Ali Akbar Estaji; Maryam Taha

The clean elements of the ring $\mathcal{R}(L)$

Czechoslovak Mathematical Journal, Vol. 74 (2024), No. 1, 211–230

Persistent URL: http://dml.cz/dmlcz/152276

Czechoslovak Mathematical Journal, 74 (149) (2024), 211–230

THE CLEAN ELEMENTS OF THE RING $\mathcal{R}(L)$

Ali Akbar Estaji, Maryam Taha, Sabzevar

Received February 8, 2023. Published online October 27, 2023.

Abstract. We characterize clean elements of $\mathcal{R}(L)$ and show that $\alpha \in \mathcal{R}(L)$ is clean if and only if there exists a clopen sublocale U in L such that $\mathfrak{c}_L(\mathrm{coz}(\alpha - \mathbf{1})) \subseteq U \subseteq \mathfrak{o}_L(\mathrm{coz}(\alpha))$. Also, we prove that $\mathcal{R}(L)$ is clean if and only if $\mathcal{R}(L)$ has a clean prime ideal. Then, according to the results about $\mathcal{R}(L)$, we immediately get results about $\mathcal{C}_c(L)$.

Keywords: frame; ring of real-valued continuous function; strongly zero-dimensional; clean element; sublocale

MSC 2020: 06D22, 54C05, 54C30

1. Introduction

Prior to 1996, the ring $\mathcal{R}(L)$ of real-valued continuous functions on a frame, namely, the pointfree version of the ring $C(X)$, had been studied by some authors such as Ball and Hager, see [3]. In 1996, a systematic study of the ring of real-valued continuous functions in the pointfree topology was undertaken by Banaschewski, see [6]. Also, [4], [5], [9], [15], [22] are good references on the subject of frames and the ring $\mathcal{R}(L)$.

Johnstone in [16], [17] introduced the notion of strongly zero-dimensional locales, but Hui and Kang in [20] introduced a different definition for strongly zero-dimensional and proved that for every strongly zero-dimensional locale L and $a, b \in \mathrm{Coz}[L]$ with $a \vee b = \top$ there exists a complemented element c in L such that $c' \leqslant a$ and $c \leqslant b$. In this article, we adopt Hui-Kangs definition rather than Johnstones one.

We recall that a frame is strongly zero-dimensional if its Stone-Čech compactification is generated by its complemented elements. Banaschewski and Brummer in [7] showed that L is strongly zero-dimensional if and only if $a \prec\!\prec b$ in L implies the existence of a complemented element c such that $a \leqslant c \leqslant b$.

DOI: 10.21136/CMJ.2023.0062-23

Dube in [10] showed that L is strongly zero-dimensional if and only if for all $a, b \in \mathrm{Coz}[L]$ such that $a \vee b = \top$, there exist complemented elements c and d such that $c \leqslant a$, $d \leqslant b$ and $c \vee d = \top$ for each completely regular frame.

An element of the ring R is clean if it can be written as the sum of an idempotent element and a unit element. We say that R is a clean ring if every element of R is clean. A characterization of clean s of $C(X)$ was given in [2]. Azarpanah in [2] also proved that $C(X)$ is clean if and only if X is strongly zero-dimensional if and only if there exists a clean prime ideal in $C(X)$.

Since $C_c(X)$ is the largest subring of $C(X)$ whose elements are countable images, it led Estaji et al. in [12], [19], [23] to research the pointfree version of $C_c(X)$, that is, $\mathcal{R}_c(L)$. On the other hand, the concepts of localization have been considered recently, see [11]. In particular, in [18], $\mathbb{R}$-subalgebras $L_c(X)$, $L_f(X)$, and $L_1(X)$ of $C(X)$, include, respectively, functions with locally countable image, functions with locally finite image, and functions with locally constant image. In [21], we can see the results of recent studies in $L_c(X)$.

In 2020, Estaji et al. in [13] introduced R_α for the first time. Also, in 2021, Rezaei Aliabadi and Mahmodi in [1] studied $\mathrm{pim}(\alpha)$ and proved that $\mathrm{pim}(\alpha) = R_\alpha$. In [24], the authors introduced another pointfree version of $\mathcal{C}_c(X)$ via the range of functions, by using the definition of R_α and denoted it by $\mathcal{C}_c(L)$. This version is more compatible with the functions of countable range. Moreover, $\mathcal{C}_c(X)$ contains functions that have countable range but not countable overlap. One of the purposes of this article is to characterize the clean elements of $\mathcal{R}(L)$. Dube in [10] showed that L is strongly zero-dimensional if and only if $\mathcal{R}(L)$ is clean. In this paper, we also present a different proof for this statement and give other characterizations for $\mathcal{R}(L)$ and we prove when $\mathcal{C}_c(L)$ is clean.

2. Preliminaries

2.1. The frame of reals. A complete lattice L is said to be a *frame* (or *locale*) if for any $a \in L$ and $B \subseteq L$ we have $a \wedge \bigvee B = \bigvee_{b \in B} (a \wedge b)$. We denote the top element and the bottom element of a frame L by $\top$ and $\bot$, respectively. For every element $a, b \in L$ we have the Heyting operation $a \to b = \bigvee \{x \in L \colon a \wedge x \leqslant b\}$, and $a^* = a \to \bot$. A *frame homomorphism* is a map f from a frame L to a frame L' such that it preserves finite meets and arbitrary joins.

Let L be a frame. An element a of a frame L is said to be *rather below* an element b or a is *well inside* b, written as $a \prec b$, if there is an element x such that $a \wedge x = \bot$ and $x \vee b = \top$. In other words, $a^* \vee b = \top$. A frame L is said to be *regular* if $a = \bigvee_{x \prec a} x$ for every $a \in L$.

For any a and b in a frame L, we say that a is *completely below* b in L and write $a \prec\!\!\prec b$ if there exists a trail $\{a_i\}_{i\in[0,1]\cap\mathbb{Q}} \subseteq L$ such that $a_0 = a$, $a_1 = b$, and for every $p, q \in [0,1] \cap \mathbb{Q}$ with $p < q$, $a_p \prec a_q$. A frame L is said to be *completely regular* if $a = \bigvee_{x \prec\!\prec a} x$ for every $a \in L$. Throughout this article, we consider all frames to be completely regular.

Recall that the *frame of reals* is the frame $\mathcal{L}(\mathbb{R})$ generated by all ordered pairs (p, q), with $p, q \in \mathbb{Q}$, subject to the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,
(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leqslant r < q \leqslant s$,
(R3) $(p, q) = \bigvee\{(r, s)\colon\ p < r < s < q\}$, and
(R4) $\top = \bigvee\{(p, q)\colon\ p, q \in \mathbb{Q}\}$.

A *continuous real function* on a frame is a homomorphism $\mathcal{L}(\mathbb{R}) \to L$. The set of all continuous real functions on a frame L is denoted by $\mathcal{R}(L)$ and the subring of $\mathcal{R}(L)$ consisting of all bounded elements is denoted by $\mathcal{R}^*(L)$. We have $\tau\colon\ \mathcal{L}(\mathbb{R}) \to \mathcal{O}(\mathbb{R})$ an isomorphism frame by $\tau(p, q) = \{x \in \mathbb{R}\colon\ p < x < q\}$ and use this notation throughout the article. Also, we set $\tau(p, q) \diamond \tau(r, s) := \{x \diamond y\colon\ x \in \tau(p, q),\ y \in \tau(r, s)\}$ for every $\diamond \in \{+, \cdot, \vee, \wedge\}$.

