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MORE ON THE STRONGLY 1-ABSORBING PRIMARY IDEALS OF COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring with identity. We study the concept of strongly 1-absorbing primary ideals which is a generalization of n-ideals and a subclass of 1-absorbing primary ideals. A proper ideal I of R is called strongly 1-absorbing primary if for all nonunit elements $a, b, c \in R$ such that $abc \in I$, it is either $ab \in I$ or $c \in \sqrt{0}$. Some properties of strongly 1-absorbing primary ideals are studied. Finally, rings R over which every semi-primary ideal is strongly 1-absorbing primary, and rings R over which every strongly 1-absorbing primary ideal is prime (or primary) are characterized. Many examples are given to illustrate the obtained results.

Keywords: strongly 1-absorbing primary ideal; n-ideal; primary ideal; semi-primary ideal

MSC 2020: 13A15, 13C05

1. Introduction

We assume throughout this paper that all the rings are commutative with identity. Let R be a ring and I be an ideal of R. The set of nilpotent elements of R, the set of zero-divisors of R, the set of integers, and integers modulo n are denoted by $\sqrt{0}$, $Z(R)$, $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively. By a proper ideal I of R we mean an ideal with $I \neq R$. A semi-primary ideal is an ideal with prime radical. For any undefined notation or terminology in commutative and semicommutative ring theory, we refer the reader to [12] and [15].

It is reasonable to take the view that prime ideals form the most important class of ideals in commutative ring theory. Various generalizations of prime ideals have been studied. For instance, 2-prime ideals; indeed, a nonzero proper ideal I of R is called 2-*prime* if whenever $a, b \in R$ and $ab \in I$, then $a^2 \in I$ or $b^2 \in I$, see [6] and [14] for more details. The notion of 2-absorbing ideals was introduced and investigated in 2007 by Badawi, see [3]. A nonzero proper ideal I of R is called 2-*absorbing* if

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whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. According to [4], a nonzero proper ideal I of R is called 2-*absorbing primary* of R if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. In [5], the 1-absorbing primary ideal was introduced and studied. A proper ideal I of R is called 1-*absorbing primary* if for all nonunits $a, b, c \in R$ such that $abc \in I$, it is either $ab \in I$ or $c \in \sqrt{I}$. Recall from [18] that a nonzero proper ideal I of R is said to be 1-absorbing prime if for all nonunits $a, b, c \in R$ such that $abc \in I$, it is either $ab \in I$ or $c \in I$. It is interesting that there are some classes of ideals which are not generalizations of prime ideals but they have some properties similar to primes. For instance, n-ideals; recently, in [17], Tekir et al. introduced the concept of n-ideals. A proper ideal I of R is called an *n-ideal* if whenever $a, b \in R$ and $ab \in I$ such that $a \notin \sqrt{0}$, then $b \in I$. One can see that every n-ideal is primary but it is not prime, and a prime ideal need not be an n-ideal, see [17], Example 3.2. Then Tamekkante et al. in [16] generalized n-ideals to the concept of $(2, n)$-ideals which is a subclass of 2-absorbing primary ideals. A proper ideal I of R is called *$(2, n)$-ideal* if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{0}$ or $bc \in \sqrt{0}$. Recently, in [1], Almahdi et al. introduced the concept of strongly 1-absorbing primary ideals of commutative rings. This paper is concerned with this theme and it is devoted to studying a generalization of n-ideals which is a subclass of 1-absorbing primary ideals called the strongly 1-absorbing primary ideal.

A proper ideal I of R is called *strongly* 1-*absorbing primary* if for all nonunit elements $a, b, c \in R$ such that $abc \in I$, it is either $ab \in I$ or $c \in \sqrt{0}$, see [1], Definition on page 2. The concept of 1-absorbing primary ideals was first introduced and studied by Badawi and Yetkin and they investigated some interesting properties of 1-absorbing primary ideals. Clearly, every n-ideal is a strongly 1-absorbing primary ideal and every strongly 1-absorbing primary ideal is a 1-absorbing primary ideal. However, the converse is not true, see Example 2.2.

In this paper, first we give the definition of strongly 1-absorbing primary ideals, see Definition 2.1. We give examples (see Example 2.2) of a strongly 1-absorbing primary ideal of R that is not a 1-absorbing primary (prime) ideal. In Section 3, rings R over which every semi-primary ideal is strongly 1-absorbing primary are characterized, and we also characterize rings R over which every strongly 1-absorbing primary ideal is prime (or primary). Many examples are given in this paper to illustrate the results obtained.

