal ideal of $\mathbb{Z}_{3^m}[\alpha]$, we get

$$J(\mathbb{Z}_{3^m}[\alpha]) \subseteq \mathrm{Ker}(\varphi).$$

As $3 = (1 - \alpha)^2(1 + \alpha) = -(1 - \alpha)^2\alpha^2$, we get $3^{3m} = (-1)^{3m}(1 - \alpha)^{6m}(\alpha^3)^2$. It follows that $(1 - \alpha)^{6m} = 0$, and so $1 - \alpha \in \mathbb{Z}_{3^m}[\alpha]$ is nilpotent. This shows that $1 - \alpha \in J(\mathbb{Z}_{3^m}[\alpha])$. Hence,

$$\mathrm{Ker}(\varphi) \subseteq J(\mathbb{Z}_{3^m}[\alpha]).$$

Clearly, $J(\mathbb{Z}_{3^m}[\alpha])$ is nil, and thus, $P(\mathbb{Z}_{3^m}[\alpha]) = \mathrm{Ker}(\varphi) = (1 - \alpha)$. Accordingly, $\mathbb{Z}_{3^m}[\alpha]/P(\mathbb{Z}_{3^m}[\alpha]) \cong \mathbb{Z}_3$. In view of Theorem 2.6, $\mathbb{Z}_{3^m}[\alpha]$ is a 2-primal feebly nil-clean ring. This completes the proof by Theorem 3.3. □

Lemma 3.8 ([11], Lemma 6.6)**.** *Let R be of bounded index. If $J(R)$ is nil, then $M_n(R)$ is nil for all $n \in \mathbb{N}$.*

Theorem 3.9. *Let R be of bounded index. If R is feebly nil-clean, then $M_n(R)$ is feebly nil-clean for all $n \in \mathbb{N}$.*

Proof. By virtue of Lemma 3.8, $M_n(J(R))$ is nil. In view of Lemma 2.6, $R/J(R)$ is tripotent. Thus, $M_n(R/J(R))$ is feebly nil-clean, in terms of Lemma 3.2. Since $M_n(R/J(R))/J(M_n(R)) \cong M_n(R/J(R))$, according to Theorem 2.3, $M_n(R)$ is feebly nil-clean. $\square$

Corollary 3.10. *Let R be a ring, and let $m \in \mathbb{N}$. If $(a - a^3)^m = 0$ for all $a \in R$, then $M_n(R)$ is feebly nil-clean for all $n \in \mathbb{N}$.*

Proof. Let $x \in J(R)$. Then $(x - x^3)^m = 0$, and so $x^m = 0$. This implies that $J(R)$ is nil. In light of [9], Theorem A.1, $N(R)$ forms an ideal of R, and so $N(R) \subseteq J(R)$. Hence, $J(R) = N(R)$ is nil. Further, $R/J(R)$ is tripotent. In light of Lemma 2.6, R is feebly nil-clean. If $a^k = 0$ ($k \in \mathbb{N}$), then $1 - a, 1 + a \in U(R)$, and so $1 - a^2 = (1 - a)(1 + a) \in U(R)$. By hypothesis, $a^m(1 - a^2)^m = 0$. Hence, $a^m = 0$, and so R is of bounded index. This completes the proof by Theorem 3.9. $\square$

A ring R is a 2-Boolean ring provided that a^2 is an idempotent for all $a \in R$.

Corollary 3.11. *Let R be a 2-Boolean ring. Then $M_n(R)$ is feebly nil-clean for all $n \in \mathbb{N}$.*

Proof. Let $a \in R$. Then $a^2 = a^4$. Hence, $a^2(1 - a^2) = 0$. This shows that $(1 - a^2)^2 a^2 (1 - a^2) a = 0$, i.e., $(a - a^3)^3 = 0$. In light of Corollary 3.10, the result follows. $\square$

Let $n \geqslant 2$ be a fixed integer. Following Tominaga and Yaqub, a ring R is said to be generalized n-like provided that for any $a, b \in R$, $(ab)^n - ab^n - a^n b + ab = 0$, see [12].

Corollary 3.12. *Let R be a generalized 3-like ring. Then $M_n(R)$ is feebly nil-clean for all $n \in \mathbb{N}$.*

Proof. Let $a \in R$. Then $(a - a^3)^2 = 0$, hence the result by Corollary 3.10. $\square$

Recall that a ring R is strongly SIT-ring if every element in R is the sum of an idempotent and a tripotent that commute, see [13]. We have:

Corollary 3.13. *Let R be a strongly SIT-ring. Then $M_n(R)$ is feebly nil-clean for all $n \in \mathbb{N}$.*

Proof. Let R be a strongly SIT-ring, and let $a \in R$. In view of [13], Theorem 3.10, we see that $a^6 = a^4$, hence, $a^4(1 - a^2) = 0$. This implies that $(a - a^3)^5 = 0$. In light of Corollary 3.10, we complete the proof. $\square$

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