For every $\alpha \in C(L)$, it holds $\mathrm{pim}(\alpha) = \bigcap\{w \in O\mathbb{R}\colon\ \alpha(w) = \top\}$, see [1]. A *cozero element* of L is an element of the form $\alpha(-, 0) \vee \alpha(0, -)$ for some $\alpha \in \mathcal{R}(L)$. For any such α, we refer to $\alpha(-, 0) \vee \alpha(0, -)$ as $\mathrm{coz}(\alpha)$. Estaji at al. in [13] put $R_\alpha = \{r \in \mathbb{R}\colon\ \mathrm{coz}(\alpha - r) \neq \top\}$ for every $\alpha \in C(L)$, and they studied some of its properties. By [1], Proposition 2.3, it is evident that $R_\alpha = \mathrm{pim}(\alpha)$. We recall that $\mathcal{C}_c(L) = \{\alpha \in \mathcal{R}(L)\colon\ R_\alpha$ is a countable subset of $\mathbb{R}\}$ as the pointfree topology version of the ring $C_c(X)$ and $\mathrm{Coz}_c[L] := \{\mathrm{coz}(\alpha)\colon\ \alpha \in \mathcal{C}_c(L)\}$.

2.2. Sublocales. For a locale L, a subset $S \subseteq L$ is a *sublocale* if and only if

$$M \subseteq L \Rightarrow \bigwedge M \in S \quad \text{and} \quad (x \in L, s \in S) \Rightarrow x \to s \in S.$$

The subset S is a frame in the order of L and inherits its Heyting structure. The smallest sublocale of L is $\mathsf{O} = \{\top\}$, and is called the *void* sublocale, and the largest sublocale of L is L. We say sublocales A and B are *disjoint* if $A \cap B = \mathsf{O}$. The set of all sublocales of a frame L is denoted by $\mathcal{S}(L)$, see [22]. For every $\{S_i \in \mathcal{S}(L)\colon\ i \in I\}$ there is the formula $\bigvee_{i\in I} S_i = \left\{\bigwedge A\colon\ A \subseteq \bigcup_{i\in I} S_i\right\}$ and $\bigwedge_{i\in I} S_i = \bigcap_{i\in I} S_i$. By [14], Proposition 4.1 (4), $L \backslash B = \bigvee\{S \in \mathcal{S}(L)\colon\ B \cap S = \mathsf{O}\}$ for every $B \in \mathcal{S}(L)$. (There are many facts about $L \setminus (-)$ in [14].) The *open* and the *closed* sublocales corresponding to each $a \in L$ are, respectively, the sublocales

$$\mathfrak{o}_L(a) = \{a \to x\colon\ x \in L\} = \{x\colon\ x = a \to x\} \quad \text{and} \quad \mathfrak{c}_L(a) = {\uparrow} a = \{x \in L\colon\ x \geqslant a\}.$$

A sublocale U of a locale L is a clopen sublocale if and only if there exists a complemented element $a \in L$ such that $U = \mathfrak{c}_L(a)$.

Some of their properties that we shall freely use are as follows:

▷ $\mathfrak{o}_L(\bot) = \mathfrak{c}_L(\top) = \mathsf{O}$ and $\mathfrak{o}_L(\top) = \mathfrak{c}_L(\bot) = L$,

▷ $\mathfrak{c}_L(a) \subseteq \mathfrak{o}_L(b)$ if and only if $a \vee b = \top$,

▷ $\mathfrak{o}_L(a) \cap \mathfrak{o}_L(b) = \mathfrak{o}_L(a \wedge b)$ and $\mathfrak{c}_L(a) \vee \mathfrak{c}_L(b) = \mathfrak{c}(a \wedge b)$,

▷ $\bigvee\limits_i \mathfrak{o}_L(a_i) = \mathfrak{o}_L\Big(\bigvee\limits_i a_i\Big)$ and $\bigcap\limits_i \mathfrak{c}_L(a_i) = \mathfrak{c}_L\Big(\bigvee\limits_i a_i\Big)$.

We know that for every sublocale M of L there exists a frame homomorphism $\nu_A\colon L \to M$ such that $\nu_A(a) = \bigwedge\{x \in M\colon\ a \leqslant x\}$ for every $a \in L$.

3. Characterizations for clean elements of $\mathcal{R}(L)$

In this section, we characterize clean elements of $\mathcal{R}(L)$. We prove $\alpha \in \mathcal{R}(L)$ is clean if and only if there exists a clopen sublocale $U \in \mathcal{S}(L)$ such that $\mathfrak{c}_L(\mathrm{coz}(\alpha - \mathbf{1})) \subseteq U \subseteq \mathfrak{o}_L(\mathrm{coz}(\alpha))$. Also, we characterize clean elements of $\mathcal{R}^*(L)$.

Let $\gamma \in \mathcal{R}(L)$ and $a \in L$ be given. For every $p, q \in \mathbb{Q}$, we define $\gamma|^a\colon\ \mathcal{L}(\mathbb{R}) \to {\uparrow}a$ given by $(p, q) \mapsto \gamma(p, q) \vee a$.

Lemma 3.1. *Suppose that a is a complemented element of L and that $\alpha \in \mathcal{R}(L)$. Then there exists a unique element γ in $\mathcal{R}(L)$ such that*

$$\gamma(r, s) = [a \vee \alpha(r, s)] \wedge [a \to \alpha(r + 1, s + 1)]$$

for every $r, s \in \mathbb{Q}$.

P r o o f. Suppose that $A := \mathfrak{c}_L(a)$ and $B := \mathfrak{o}_L(a)$. Let $h_a = \nu_A\alpha$ and $h'_a = \nu_B(\alpha - \mathbf{1})$ be given. It is evident that $h_a(s) \vee a \vee a' = h'_a(s) \vee a \vee a'$ for every $s \in \mathcal{L}(\mathbb{R})$. Thus, by [8], Proposition 1.7, there exists a unique element γ in $\mathcal{R}(L)$ such that $\gamma|^a = h_a$ and $\gamma|^{a'} = h'_a$. Hence,

$$\begin{aligned}\gamma(r, s) &= [\gamma(r, s) \vee a] \wedge [\gamma(r, s) \vee a'] = h_a(r, s) \wedge h'_a(r, s)\\ &= [a \vee \alpha(r, s)] \wedge [a \to \alpha(r + 1, s + 1)]\end{aligned}$$

for every $r, s \in \mathbb{Q}$. □

Proposition 3.2. *It holds that $\alpha \in \mathcal{R}(L)$ is clean if and only if there exists a clopen sublocale U in L such that $\mathfrak{c}_L(\mathrm{coz}(\alpha - \mathbf{1})) \subseteq U \subseteq \mathfrak{o}_L(\mathrm{coz}(\alpha))$.*

P r o o f. *Necessity.* Let $\alpha \in \mathcal{R}(L)$ be clean. Then there exist a unit element u and an idempotent e in $\mathcal{R}(L)$ such that $\alpha = u + e$. Since $\operatorname{coz}(e)$ has a complement in L, we conclude that $\mathfrak{c}_L(\operatorname{coz}(e))$ is a clopen sublocale of L. Since

$$\operatorname{coz}(e) \wedge \operatorname{coz}(\alpha - \mathbf{1}) = \operatorname{coz}(eu + e^2 - e) = \operatorname{coz}(eu) = \operatorname{coz}(e) \wedge \operatorname{coz}(u) = \operatorname{coz}(e)$$

and

$$u = e - \alpha \Rightarrow \top = \operatorname{coz}(u) = \operatorname{coz}(e - \alpha) \leqslant \operatorname{coz}(e) \vee \operatorname{coz}(\alpha),$$

we conclude that $\mathfrak{c}_L(\operatorname{coz}(\alpha - \mathbf{1})) \subseteq \mathfrak{c}_L(\operatorname{coz}(e)) \subseteq \mathfrak{o}_L(\operatorname{coz}(\alpha))$.