2. CHARACTERIZATION OF STRONGLY 1-ABSORBING PRIMARY IDEALS

In this section, basic properties of strongly 1-absorbing primary ideals are studied and some nontrivial examples of strongly 1-absorbing primary ideals are given.

Definition 2.1 ([1], Definition on page 2)**.** Let R be a ring. A proper ideal I of R is called strongly 1-absorbing primary if for all nonunit elements $a, b, c \in R$ such that $abc \in I$, it is either $ab \in I$ or $c \in \sqrt{0}$.

It is easy to see that every n-ideal is strongly 1-absorbing primary and every strongly 1-absorbing primary ideal is 2-absorbing primary, but the converse need not be true in general. Also, the classes of prime ideals and strongly 1-absorbing primary ideals do not coincide. We can also see that every strongly 1-absorbing primary ideal is 1-absorbing primary. However, the converse also is not true, see the following example.

Example 2.2.

(1) $3\mathbb{Z}$ is a prime (primary) ideal of $\mathbb{Z}$, but it is not a strongly 1-absorbing primary ideal, since $2 \cdot 2 \cdot 3 \in 3\mathbb{Z}$ and neither $2 \cdot 2 \in 3\mathbb{Z}$ nor $3 \in \sqrt{0}$.

(2) (9) is a strongly 1-absorbing primary ideal of $\mathbb{Z}_{27}$, but it is not a prime ideal. Note that if I is a prime ideal of a ring R such that $I = \sqrt{0}$, then I is a strongly 1-absorbing primary ideal, since if $abc \in I$ is such that $c \notin \sqrt{0}$ for some nonunits $a, b, c \in R$, then $ab \in \sqrt{0} = I$.

(3) Suppose that $R = \mathbb{Z}[X, Y]/\langle Y^4\rangle$ and $I = \langle xy, y^2\rangle$ is an ideal of R such that $x = X + \langle Y^4\rangle$ and $y = Y + \langle Y^4\rangle$. Clearly, I is a strongly 1-absorbing primary ideal of R, but it is not n-ideal, since $y(x + y) \in I$, $x + y \notin \sqrt{0_R}$ and $y \notin I$.

(4) It follows from [16], Example 2.3 that $(\overline{0})$ is a $(2, n)$-ideal of the reduced ring $R = \mathbb{Z}_6$, but $(\overline{0})$ is not a strongly 1-absorbing primary ideal of R, since $\overline{2}\cdot\overline{5}\cdot\overline{3} \in (\overline{0})$ and neither $\overline{2} \cdot \overline{5} \in (\overline{0})$ nor $\overline{3} \in \sqrt{\overline{0}}$.

(5) Suppose that $D = \mathbb{Z}_2[x^2, x^3, y]$ is the ring of polynomials in indeterminates x^2, x^3, y as in [5], Example 5. Then $H = (x^2, x^3, y)$ is a maximal ideal of D and $R = D_H$ is a quasilocal ring with the maximal ideal $M = H_H = (x^2, x^3, y)_H$, where x^2 is an irreducible element of R that is not a prime element of R ($x \notin R$). It follows from [5], Example 5 that $x^2M = (x^4, x^5, x^2y)_H$ is a 1-absorbing primary ideal of R, but since $x^3yx^2 \in x^2M$ and neither $x^3y \in x^2M$ nor $x^2 \in \sqrt{0_R}$, we conclude that x^2M is not a strongly 1-absorbing primary ideal of R.

(6) Suppose that $R = k[X, Y]/\langle X^2, XY\rangle$ where k is a field and $P = \langle X, Y\rangle/\langle X^2, XY\rangle$ a prime ideal of R. Then R_P is a quasilocal ring with the maximal ideal $M = P_P$ and it is easy to see that R_P has exactly one nonmaximal prime ideal for which it is $\sqrt{0_{R_P}} = Q = (\langle X\rangle/\langle X^2, XY\rangle)_P$ such that $QM = 0_{R_P}$. It is easy to see that the only maximal ideal M of R_P is a strongly 1-absorbing primary ideal, since for all nonunits $a, b, c \in R_P$ such that $abc \in M$, $ab \in M$, and so M is a strongly 1-absorbing primary ideal of R_P. But $\frac{\overline{1}}{1} \cdot \frac{\overline{1}}{1} \cdot \frac{\overline{Y}}{1} = \frac{\overline{Y}}{1} \in M$, and neither $\frac{\overline{1}}{1} \in M$ nor $\frac{\overline{Y}}{1} \in \sqrt{0_{R_P}}$. Thus, M is not a $(2, n)$-ideal.