Sufficiency. Let U be a clopen sublocale in L such that

$$\mathfrak{c}_L(\operatorname{coz}(\alpha - \mathbf{1})) \subseteq U \subseteq \mathfrak{o}_L(\operatorname{coz}(\alpha)).$$

Then there exists a complemented element $a \in L$ such that

$$U = \mathfrak{c}_L(a) = \mathfrak{c}_L(\operatorname{coz}(f_a)).$$

Thus, it follows from Lemma 3.1 that there exists a unique element γ in $\mathcal{R}(L)$ such that $\gamma = h_a \wedge h'_a$. Since

$$\operatorname{coz}(h_a) = \nu_A \operatorname{coz}(\alpha) = \bigwedge (A \cap \mathfrak{c}_L(\operatorname{coz}(\alpha))) = \bigwedge \mathsf{O} = \top$$

and

$$\operatorname{coz}(h'_a) = \nu_B \operatorname{coz}(\alpha - \mathbf{1}) = \bigwedge (B \cap \mathfrak{c}_L(\operatorname{coz}(\alpha - \mathbf{1}))) = \bigwedge \mathsf{O} = \top,$$

we conclude that

$$\operatorname{coz}(\gamma) = \operatorname{coz}(\gamma) \vee (a \wedge a') = (\operatorname{coz}(\gamma) \vee a) \wedge (\operatorname{coz}(\gamma) \vee a') = \operatorname{coz}(h_a) \wedge \operatorname{coz}(h'_a) = \top,$$

which implies that γ is a unit element in $\mathcal{R}(L)$. Also, we have

$$\begin{aligned}(f_a + \gamma)(r, -) &= \bigvee_{t \in \mathbb{Q}} f_a(t, -) \wedge \gamma(r - t, -) \\ &= \bigvee_{\substack{t<0 \\ t \in \mathbb{Q}}} \top \wedge \gamma(r - t, -) \vee \bigvee_{\substack{0 \leqslant t < 1 \\ t \in \mathbb{Q}}} a \wedge \gamma(r - t, -) \vee \bigvee_{\substack{t \geqslant 1 \\ t \in \mathbb{Q}}} \bot \wedge \gamma(r - t, -) \\ &= \gamma(r, -) \vee (a \wedge \gamma(r - 1, -)) \leqslant \alpha(r, -)\end{aligned}$$

and

$$\begin{aligned}(f_a + \gamma)(-, r) &= \bigvee_{t \in \mathbb{Q}} f_a(-, t) \wedge \gamma(-, r - t) \\ &= \bigvee_{\substack{t>1 \\ t \in \mathbb{Q}}} \top \wedge \gamma(-, r - t) \vee \bigvee_{\substack{0 < t \leqslant 1 \\ t \in \mathbb{Q}}} a' \wedge \gamma(-, r - t) \vee \bigvee_{\substack{t \leqslant 0 \\ t \in \mathbb{Q}}} \bot \wedge \gamma(-, r - t) \\ &= \gamma(-, r - 1) \vee (a' \wedge \gamma(-, r)) \leqslant \alpha(-, r)\end{aligned}$$

for every $r \in \mathbb{Q}$, which implies that $(f_a + \gamma)(r, s) \leqslant \alpha(r, s)$ for every $r, s \in \mathbb{Q}$. Since $\mathcal{L}(\mathbb{R})$ is regular, we conclude that $f_a + \gamma = \alpha$. Therefore, α is clean. □

Since in any ring R, $a \in \mathbf{R}$ is clean if and only if $1 - a$ is clean, we have the following result.

Corollary 3.3. *It holds that $\alpha \in \mathcal{R}(L)$ is clean if and only if there exists a clopen sublocale U in L such that*

$$\mathfrak{c}_L(\operatorname{coz}(\alpha)) \subseteq U \subseteq \mathfrak{o}_L(\operatorname{coz}(\alpha - \mathbf{1})) \quad \textit{or} \quad \mathfrak{c}_L(\operatorname{coz}(\alpha - \mathbf{1})) \subseteq U \subseteq \mathfrak{o}_L(\operatorname{coz}(\alpha)).$$

Example 3.4. The above lemma is an effective criterion for recognizing the clean elements of $\mathcal{R}(L)$.

▷ Clearly every unit and every idempotent are clean.

▷ If $0 \leqslant \alpha \in \mathcal{R}(L)$ and $n \in \mathbb{N}$, then

$$\begin{aligned}\operatorname{coz}(\alpha^n - \mathbf{1}) &= \operatorname{coz}\Bigl((\alpha - \mathbf{1})\sum_{i=0}^{n-1}\alpha^i\Bigr) = \operatorname{coz}(\alpha - \mathbf{1}) \wedge \operatorname{coz}\Bigl(\sum_{i=0}^{n-1}\alpha^i\Bigr)\\ &= \operatorname{coz}(\alpha - \mathbf{1}) \wedge \top = \operatorname{coz}(\alpha - 1).\end{aligned}$$

Hence, every positive power integer of nonnegative clean elements of $\mathcal{R}(L)$ is also clean.

▷ Let $n \in \mathbb{N}$ with $2 \nmid n$ be given, and let $\alpha \in \mathcal{R}(L)$ be given. If $\alpha \in \mathcal{R}(L)$ and $\varrho_n\colon \mathcal{L}(\mathbb{R}) \to \mathcal{L}(\mathbb{R})$ is defined by $\varrho_n(p, q) = (p^n, q^n)$, then $\alpha \circ \varrho_n \in \mathcal{R}(L)$, $(\alpha \circ \varrho_n)^n = \alpha$ and $\operatorname{coz}(\alpha) = \operatorname{coz}(\alpha \circ \varrho_n)$. Since $\operatorname{coz}(\alpha \circ \varrho_n - \mathbf{1}) = \operatorname{coz}(\alpha - \mathbf{1})$, we conclude that $\alpha \in \mathcal{R}(L)$ is clean if and only if $\alpha^{1/n} = \alpha \circ \varrho_n$ is clean.

▷ Let $n \in \mathbb{N}$ with $2 \mid n$ and let $0 \leqslant \alpha \in \mathcal{R}(L)$. If $\pi\colon \mathcal{L}(\mathbb{R}) \to \uparrow(-, 0)$ such that

$$\pi(p, q) \begin{cases} (p^n, q^n) \vee (-, 0), & 0 \leqslant p < q,\\ (-, q^n), & p < 0 < q,\\ (-, 0), & p < q \leqslant 0,\end{cases}$$

then $\pi^n((p, q) \mapsto (p, q) \vee (-, 0))\colon \mathcal{L}(\mathbb{R}) \to \uparrow(-, 0)$ is the quotient map. Hence, by [4], Lemma 2.1.1, there exists an element $\overline{\alpha}$ in $\mathcal{R}(\uparrow(-, 0))$ such that $\alpha = \overline{\alpha}\pi^n$. If $\beta = \overline{\alpha}\pi$, then $\beta^n = \alpha$, see [6], Proposition 11 (3). Therefore, $\alpha \in \mathcal{R}(L)$ is clean if and only if $\alpha^{1/n} := \beta$ is clean for $\operatorname{coz}(\beta - \mathbf{1}) = \operatorname{coz}(\alpha - \mathbf{1})$.

▷ For every $\alpha \in \mathcal{R}(L)$, if $\operatorname{coz}(\alpha - \mathbf{1}) = \top$, then α is clean, because

$$\mathsf{O} = \mathfrak{c}_L(\operatorname{coz}(\alpha - \mathbf{1})) \subseteq \mathsf{O} \subseteq \mathfrak{o}_L(\operatorname{coz}(\alpha)).$$

▷ If $\alpha \in \mathcal{R}(L)$ with $|\alpha| < 1$, then it is clean.

▷ Corresponding to any $\alpha \in \mathcal{R}(L)$, $\alpha^2/(1+\alpha^2)$ is clean for $\alpha^2/(1+\alpha^2) = 1 - 1/(1+\alpha^2)$, where 1 is idempotent and $1/(1+\alpha^2)$ is a unit.

So, $\mathcal{R}(L)$ is rich in clean members.

Remark 3.5. The sum or product of two clean elements of $\mathcal{R}(L)$ need not be clean. For example, put $f(x \mapsto x^2/(4+x^2))\colon \mathbb{R} \to \mathbb{R}$ and $g(x \mapsto 4+x^2)\colon \mathbb{R} \to \mathbb{R}$. If $\alpha = \mathcal{O}(f)$ and $\beta = \mathcal{O}(g)$, then they are clean for $\mathfrak{c}_L(\mathrm{coz}(\alpha-\mathbf{1})) = \mathfrak{c}_L(\mathrm{coz}(\beta-\mathbf{1})) = \mathsf{O}$. Since

$$\begin{aligned}\mathfrak{c}_L(\mathrm{coz}(2\alpha-\mathbf{1})) &= \mathfrak{c}_L\Big(\mathbb{R} - \Big\{\frac{2}{\sqrt{3}}, \frac{-2}{\sqrt{3}}\Big\}\Big)\\ &= \Big\{\mathbb{R}, \mathbb{R} - \Big\{\frac{2}{\sqrt{3}}\Big\}, \mathbb{R} - \Big\{\frac{-2}{\sqrt{3}}\Big\}, \mathbb{R} - \Big\{\frac{2}{\sqrt{3}}, \frac{-2}{\sqrt{3}}\Big\}\Big\}\\ &\nsubseteq \mathfrak{o}_L(\mathbb{R} - \{0\}) = \mathfrak{o}_L(\mathrm{coz}(\alpha)) = \mathfrak{o}_L(\mathrm{coz}(2\alpha)),\end{aligned}$$

we conclude that 2α is not clean. One can easily see that $\alpha\beta$ is not clean. The idea of this example is taken from [2], Remark 2.3.

Lemma 3.6. *Let $\alpha \in \mathcal{R}(L)$ be idempotent and let β be a nonnegative element of $\mathcal{R}(L)$. Then the following statements are true:*

(1) $\mathrm{coz}(\alpha\beta - \mathbf{1}) = \mathrm{coz}(\alpha - \mathbf{1}) \vee \mathrm{coz}(\beta - \mathbf{1})$,

(2) $\mathrm{coz}(\alpha + \beta - \mathbf{1}) = (\mathrm{coz}(\alpha - \mathbf{1}) \vee \mathrm{coz}(\beta)) \wedge (\mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta - \mathbf{1}))$.

P r o o f. It is evident that

$$\alpha(x,-) = \begin{cases} \top, & x < 0,\\ \mathrm{coz}(\alpha), & 0 \leqslant x < 1,\\ \bot, & x \geqslant 1,\end{cases} \quad \text{and} \quad \alpha(-,x) = \begin{cases} \bot, & x \leqslant 0,\\ (\mathrm{coz}(\alpha))', & 0 < x \leqslant 1,\\ \top, & x > 1.\end{cases}$$

(1) We have

$$\begin{aligned}\mathrm{coz}(\alpha\beta - \mathbf{1}) &= \alpha\beta(-,1) \vee \alpha\beta(1,-)\\ &= \bigvee_{s>0}\Big(\alpha(-,s) \wedge \beta\Big(-,\frac{1}{s}\Big)\Big) \vee \bigvee_{s\geqslant 0}\Big(\alpha(s,-) \wedge \beta\Big(\frac{1}{s},-\Big)\Big)\\ &= [(\mathrm{coz}(\alpha))' \vee \beta(-,1)] \vee [\mathrm{coz}(\alpha) \wedge \beta(1,-)] = \mathrm{coz}(\beta - \mathbf{1}) \vee \mathrm{coz}(\alpha - \mathbf{1}).\end{aligned}$$

(2) We have

$$\begin{aligned}\mathrm{coz}(\alpha + \beta - \mathbf{1}) &= (\alpha+\beta)(-,1) \vee (\alpha+\beta)(1,-)\\ &= \bigvee_{t\in\mathbb{Q}} (\alpha(-,t) \wedge \beta(-,1-t)) \vee \bigvee_{t\in\mathbb{Q}} (\alpha(t,-) \wedge \beta(1-t,-))\\ &= [(\mathrm{coz}(\mathbf{1} - \alpha) \wedge \beta(-,1)) \vee \beta(-,0)] \vee [\beta(1,-) \vee (\mathrm{coz}(\alpha) \wedge \beta(0,-))].\end{aligned}$$

Also,

$$(\operatorname{coz}(\mathbf{1}-\alpha)\wedge\beta(-,1))\vee\beta(-,0)\vee\beta(1,-)\vee\operatorname{coz}(\alpha)=\operatorname{coz}(\beta-\mathbf{1})\vee\operatorname{coz}(\alpha),$$

and

$$(\operatorname{coz}(\mathbf{1}-\alpha)\wedge\beta(-,1))\vee\beta(-,0)\vee\beta(1,-)\vee\beta(0,-)=\operatorname{coz}(\mathbf{1}-\alpha)\vee\operatorname{coz}(\beta).$$

Hence,

$$\operatorname{coz}(\alpha+\beta-\mathbf{1})=(\operatorname{coz}(\alpha-\mathbf{1})\vee\operatorname{coz}(\beta))\wedge(\operatorname{coz}(\alpha)\vee\operatorname{coz}(\mathbf{1}-\beta)).$$

□

According to the intersection and the union of two clopen sublocales is a clopen sublocale, we have the following lemma.

Lemma 3.7. *In $\mathcal{R}(L)$, the following statements are true:*
(1) *The product of an idempotent and a nonnegative clean element is clean.*
(2) *The sum of an idempotent and a nonnegative clean element is clean.*

P r o o f. Let e be an idempotent element of $\mathcal{R}(L)$ and let α be a clean nonnegative element of $\mathcal{R}(L)$.