Consequently, in view of the last display, we have

$$\begin{aligned}\{n\text{-ideal}\} &\subsetneq \{\text{strongly 1-absorbing primary}\} \subsetneq \{\text{1-absorbing primary}\}\\ &\subsetneq \{\text{2-absorbing primary}\}\\ \{n\text{-ideal}\} &\subsetneq \{(2,n)\text{-ideal}\} \nsubseteq \{\text{strongly 1-absorbing primary}\} \subsetneq \{\text{2-absorbing primary}\}\\ \{n\text{-ideal}\} &\subsetneq \{\text{strongly 1-absorbing primary}\} \nsubseteq \{(2,n)\text{-ideal}\} \subsetneq \{\text{2-absorbing primary}\}\end{aligned}$$

Theorem 2.3. *Let I be a strongly 1-absorbing primary ideal of a ring R. Then $\sqrt{I}$ is a prime ideal of R. Moreover, $(I : c) = \{x \in R\colon cx \in I\}$ is an n-ideal of R for every nonunit element $c \in R \setminus \sqrt{I}$.*

Proof. It follows from [5], Theorem 2 that $\sqrt{I}$ is a prime ideal of R. For the second statement suppose that $ab \in (I : c)$ for some elements $a, b \in R$ and a nonunit element $c \in R \setminus \sqrt{I}$ such that $a \notin \sqrt{0}$. If a and b are units, then $c \in I$, a contradiction. So suppose that b is unit and a is nonunit. Then $c^2a \in I$, which means that $a \in \sqrt{0}$, because $c \notin \sqrt{I}$ and I is a strongly 1-absorbing primary ideal of R, a contradiction. Therefore, assume that a, b are nonunit elements of R. Then $abc \in I$, and so $bc \in I$, since I is a strongly 1-absorbing primary ideal of R and $a \notin \sqrt{0}$. Thus, $(I : c)$ is an n-ideal of R. $\square$

In the following theorem, we see that if R is not a quasilocal ring, then every strongly 1-absorbing primary ideal is primary.

Theorem 2.4 ([5], Theorem 3)**.** *Let R be a ring. If I is a strongly 1-absorbing primary ideal that is not a primary for some ideal I of R, then R is a quasilocal ring.*

In the following example, one can see that if R is a quasilocal ring, then these assertions are not necessarily equivalent.

Example 2.5. Suppose that $R = k[X, Y]/\langle X^2, XY\rangle$, where k is a field and $P = \langle X, Y\rangle/\langle X^2, XY\rangle$ a prime ideal of R. Then R_P is a quasilocal ring with maximal ideal $M = P_P$ and it is easy to see that R_P has exactly one nonmaximal prime ideal which is $\sqrt{0_{R_P}} = Q = (\langle X\rangle/\langle X^2, XY\rangle)_P$ such that $QM = 0_{R_P}$. First, we show that 0_{R_P} is a strongly 1-absorbing primary ideal of R_P. Suppose that $abc \in 0_{R_P}$ for some nonunits $a, b, c \in R_P$ such that $c \notin \sqrt{0_{R_P}}$. Since $\sqrt{0_{R_P}}$ is a prime ideal of R_P, then either $a \in \sqrt{0_{R_P}}$ or $b \in \sqrt{0_{R_P}}$. Hence, $ab = 0_{R_P} \in QM$, and thus 0_{R_P} is a strongly 1-absorbing primary ideal of R_P. But $\frac{\overline{X}}{\overline{1}} \cdot \frac{\overline{Y}}{\overline{1}} = 0_{R_P} \in \sqrt{0_{R_P}}$, and neither $\frac{\overline{X}}{\overline{1}} = 0_{R_P}$ nor $\frac{\overline{Y}}{\overline{1}} \in \sqrt{0_{R_P}}$. Therefore, 0_{R_P} is not primary.

Corollary 2.6. *Let $R = R_1 \times R_2$ be a decomposable ring, where R_1 and R_2 are rings with $1 \neq 0$. Then if I is a strongly 1-absorbing primary ideal of R, then I is a primary ideal of R.*

Recall that a ring R is said to be von Neumann regular if for every $a \in R$, there exists an element x of R such that $a = a^2x$. Also a ring R is called a *boolean ring* if whenever $a = a^2$ for every $a \in R$ or, equivalently, R has the Krull dimension 0 and is reduced. Notice that every boolean ring is also von Neumann regular.