(1) By Lemma 3.3 there exists a complemented element $a\in L$ such that

$$\mathfrak{c}_L(\operatorname{coz}(\alpha-\mathbf{1}))\subseteq\mathfrak{c}_L(\operatorname{coz}(f_a))\subseteq\mathfrak{o}_L(\operatorname{coz}(\alpha)).$$

By Lemma 3.6, we have

$$\begin{aligned}\mathfrak{c}_L(\operatorname{coz}(e\alpha-\mathbf{1}))&=\mathfrak{c}_L(\operatorname{coz}(\mathbf{1}-e))\cap\mathfrak{c}_L(\operatorname{coz}(\alpha-\mathbf{1}))\subseteq\mathfrak{c}_L(\operatorname{coz}(\mathbf{1}-e))\cap\mathfrak{c}_L(\operatorname{coz}(f_a))\\&=\mathfrak{o}_L(\operatorname{coz}(e))\cap\mathfrak{c}_L(\operatorname{coz}(f_a))\subseteq\mathfrak{o}_L(\operatorname{coz}(e))\cap\mathfrak{o}_L(\operatorname{coz}(\alpha))=\mathfrak{o}_L(\operatorname{coz}(e\alpha)),\end{aligned}$$

where $\mathfrak{c}_L(\operatorname{coz}(\mathbf{1}-e))\cap\mathfrak{c}_L(\operatorname{coz}(f_a))$ is a clopen sublocale of L. Therefore, by Lemma 3.3, $e\alpha$ is a clean element of $\mathcal{R}(L)$.

(2) Similarly, there exists a complemented element $a\in L$ such that

$$\mathfrak{c}_L(\operatorname{coz}(\alpha-\mathbf{1}))\subseteq\mathfrak{c}_L(\operatorname{coz}(f_a))\subseteq\mathfrak{o}_L(\operatorname{coz}(\alpha)).$$

By Lemma 3.6, we have

$$\operatorname{coz}(e+\alpha-\mathbf{1})=(\operatorname{coz}(\alpha-\mathbf{1})\vee\operatorname{coz}(e))\wedge(\operatorname{coz}(\alpha)\vee\operatorname{coz}(\mathbf{1}-e)).$$

Hence,

$$\begin{aligned}\mathfrak{c}_L(\operatorname{coz}(e+\alpha-\mathbf{1})) &= \mathfrak{c}_L(\operatorname{coz}(\alpha-\mathbf{1})\vee\operatorname{coz}(e))\vee\mathfrak{c}_L(\operatorname{coz}(\alpha)\vee\operatorname{coz}(\mathbf{1}-e))\\ &\subseteq (\mathfrak{c}_L(\operatorname{coz}(f_a))\cap\mathfrak{c}_L(\operatorname{coz}(e)))\vee(\mathfrak{c}_L(\operatorname{coz}(\mathbf{1}-f_a))\cap\mathfrak{c}_L(\operatorname{coz}(\mathbf{1}-e)))\\ &= \mathfrak{c}_L(\operatorname{coz}(\beta)),\end{aligned}$$

in which $\mathfrak{c}_L(\operatorname{coz}(\beta))$ is a clopen sublocale. Since

$$\mathfrak{c}_L(\operatorname{coz}(\alpha+e)) = \mathfrak{c}_L(\operatorname{coz}(\alpha)\vee\operatorname{coz}(e)) = \mathfrak{c}_L(\operatorname{coz}(\alpha))\cap\mathfrak{c}_L(\operatorname{coz}(e)),$$

then

$$\begin{aligned}\mathfrak{c}_L(\operatorname{coz}(\beta))\cap\mathfrak{c}_L(\operatorname{coz}(\alpha+e)) &= \mathfrak{c}_L(\operatorname{coz}(\beta))\cap\mathfrak{c}_L(\operatorname{coz}(\alpha))\cap\mathfrak{c}_L(\operatorname{coz}(e))\\ &= \mathfrak{c}_L(\operatorname{coz}(f_a))\cap\mathfrak{c}_L(\operatorname{coz}(e))\cap\mathfrak{c}_L(\operatorname{coz}(\alpha))\cap\mathfrak{c}_L(\operatorname{coz}(e))\\ &\subseteq \mathfrak{c}_L(\operatorname{coz}(f_a))\cap\mathfrak{c}_L(\operatorname{coz}(e))\cap\mathfrak{c}_L(\operatorname{coz}(\mathbf{1}-f_a)) = \mathsf{O}.\end{aligned}$$

By Example 3.4, $\alpha+e$ is clean. □

It is well known that $\alpha\in\mathcal{R}(L)$ is bounded if and only if there exists $n\in\mathbb{N}$ such that $\alpha(-n,n)=\top$.

Throughout this article for every $\alpha\in\mathcal{R}(L)$ and $p\in\mathbb{R}$ we consider

$$A_{(\alpha,p)} := \mathfrak{c}_L(\operatorname{coz}((\alpha-p)\wedge\mathbf{0})).$$

Lemma 3.8. *Let $\alpha\in\mathcal{R}^*(L)$ be given. Let $p,q\in\mathbb{R}$ with $0<p<q<1$ be given. If there exists an idempotent element e in $\mathcal{R}(L)$ such that $A_{(\alpha,q)}\subseteq\mathfrak{c}_L(\operatorname{coz}(e))\subseteq A_{(\alpha,p)}$, then α is a clean element in $\mathcal{R}^*(L)$.*

P r o o f. Let $\alpha\in\mathcal{R}^*(L)$ and $p,q\in\mathbb{R}$ with $0<p<q<1$ be given, and let e be an idempotent element such that $A_{(\alpha,q)}\subseteq\mathfrak{c}_L(\operatorname{coz}(e))\subseteq A_{(\alpha,p)}$. Let $A:=\mathfrak{c}_L(a)$ and $B:=\mathfrak{o}_L(a)$ in which $a=\operatorname{coz}(e)$. Let $h=\nu_A\alpha$ and $h'=\nu_B(\alpha-\mathbf{1})$ be given. If we define

$$\gamma(r,s) = [a\vee\alpha(r,s)]\wedge[a\to\alpha(r+1,s+1)]$$

for every $r,s\in\mathbb{Q}$, then by Lemma 3.1, $\gamma\in\mathcal{R}(L)$. Now, we prove that γ is a unit and bounded. Since $\mathfrak{c}_L(\operatorname{coz}(e))\subseteq A_{(\alpha,p)}$, we conclude that

$$\top = \alpha(-,p)\vee\operatorname{coz}(\alpha) = \operatorname{coz}((\alpha-p)\wedge\mathbf{0})\vee\operatorname{coz}(\alpha) \leqslant \operatorname{coz}(e)\vee\operatorname{coz}(\alpha) = \operatorname{coz}(h).$$

According to the assumption of the lemma, $\mathfrak{c}_L(\operatorname{coz}((\alpha-q)\wedge\mathbf{0}))\leqslant\mathfrak{c}_L(\operatorname{coz}(e))$. Then

$$\operatorname{coz}(e)\leqslant\operatorname{coz}((\alpha-q)\wedge\mathbf{0})\leqslant\operatorname{coz}(\alpha-\mathbf{1}),$$

and so, $\mathrm{coz}(h') = \mathrm{coz}(e) \to \mathrm{coz}(\alpha - \mathbf{1}) = \top$. Then we can conclude that γ is a unit. Now we prove h and h' are bounded. Since α is bounded, there exists $n \in \mathbb{N}$ such that $\alpha(-n, n) = \top$. Then

$$h(-n, n) = \nu_A \alpha(-n, n) = \nu_A(\top) = \top.$$

Also, α is bounded, so is $\alpha - \mathbf{1}$. Thus, there exists $n \in \mathbb{N}$ such that $\alpha(-n, n) = \top$. Then

$$h'(-n, n) = \nu_B \alpha(-n, n) = \nu_B(\top) = \top.$$

Therefore, γ is bounded. Now we claim that the inverse of γ is

$$\beta := \gamma \circ \tau^{-1} \circ \mathcal{O}f \circ \tau\colon \ \mathcal{L}(\mathbb{R}) \to L,$$

where $f(x \mapsto x^{-1})\colon \ \mathbb{R} \setminus \{0\} \to \mathbb{R}$. It is evident that β is a homomorphism frame and that $\mathrm{coz}(\beta) = \mathrm{coz}(\gamma)$. Then

$$\mathrm{coz}(\beta\gamma) = \mathrm{coz}(\beta) \wedge \mathrm{coz}(\gamma) = \mathrm{coz}(\gamma) = \top = \mathrm{coz}(\mathbf{1}).$$

Similarly, $\mathrm{coz}(\gamma\beta) = \top = \mathrm{coz}(\mathbf{1})$. Let $p, q \in \mathbb{Q}$ be given. Then

$$\begin{aligned} \beta\gamma(p, q) &= \bigvee \{\beta(u, v) \wedge \gamma(r, s)\colon \ \tau(u, v)\tau(r, s) \subseteq \tau(p, q)\} \\ &= \gamma\Big(\bigvee \{\tau^{-1} \circ \mathcal{O}f \circ \tau(u, v) \wedge (r, s)\colon \ \tau(u, v)\tau(r, s) \subseteq \tau(p, q)\}\Big). \end{aligned}$$

Let $u, v, r, s \in \mathbb{Q}$ with $\tau(u, v)\tau(r, s) \subseteq \tau(p, q)$ be given. If $\tau^{-1} \circ \mathcal{O}f \circ \tau(u, v) \wedge (r, s) \neq \emptyset$, then $\mathbf{1}(p, q) = \top \geqslant \beta\gamma(p, q)$. Also, if $\tau^{-1} \circ \mathcal{O}f \circ \tau(u, v) \wedge (r, s) = \emptyset$, then

$$\beta\gamma(p, q) = \gamma\Big(\bigvee \{\emptyset\colon \ \tau(u, v)(r, s) \subseteq \tau(p, q)\}\Big) = \gamma(\emptyset) = \bot \leqslant \mathbf{1}(p, q).$$

Then $\beta\gamma(p, q) \leqslant \mathbf{1}(p, q)$ for every $(p, q) \in \mathcal{L}(\mathbb{R})$. Since $\mathcal{L}(\mathbb{R})$ is regular, we can conclude $\beta\gamma = \mathbf{1}$. Now we show that β is bounded. Let $\varepsilon \in \{p, 1 - q\}$ be given. Then

$$\begin{aligned} \gamma(-\varepsilon, \varepsilon) &= h(-\varepsilon, \varepsilon) \wedge h'(-\varepsilon, \varepsilon) \\ &= [\mathrm{coz}(e) \wedge (\alpha - \mathbf{1})(-\varepsilon, \varepsilon))] \vee [\alpha(-\varepsilon, \varepsilon) \wedge (\mathrm{coz}(e) \to (\alpha - \mathbf{1})(-\varepsilon, \varepsilon))]. \end{aligned}$$

If $\varepsilon = p$, then

$$\begin{aligned} \mathrm{coz}(e) \wedge \alpha(-p + 1, p + 1) &= \mathrm{coz}(e) \wedge \alpha(-p + 1, p + 1) \\ &\leqslant \alpha(-, q) \wedge \alpha(-p + 1, p + 1) = \bot, \end{aligned}$$

and

$$\begin{aligned} \alpha(-\varepsilon, \varepsilon) \wedge (\mathrm{coz}(e) \to (\alpha - \mathbf{1}))(-\varepsilon, \varepsilon) &= \alpha(-p, p) \wedge (\mathrm{coz}(e) \to \alpha(1 - p, 1 + p)) \\ &\leqslant \alpha(-p, p) \wedge (\alpha(-, p) \to \alpha(1 - p, 1 + p)) \\ &\leqslant \alpha(-, p) \wedge (\alpha(-, p))^*. \end{aligned}$$

Therefore, $\gamma(-\varepsilon,\varepsilon)=\bot$ and

$$\bot=\gamma(-\varepsilon,\varepsilon)=\gamma\circ\tau^{-1}\circ\mathcal{O}f\circ\tau\Big(\Big(-,\frac{-1}{\varepsilon}\Big)\vee\Big(\frac{1}{\varepsilon},-\Big)\Big)=\beta\Big(-,\frac{-1}{\varepsilon}\Big)\vee\beta\Big(\frac{1}{\varepsilon},-\Big).$$

Let $r\in\mathbb{Q}$ with $r>\varepsilon^{-1}$ be given. Then

$$\begin{aligned}\beta(-r,r)&=\beta(-r,r)\vee\bot=\beta(-r,r)\vee\beta\Big(-,\frac{1}{\varepsilon}\Big)\vee\beta\Big(\frac{1}{\varepsilon},-\Big)\\&=\beta\Big((-r,r)\vee\Big(-,\frac{1}{\varepsilon}\Big)\vee\Big(\frac{1}{\varepsilon},-\Big)\Big)=\top.\end{aligned}$$

Thus, β for $\varepsilon=p$ is bounded. Suppose that $\varepsilon=1-q$ is given. Similarly to the proof above, it is proved that β is bounded. □

4. Strongly zero-dimensional frames

In this section, we prove that a frame L is strongly zero-dimensional if and only if for every two completely separated sublocales A and B there exists a clopen sublocale $U\in\mathcal{S}(L)$ such that $A\subseteq U\subseteq L\setminus B$. Also, we prove under which conditions is $\mathcal{R}(L)$ a clean ring. Then we introduce the concept of c-strongly zero-dimensional frame and according to the results about $\mathcal{R}(L)$, we immediately get results about when $\mathcal{C}_c(L)$ is a clean ring.

By [20], Definition 1.1, we know that L is a strongly zero-dimensional frame if for every $\alpha_1,\dots,\alpha_n\in\mathcal{R}(L)$ with $\bigvee_{i=1}^{n}\operatorname{coz}(\alpha_i)=\top$, there exists $\{a_i\}_{i=1}^{m}$ such that

(1) $\bigvee_{i=1}^{m}a_i=\top$,

(2) for every $i\neq j$, $a_i\wedge a_j=\bot$, and

(3) for every $1\leqslant i\leqslant m$, there exists $1\leqslant j\leqslant n$ such that $a_i\leqslant\operatorname{coz}(\alpha_j)$.

Similarly to the previous definition, we define the c-strongly zero-dimensional frame. A frame L is *c-strongly zero-dimensional* if for every $\alpha_1,\dots,\alpha_n\in\mathcal{C}_c(L)$ with $\bigvee_{i=1}^{n}\operatorname{coz}(\alpha_i)=\top$, there exists $\{a_i\}_{i=1}^{m}$ in L satisfying the conditions of the previous definition.

Obviously, for every $A,B\in\mathcal{S}(L)$ and $\alpha\in\mathcal{R}(L)$, $A\subseteq\mathfrak{c}_L(\operatorname{coz}(\alpha))$ and $B\subseteq\mathfrak{c}_L(\operatorname{coz}(\alpha-\mathbf{1}))$ if and only if $\nu_A\alpha(-,0)\vee(0,-)=\bot_A$ and $\nu_B\alpha(-,0)\vee(0,-)=\bot_B$. From [4], Definition 6.2.1 if $m_0:=\nu_A$ and $m_1:=\nu_B$, then A and B are completely separated if and only if there exists $\alpha\in\mathcal{R}(L)$ such that $A\subseteq\mathfrak{c}_L(\operatorname{coz}(\alpha))$ and $B\subseteq\mathfrak{c}_L(\operatorname{coz}(\alpha-\mathbf{1}))$.