Theorem 2.7. *Suppose that R is a ring. Then the following statements hold:*

(1) *R is a field if and only if R is a von Neumann regular ring and 0 is a strongly 1-absorbing primary ideal.*

(2) *If R is a boolean ring, then R is a field if and only if 0 is a strongly 1-absorbing primary ideal. In particular $R \cong \mathbb{Z}_2$.*

Proof. (1) Suppose that R is a field. Then, it is easy to see that R is von Neumann regular. Now, let $abc = 0$ for some nonunits $a, b, c \in R$ such that $c \notin \sqrt{0}$. It is easy to see that $\sqrt{0} = 0$. Since every field is an integral domain, $ab = 0$. Hence, 0 is a strongly 1-absorbing primary ideal. Conversely, suppose that R is a von Neumann regular ring and 0 is a strongly 1-absorbing primary ideal. Then R has the Krull dimension 0 and is reduced. Let a be a nonzero element of R. If R is nonlocal, it follows from Theorem 2.4 that 0 is primary. Since R is von Neumann regular, $a = a^2x$ for some $x \in R$. Also since $a(1 - ax) = 0$ and $a \neq 0$ and 0 is a primary ideal of R, $1 - ax = 0$. Hence, $ax = 1$, and thus a is unit. Consequently, R is a field. Now, assume that R is a quasilocal ring with the maximal ideal M and a is nonunit. Since R is von Neumann regular, $a = a^2x$ for some $x \in R$. Also since $a^2(1 - ax) = 0$ and $a \in M$ and 0 is a strongly 1-absorbing primary ideal of R, $1 - ax = 0$. Hence, $ax = 1$, and thus a is unit, a contradiction. Consequently, R is a field.

(2) Let R be a boolean ring. Then R is a von Neumann regular ring. Therefore, it follows from part (1) that R is a field if and only if 0 is a strongly 1-absorbing primary ideal. Hence, it can be easily seen that $R \cong \mathbb{Z}_2$. □

Note that, for a multiplicatively closed subset S of R and an ideal I of R, if $S^{-1}I$ is a strongly 1-absorbing primary ideal of $S^{-1}R$ and $S \cap Z_I(R) = \emptyset$, then I need not be a strongly 1-absorbing primary ideal of R, see the following example.

Example 2.8. Let $R = \mathbb{Z}$ and p be a prime number. Then $S = R \setminus p\mathbb{Z}$ is a multiplicatively closed set. Take the ideal $I = p^2\mathbb{Z}$ and choose a prime number $q \neq p$. Since $pqp = p^2q \in I$, $pq \notin I$ and $p \notin \sqrt{0}$, I is not a strongly 1-absorbing primary ideal of R. Also note that $Z_I(R) = p\mathbb{Z}$ and $S \cap Z_I(R) = \emptyset$. On the other hand, note that $S^{-1}R = \mathbb{Z}_{(p)}$ and $S^{-1}I = p^2\mathbb{Z}_{(p)}$. Now, first note that $\frac{a}{s} \in \mathbb{Z}_{(p)}$ is nonunit if and only if $p|a$. Let $\frac{x}{s_1}\frac{y}{s_2}\frac{z}{s_3} \in S^{-1}I$ for some nonunits $\frac{x}{s_1}, \frac{y}{s_2}, \frac{z}{s_3} \in \mathbb{Z}_{(p)}$. Then $p|x$, $p|y$ and $p|z$. Then we have $\frac{x}{s_1}\frac{y}{s_2} \in S^{-1}I$. Therefore, $S^{-1}I$ is a strongly 1-absorbing primary ideal of $S^{-1}R$.

Let M be an R-module. The idealization $R(+)M = \{(r, m)\colon r \in R, m \in M\}$ of M is a commutative ring with the componentwise addition and multiplication $(a, m_1)(b, m_2) = (ab, am_2 + bm_1)$ for every $a, b \in R$; $m_1, m_2 \in M$. Suppose that I is an ideal of R and N is a submodule of M. Then $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$. In this case, $I(+)N$ is called a *homogeneous ideal* of $R(+)M$. The radical of a homogeneous ideal is $\sqrt{I(+)N} = \sqrt{I}(+)M$, see [2].

Theorem 2.9. *Let M be an R-module and $I(+)N$ be a homogeneous ideal of the ring $R(+)M$. If $I(+)N$ is a strongly 1-absorbing primary ideal of $R(+)M$, then I is a strongly 1-absorbing primary ideal of R.*

Proof. Suppose that $abc \in I$ for some nonunits $a, b, c \in R$ such that $c \notin \sqrt{0}$. Then $(a, 0)(b, 0)(c, 0) = (abc, 0) \in I(+)N$ since $I(+)N$ is a strongly 1-absorbing primary ideal of $R(+)M$ and $c \notin \sqrt{0}$, $(a, 0)(b, 0) = (ab, 0) \in I(+)N$ and $(c, 0) \notin \sqrt{0_{R(+)N}}$. Hence, $ab \in I$, and thus I is a strongly 1-absorbing primary ideal of R. $\square$