In this paper, the completely separated of the two subslocales is based on the previous paragraph. Now, according to this definition, we define:

Definition 4.1. Two sublocales A and B of a locale L are c-completely separated if there exists $\alpha \in \mathcal{C}_c(L)$ such that $A \subseteq \mathfrak{c}_L(\operatorname{coz}(\alpha))$ and $B \subseteq \mathfrak{c}_L(\operatorname{coz}(\alpha - \mathbf{1}))$.

It is clear that $A, B \in \mathcal{S}(L)$ are two completely separated sublocales if and only if for every $r, s \in \mathbb{Q}$ with $r < s$ there exists $\alpha \in \mathcal{R}(L)$ such that $A \subseteq \mathfrak{c}_L(\operatorname{coz}(\alpha - \mathbf{r}))$ and $B \subseteq \mathfrak{c}_L(\operatorname{coz}(\alpha - \mathbf{s}))$.

Lemma 4.2. *Let L be a strongly zero-dimensional frame. Then for every two completely separated sublocales A and B in a frame L there exists a clopen sublocale $U \in \mathcal{S}(L)$ such that $A \subseteq U \subseteq L \setminus B$.*

P r o o f. By the definition of completely separated, there exists $\alpha \in \mathcal{R}(L)$ such that $A \subseteq \mathfrak{c}_L(\operatorname{coz}(\alpha))$ and $B \subseteq \mathfrak{c}_L(\operatorname{coz}(\alpha - \mathbf{1}))$. Since

$$\operatorname{coz}((\alpha - \mathbf{1}) \wedge \mathbf{0}) \vee \operatorname{coz}((\alpha) \vee \mathbf{0}) = \alpha(-, 1) \vee \alpha(0, -) = \top,$$

we conclude that there exists a finite cover $\{a_i\}_{i \in I}$ such that
(1) $\bigvee_{i \in I} a_i = \top$,
(2) for every $i \neq j$, $a_i \wedge a_j = \bot$, and
(3) for every $i \in I$, $a_i \leqslant \operatorname{coz}((\alpha - \mathbf{1}) \wedge \mathbf{0})$ or $a_i \leqslant \operatorname{coz}(\alpha \vee \mathbf{0})$.
We put

$$U = \bigvee\{\mathfrak{o}_L(a)\colon\ a \in \{a_i\}_{i \in I},\ A \wedge \mathfrak{o}_L(a) \neq \mathsf{O}\} \text{ and } H := \{a \in \{a_i\}_{i \in I}\colon\ A \wedge \mathfrak{o}_L(a) \neq \mathsf{O}\}.$$

Since

$$L = \bigvee_{a \in H} \mathfrak{o}_L(a) \vee \bigvee_{a \in \{a_i\} - H} \mathfrak{o}_L(a) = \mathfrak{o}_L\Big(\bigvee_{a \in H} a\Big) \vee \mathfrak{o}_L\Big(\bigvee_{a \in \{a_i\} - H} a\Big)$$

and

$$\mathfrak{o}_L\Big(\bigvee_{a \in H} a\Big) \cap \mathfrak{o}_L\Big(\bigvee_{b \in \{a_i\} - H} b\Big) = \bigvee_{a \in H} \bigvee_{b \in \{a_i\} - H} \mathfrak{o}_L(a \wedge b) = \bigvee_{a \in H} \bigvee_{b \in \{a_i\} - H} \mathfrak{o}_L(\bot) = \mathsf{O},$$

we conclude that $\mathfrak{o}_L\big(\bigvee_{a \notin H} a\big)$ is the complement of $\mathfrak{o}_L\big(\bigvee_{a \in H} a\big)$. Thus, $U := \mathfrak{o}_L\big(\bigvee_{a \in H} a\big)$ is a clopen sublocale. Form

$$A = \bigvee_{a \in H} (A \cap \mathfrak{o}_L(a)) \vee \bigvee_{a \notin H} (A \cap \mathfrak{o}_L(a)) = \bigvee_{a \in H} (A \cap \mathfrak{o}_L(a)),$$

we infer that $A \subseteq \bigvee_{a \in H} \mathfrak{o}_L(a) = U$. Now we show that if $a \in H$, then $\mathfrak{o}_L(a) \cap B = \mathsf{O}$. We assume $a \in H$. Then $a \in \{a_i\}_{i \in I}$, which implies that $a \leqslant \alpha(-, 1)$ or

$a \leqslant \alpha(0,-)$. Let $a \leqslant \alpha(0,-)$. Then $\mathfrak{o}_L(a) \leqslant \mathfrak{o}_L(\alpha(0,-))$, but $\mathsf{O} \neq \mathfrak{o}_L(a) \wedge A \subseteq \mathfrak{c}_L(\operatorname{coz}(\alpha)) \cap \mathfrak{o}_L(\operatorname{coz}(\alpha)) = \mathsf{O}$, which is a contradiction. Thus, $a \leqslant \alpha(-,1)$ and $\mathfrak{o}_L(a) \leqslant \mathfrak{o}_L(\alpha(-,1))$. Also, $B \cap \mathfrak{o}_L(a) = \mathsf{O}$. Hence, $U \cap B = \mathfrak{o}_L\Big(\bigvee_{a\in H} a\Big) \cap B = \mathsf{O}$.

We know $L \setminus B = \bigvee\{S \in \mathcal{S}(L)\colon\ B \cap S = \mathsf{O}\}$. Thus, $A \subseteq U \subseteq L \setminus B$. $\square$

Also, if L is a c-strongly zero-dimensional frame, then for every two completely separated sublocales A and B in the frame L there exists clopen sublocale $U \in \mathcal{S}(L)$ such that $A \subseteq U \subseteq L \setminus B$.

Lemma 4.3. *If there exist $\alpha, \beta \in \mathcal{R}(L)$ such that $\mathfrak{c}_L(\operatorname{coz}(\alpha)) \cap \mathfrak{c}_L(\operatorname{coz}(\beta)) = \mathsf{O}$, then $\mathfrak{c}_L(\operatorname{coz}(\alpha))$ and $\mathfrak{c}_L(\operatorname{coz}(\beta))$ are completely separated.*

Proof. It is evident. $\square$

Using a similar argument, we can show that if $\alpha, \beta \in \mathcal{C}_c(L)$ with $\mathfrak{c}_L(\operatorname{coz}(\alpha)) \cap \mathfrak{c}_L(\operatorname{coz}(\beta)) = \mathsf{O}$, then $\mathfrak{c}_L(\operatorname{coz}(\alpha))$ and $\mathfrak{c}_L(\operatorname{coz}(\beta))$ are c-completely separated.