3. RINGS OVER WHICH EVERY SEMI-PRIMARY IDEAL IS STRONGLY 1-ABSORBING PRIMARY AND EVERY STRONGLY 1-ABSORBING PRIMARY IDEAL IS PRIME

A complete classification of rings R satisfying the condition that every semi-primary ideal of R is primary was obtained by Gilmer et al. in [7], [8], [9]. We have seen in Theorem 2.4 that if R is a non quasilocal ring, then every strongly 1-absorbing primary ideal of R is a primary ideal of R and in Example 2.5 that in a local ring R these assertions are not necessarily equivalent. In this section the rings R over which every semi-primary ideal is strongly 1-absorbing primary are characterized as well as the rings R in which every strongly 1-absorbing primary ideal is prime (or primary).

Theorem 3.1. *Let R be a nonlocal ring such that $I \subseteq \sqrt{0}$ for every primary ideal I of R. Then the following statements are equivalent:*

1. *Every semi-primary ideal is strongly 1-absorbing primary.*
2. *Every semi-primary ideal is primary.*
3. *Every semi-primary ideal is an n-ideal.*
4. *For every nonmaximal prime ideal P of R, $p \in P$ implies that $p \in pP$.*
5. *R is of one of the following types:*
    - (a) *a ring in which every element is nilpotent,*
    - (b) *a primary domain (i.e., a domain with at most one nonzero prime ideal),*
    - (c) *a zero-dimensional ring (i.e., every prime ideal of R is maximal),*
    - (d) *a one-dimensional ring with the following property. If P and M are proper prime ideals such that $P \subset M$, and if $p \in P$, then $p = pm$ for some $m \in M$.*

Proof. (1) $\Rightarrow$ (2) It follows from Theorem 2.4.
(2) $\Rightarrow$ (3) It follows from [17], Corollary 2.13.
(3) $\Rightarrow$ (1) It is obvious.
(2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) It follows from [8], Theorem 7 and [9], Theorem 5. $\square$

Lemma 3.2. *Let R be a ring such that every semi-primary ideal of R is strongly 1-absorbing primary. Then for every semi-primary ideal I with a nomaximal-prime radical P, $P^2 \subseteq I$.*

Proof. Let R be a ring such that every semi-primary ideal of R is strongly 1-absorbing primary and let M be a maximal ideal of R such that $P \subsetneq M$ and $m \in M \setminus P$. It is easy to see that $P = \sqrt{I + (abm)}$ for every $a, b \in P$, which means that $I + (abm)$ is strongly 1-absorbing primary. But $abm \in I + (abm)$ and $m \notin \sqrt{0} \subseteq P$, hence $ab \in I + (abm)$. It follows that $ab = x + rabm$ for some $x \in I$ and $r \in R$, and so $ab(1 - rm) \in I$. Since $1 - rm \notin \sqrt{0} \subseteq M$ and I is strongly 1-absorbing primary, $ab \in I$. Thus, $P^2 \subseteq I$. $\square$

Lemma 3.3. *Let R be a ring such that every semi-primary ideal of R is strongly 1-absorbing primary. Then every semi-primary ideal of R/I is strongly 1-absorbing primary for every proper ideal I of R.*

Proof. Let R be a ring such that every semi-primary ideal of R is strongly 1-absorbing primary and let J/I be a semi-primary ideal of R/I. Then $\sqrt{J/I} = \sqrt{J}/I$ is a prime ideal of R/I. Hence, $\sqrt{J}$ is a prime ideal of R, and consequently J is a strongly 1-absorbing primary ideal of R. Therefore, by [1], Corollary 3.2, J/I is a strongly 1-absorbing primary ideal of R/I. $\square$

Recall that if S is a set of elements of a ring R, S is said to be a multiplicative system if the product of two elements of S is always an element of S.

Proposition 3.4. *Suppose that R is a ring such that every semi-primary ideal is strongly 1-absorbing primary. Then every prime ideal of R is either minimal or maximal.*

Proof. Suppose that P is a prime ideal of R that is not minimal. Then there exists a prime ideal Q of R such that $Q \subsetneq P$. Since P/Q is a nonzero prime ideal of the ring R/Q, it follows from Lemma 3.3 that every semi-primary ideal of R/Q is strongly 1-absorbing primary. We show that P/Q is a maximal ideal of R/Q, and hence P is a maximal ideal of R. Let $0_{R/Q} \neq x \in P/Q$. Then there is a minimal prime ideal P_1/Q over (x) such that $P_1/Q \subseteq P/Q$ by [10], page 9. Suppose that P_1/Q is not maximal and $B = \{r \in R/Q\colon$ there exists $m \notin P_1/Q$ such that $mr \in (x^3)\}$.

But P_1/Q is prime. Hence, B is an ideal of R/Q and $B \subseteq P_1/Q$. For every $p \in P_1/Q$, set $N_p = \{p^i m\colon i \geqslant 0 \text{ and } m \notin P_1/Q\}$. We remark that N_p is a multiplicative system which contains $C(P_1/Q) = R/Q \setminus P_1/Q$ and p, and so $C(R/Q) \nsubseteq N_p$. Therefore, by [13], Lemma 3, page 106, $C(P_1/Q)$ is a maximal multiplicative system that does not meet (x^3), because P_1/Q is a minimal prime ideal of (x^3). This means that N_p meets (x^3) in $p^i m = rx^3$ for some positive integer $i \geqslant 1$, $r \in R/Q$ and $m \notin P_1/Q$. Thus, $p^i \in B$, and hence $p \in \sqrt{B}$, which means that $P_1/Q \subseteq \sqrt{B}$, and so $P_1/Q = \sqrt{B}$. It follows from Lemma 3.2 that $(P_1/Q)^2 \subseteq B$. Hence, $x^2 \in B$. Therefore, there exist $m \notin P_1/Q$ and $r \in R/Q$ such that $x^2 m = x^3 r$. Since x is regular, $m = xr \in P_1/Q$, which is a contradiction. Hence P_1/Q is maximal, and thus P/Q is maximal. Therefore, P is a maximal ideal of R. $\square$

A ring R is said to be a UN-ring if every nonunit element of R is a product of unit and nilpotent elements, or equivalently every element of R is either nilpotent or unit (see [17], Proposition 2.25). An example of UN-rings is $\mathbb{Z}_9$.

Now, we give the main theorem of this section.

Theorem 3.5. *Let R be a quasilocal ring with the maximal ideal M. Then the following statements are equivalent:*

(1) *Every semi-primary ideal whose radical is not maximal is strongly 1-absorbing primary.*

(2) *Either R is a UN-ring or $\operatorname{Spec}(R) = \{\sqrt{0}, M\}$ such that $\sqrt{0}M = 0$.*

Proof. (1) $\Rightarrow$ (2) Let R be a quasilocal ring with the maximal ideal M such that every semi-primary ideal whose radical is not maximal is strongly 1-absorbing primary. If R is a UN-ring, the proof is complete. So, suppose that R is not a UN-ring. Then $M \neq \sqrt{0}$, and so R has a nonmaximal prime ideal P. It is easy to see that for every M-primary ideal I, we have $\sqrt{IP} = \sqrt{I} \cap P = P$, and so $P^2 \subseteq IP \subseteq I$ by Lemma 3.2. Therefore $P^2 \subseteq \cap I_i P \subseteq \cap I_i$ for every i such that $\sqrt{I_i} = M$. Now, let $x \in \cap I_i \setminus P$. Then $P \subsetneq P + (x^2) \subseteq M$. If Q is a minimal prime ideal over $P + (x^2)$, then $P \subsetneq Q \subseteq M$ which means that Q is not a minimal prime ideal of R. Hence, Q is a maximal ideal, by Proposition 3.4. Thus, $Q = M$. This means that M is the unique minimal prime ideal over $P + (x^2)$, and so $\sqrt{P + (x^2)} = M$. Hence, $x \in \cap I_i \subseteq P + (x^2)$, and so $x = p + rx^2$ for some $p \in P$ and $r \in R$. From this it follows that $1 - rx \in P \subseteq M$, which is a contradiction. Thus, $P^2 \subseteq \cap I_i \subseteq P$. Now, suppose that R has another nonmaximal prime ideal $Q \neq P$. Then $Q^2 \subseteq \cap I_i \subseteq Q$ and $P^2 \subseteq \cap I_i \subseteq P$ imply $P = Q$, a contradiction. Since R is not a UN-ring, we conclude that $0 \neq M \neq \sqrt{0}$ and $\operatorname{Spec}(R) = \{\sqrt{0}, M\}$. By an argument entirely similar to that used in Lemma 3.2, one can see that $\sqrt{0}M = 0$.

(2) $\Rightarrow$ (1) First suppose that R is a UN-ring. Then $M=\sqrt{0}$ and $\sqrt{0}$ is the only prime ideal of R. Let I be a semi-primary ideal of R. Then $\sqrt{I}=\sqrt{0}$, and hence I is a primary ideal of R. Thus, I is a strongly 1-absorbing primary ideal of R by [17], Corollary 2.13. Now, suppose that $\operatorname{Spec}(R)=\{\sqrt{0},M\}$ is such that $\sqrt{0}M=\{0\}$ and I is a semi-primary ideal of R with no maximal radical. Then $\sqrt{I}=\sqrt{0}$. Let $abc\in I$ for some nonunits $a,b,c\in R$ such that $c\notin\sqrt{0}$. Then either $a\in\sqrt{0}$ or $b\in\sqrt{0}$. Hence, $ab=0\in I$, since $\sqrt{0}M=0$. Thus every semi-primary ideal with no maximal radical is strongly 1-absorbing primary. □

Corollary 3.6. *Let R be a ring. Then the following statements are equivalent:*

(1) *Every ideal whose radical is not maximal is strongly* 1*-absorbing primary.*

(2) *R is local and either R is a UN-ring or* $\operatorname{Spec}(R)=\{\sqrt{0},M\}$ *such that* $\sqrt{0}M=0$.

Proof. (1) $\Rightarrow$ (2) Let R be a ring such that every ideal with no maximal radical is strongly 1-absorbing primary. If R is quasilocal, the proof follows from Theorem 3.5, and so in order to complete the proof, it is sufficient to show that R is local. Let P and Q be prime ideals of R. If $P\cap Q=\sqrt{P\cap Q}$ is maximal, then either $P\subseteq Q$ or $Q\subseteq P$. Assume that $P\cap Q=\sqrt{P\cap Q}$ is not maximal. Then $P\cap Q$ is strongly 1-absorbing primary, and so $P\cap Q$ is prime. Hence, either $P\subseteq Q$ or $Q\subseteq P$. Therefore, prime ideals of R are comparable, and thus R is local.

(2) $\Rightarrow$ (1) By Theorem 3.5, it suffices to show that every ideal is semi-primary. Suppose that I is an ideal of R. If R is a UN-ring, then $M=\sqrt{0}$ and $\sqrt{0}$ is the only prime ideal of R. So $\sqrt{I}=\sqrt{0}$. Hence, every ideal of R is strongly 1-absorbing primary by [17], Corollary 2.13. If $\operatorname{Spec}(R)=\{\sqrt{0},M\}$ is such that $\sqrt{0}M=0$, then either $\sqrt{I}=\sqrt{0}$ or $\sqrt{I}=M$. Thus, every ideal of R is semi-primary. □

Corollary 3.7. *Let R be a ring that admits a nonunit regular element. Then the following statements are equivalent:*

(1) *Every ideal whose radical is not maximal is strongly* 1*-absorbing primary.*

(2) *R is a domain with unique nonzero prime ideal.*

(3) *R is quasilocal and every semi-primary ideal is primary.*

Proof. (1) $\Rightarrow$ (2) Let R be a ring such that every ideal whose radical is not maximal is strongly 1-absorbing primary. It is easy to see that R is not a UN-ring. Then (R,M) is local and $\operatorname{Spec}(R)=\{\sqrt{0},M\}$ such that $\sqrt{0}M=0$, by Corollary 3.6. Suppose that x is a nonunit regular element of R. Then $x\sqrt{0}=0$, and so $\sqrt{0}=0$. Hence, R is a domain with a unique nonzero prime ideal.

(2) $\Rightarrow$ (3) Assume that I is a semi-primary ideal of R. If $I=0$, then I is prime because R is a domain. If $I\neq 0$, then $\sqrt{I}$ is maximal, and hence I is primary.

(3) $\Rightarrow$ (1) It follows from Theorem 3.5 and Corollary 3.6. □

There are many examples of domains with a unique nonzero prime ideal such as discrete valuation domain. A simple example of such rings is $\mathbb{Z}_{(p)}$ for any prime integer p, see in Example 2.8.

We conclude this paper by characterizing rings R in which every strongly 1-absorbing primary ideal is prime (or primary).

Theorem 3.8. *Let R be a ring such that $I \subseteq \sqrt{0}$ for every primary ideal I of R. Then the following statements are equivalent:*

(1) *Every strongly* 1*-absorbing primary ideal is prime.*

(2) *Every n-ideal is prime.*

(3) *The localizations of R at its maximal ideals are all fields.*

(4) *R is reduced (no nonzero nilpotents), and K-dim $R = 0$.*

(5) *R is a von Neumann regular ring.*

Proof. (1) $\Rightarrow$ (2) It is obvious.

(2) $\Rightarrow$ (3) Suppose that P is a prime ideal of R. Then it follows from [17], Proposition 2.20 (i) that, if I is an n-ideal of R, then $S^{-1}I$ is an n-ideal of $S^{-1}R$, where $S = R \setminus P$. If A is an n-ideal of $S^{-1}R$, then A has the form $S^{-1}I$, and so $S^{-1}I$ is a primary of the ideal of $S^{-1}R$. Hence, I is a primary of ideal of R. Since $I \subseteq \sqrt{0}$, it follows from [17], Corollary 2.13 that I is an n-ideal of R, and thus I is prime. Therefore, $S^{-1}I$ is a prime of ideal of $S^{-1}R$. A consequence is that every n-ideal of $S^{-1}R$ is prime, and thus $S^{-1}R$ satisfies (2). We claim that the prime ideals of R are the maximal ideals. First, suppose that R is quasilocal with maximal ideal M and P is a prime ideal of R which is not maximal. Suppose also that $x \in M \setminus P$ and Q is a minimal prime ideal over $I = P + (x)$. It easy to see that if $S = R \setminus Q$, then the maximal ideal of $S^{-1}R$ is $QS^{-1}R$ which is the unique prime ideal of $S^{-1}R$ containing $S^{-1}I$. It follows that $\sqrt{S^{-1}I} = QS^{-1}R$ is a maximal ideal of $S^{-1}R$, and this means that $S^{-1}I$ is a primary ideal of $S^{-1}R$. Therefore, I is a primary ideal of R. Note that the hypothesis implies that $I \subseteq \sqrt{0}$, and thus I is an n-ideal by [17], Corollary 2.13. Hence, I is a prime ideal of R, and thus $S^{-1}I$ is a prime ideal of $S^{-1}R$. So $S^{-1}I = QS^{-1}R$ and $(S^{-1}I)^2 = S^{-1}I$, since $QS^{-1}R$ is maximal. This means that $\frac{x}{1} \in (S^{-1}I)^2$. Then $xs = p + rx^2$ for some $p \in P$, $r \in R$ and $s \in S$. Therefore, $p = x(rx - s)$ and since $x \notin P$, we conclude that $rx - s \in P$, and so $s \in P \subseteq Q$, a contradiction. Hence, the unique prime ideal of R is M. Thus, $M = \sqrt{0}$ and M is an n-ideal by [17], Corollary 2.13, and so prime. Therefore, $M = 0$ and R is a field. Now, let R be a non quasilocal ring. Then for every maximal ideal M of R, R_M is a quasilocal ring which satisfies (2). Hence, R_M is a field, and thus the localizations of R at its maximal ideals are all fields.

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) Follows from [11], Exercise 4.15.

(5) ⇒ (1) Suppose that I is a strongly 1-absorbing primary ideal of R and $x \in \sqrt{I}$. Then $x^n \in I$ for some positive integer n. But R is von Neumann regular. Hence, there exists $y \in R$ such that $x^2y = x$. Therefore, $x^{n+1}y^n = x \in I$. This means that $\sqrt{I} = I$ is a prime ideal of R by Theorem 2.3. Thus, every strongly 1-absorbing primary ideal is prime. □

Note that strongly 1-absorbing primary ideals are not necessary primary. We saw in Example 2.2 (3) that the ideal $I = \langle xy, y^2\rangle$ of the ring $R = \mathbb{Z}[X, Y]/\langle Y^4\rangle$ is a strongly 1-absorbing primary ideal, but not primary, since $xy \in I$ and neither $x \in I$ nor $y \in \sqrt{I}$. In the following, rings over which every strongly 1-absorbing primary ideal is a primary ideal (respectively an n-ideal) are characterized.

Theorem 3.9. *Suppose that (R, M) is a quasilocal ring with a principal maximal ideal M. Then every strongly 1-absorbing primary ideal of R is primary.*

Proof. Suppose that (R, M) is a quasilocal ring with a principal maximal ideal $M = (x)$ and I is a strongly 1-absorbing primary ideal of R. Suppose also $ab \in I$ for some $a, b \in R$. If a is unit, then $b \in I \subseteq \sqrt{I}$, and hence I is a primary ideal of R. So, suppose that a and b are nonunits such that $b \notin \sqrt{I}$. Since $\sqrt{I}$ is prime, $a \in \sqrt{I}$. But a is nonunit, hence $a = rx$ for some $r \in R$. If r is unit, then $M \subseteq (a) \subseteq \sqrt{I}$, which means that $M = \sqrt{I}$. Therefore, I is a primary ideal of R. So, suppose that r is nonunit. Then $xrb \in I$ and $b \notin \sqrt{0}$. Since I is a strongly 1-absorbing primary ideal of R, we deduce that $a = xr \in I$, and thus I is a primary ideal of R. □

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