Lemma 4.4. *For every two completely separated sublocales A and B in a frame L, let there be a clopen sublocale $U \in \mathcal{S}(L)$ such that $A \subseteq U \subseteq L \setminus B$. Then for every $\{\alpha_1, \dots, \alpha_k\} \subseteq \mathcal{R}(L)$, where $\top = \bigvee_{i=1}^{k} \operatorname{coz}(\alpha_i)$, there exists $\{a_i\}_{i=1}^{k} \subseteq L$ such that*
(1) $\bigvee_{i=1}^{k} a_i = \top$,
(2) *for every $i \neq j$, $a_i \wedge a_j = \bot$, and*
(3) *for every i, $a_i \leqslant \operatorname{coz}(\alpha_i)$.*

Proof. We use the induction method on k. If $k = 1$, then $\alpha \in \mathcal{R}L$ and $\operatorname{coz}(\alpha) = \top$. We put $a_1 = \operatorname{coz}(\alpha)$. Let the result for all k with $k < m$ and $1 < m$ be valid. Now we assume that there exist $\alpha_1, \dots, \alpha_m \in \mathcal{R}(L)$ such that $\bigvee_{i=1}^{m} \operatorname{coz}(\alpha_i) = \top$. Since

$$\bigvee_{i=1}^{m-2} \operatorname{coz}(\alpha_i) \vee \operatorname{coz}(\alpha_{m-1}^2 + \alpha_m^2) = \top,$$

by induction, there exists $\{a_i\}_{i=1}^{m-1} \subseteq L$ such that
(1) $\bigvee_{i=1}^{m-1} a_i = \top$,
(2) for every $i \neq j$, $a_i \wedge a_j = \bot$, and
(3) for every $1 \leqslant i \leqslant m-1$, $a_i \leqslant \operatorname{coz}(\alpha_i)$ and $a_{m-1} \leqslant \operatorname{coz}(\alpha_{m-1}^2 + \alpha_m^2)$.
Then

$$\mathfrak{o}_L(a_{m-1}) \leqslant \mathfrak{o}_L(\operatorname{coz}(\alpha_{m-1})) \vee \mathfrak{o}_L(\operatorname{coz}(\alpha_m)),$$

and

$$\mathfrak{o}_L(a_{m-1}) \wedge \mathfrak{c}_L(\operatorname{coz}(\alpha_{m-1}) \vee \operatorname{coz}(\alpha_m)) = \mathsf{O}.$$

By the properties of $\{a_i\}_{i=1}^{m-1}$, we have

$$a_i \wedge \bigvee_{i \neq j} a_j = \bot \quad \text{and} \quad a_i \vee \bigvee_{i \neq j} a_j = \top.$$

Therefore, a_i has a complement and we can write $a_i = \mathrm{coz}(f_{a_i})$. Hence,

$$\mathfrak{o}_L(a_{m-1}) = \mathfrak{o}_L(\mathrm{coz}(f_{a_{m-1}})) = \mathfrak{c}_L(\mathrm{coz}(f_{a_{m-1}} - \mathbf{1}))$$

implies that

$$\mathfrak{o}_L(a_{m-1}) \setminus \mathfrak{o}_L(\mathrm{coz}(\alpha_{m-1})) = \mathfrak{c}_L(\mathrm{coz}((f_{a_{m-1}} - \mathbf{1})^2 + \alpha_{m-1}^2)).$$

Similarly,

$$\mathfrak{o}_L(a_{m-1}) \setminus \mathfrak{o}_L(\mathrm{coz}(\alpha_m)) = \mathfrak{c}_L(\mathrm{coz}((f_{a_{m-1}} - \mathbf{1})^2 + \alpha_m^2)).$$

With the above results, we conclude that

$$\mathrm{coz}((f_{a_{m-1}} - \mathbf{1})^2 + \alpha_{m_1}^2) \vee \mathrm{coz}((f_{a_{m-1}} - \mathbf{1})^2 + \alpha_m^2)) = \top.$$

For $\alpha = (f_{a_{m-1}} - \mathbf{1})^2 + \alpha_{m_1}^2$ and $\beta = (f_{a_{m-1}} - \mathbf{1})^2 + \alpha_m^2)$, by Lemma 4.3, $\mathfrak{c}_L(\mathrm{coz}(\alpha))$ and $\mathfrak{c}_L(\mathrm{coz}(\beta))$ are completely separated, and by our hypothesis, there exists a clopen sublocale $U = \mathfrak{o}_L(a)$ such that

$$\mathfrak{o}_L(\alpha) \subseteq U \subseteq L \setminus \mathfrak{c}_L(\beta) = \mathfrak{o}_L(\beta).$$

It is evident that

$$\mathfrak{o}_L(a_{m-1}) \setminus U \subseteq \mathfrak{o}_L(a_{m-1})) \setminus \mathfrak{c}_L(\mathrm{coz}(\alpha)) \subseteq \mathfrak{o}_L(\mathrm{coz}(\alpha_{m-1}))$$

and that

$$\mathfrak{o}_L(a_{m-1}) \cap U \subseteq \mathfrak{o}_L(a_{m-1})) \cap \mathfrak{o}_L(\mathrm{coz}(\beta)) \subseteq \mathfrak{o}_L(\mathrm{coz}(\alpha_m)).$$

Since $U \in \mathcal{S}(L)$ is a clopen sublocale, there exists a complemented element a such that $U = \mathfrak{c}_L(a) = \mathfrak{o}_L(a')$, then

$$\mathfrak{o}_L(a_{m-1}) \setminus U = \mathfrak{o}_L(a_{m-1}) \setminus \mathfrak{c}_L(a) = \mathfrak{o}_L(a_{m-1}) \wedge \mathfrak{o}_L(a) = \mathfrak{o}_L(a_{m-1} \wedge a),$$

and

$$\mathfrak{o}_L(a_{m-1}) \cap U = \mathfrak{o}_L(a_{m-1}) \cap \mathfrak{c}_L(a') = \mathfrak{o}_L(a_{m-1} \wedge a').$$

Let for every $1 \leqslant i \leqslant m-2$, $b_i = a_i$, also $b_{m-1} = a_{m-1} \wedge a$, and $b_m = a_{m-1} \wedge a'$. Then we have the following properties:

(1) $\bigvee_{i=1}^{m} b_i = \bigvee_{i=1}^{m-2} a_i \vee (a_{m-1} \wedge (a \vee a')) = \bigvee_{i=1}^{m-1} a_i = \top$.

(2) If $1 \leqslant i \neq j \leqslant m-2$, then $b_i \wedge b_j = a_i \wedge a_j = \bot$. If $1 \leqslant i \neq j \leqslant m-1$, then

$$b_{m-1} \wedge b_i = a_{m-1} \wedge a \wedge a_i = \bot, \quad b_m \wedge b_i = a_{m-1} \wedge a' \wedge a_i = \bot, \quad \text{and}$$
$$b_m \wedge b_{m-1} = a_{m-1} \wedge a \wedge a' = \bot.$$

Therefore, $b_i \wedge b_j = \bot$ for every $1 \leqslant i \neq j \leqslant m$.

(3) For every $1 \leqslant i \leqslant m-2$, $b_i = a_i \leqslant \operatorname{coz}(\alpha_i)$. Since $b_{m-1} = a_{m-1} \wedge a$, we conclude that

$$\mathfrak{o}_L(b_{m-1}) = \mathfrak{o}_L(a_{m-1} \wedge a) = \mathfrak{o}_L(a_{m-1}) \setminus U \subseteq \mathfrak{o}_L(\operatorname{coz}(\alpha_{m-1})).$$

Hence, $b_{m-1} \leqslant \operatorname{coz}(\alpha_{m-1})$. Using a similar argument, we obtain that if $b_m = a_{m-1} \wedge a'$, then

$$\mathfrak{o}_L(b_m) = \mathfrak{o}_L(a_{m-1} \wedge a') = \mathfrak{o}_L(a_{m-1}) \cap U \subseteq \mathfrak{o}_L(\operatorname{coz}(\alpha_m)),$$

and so $b_m \leqslant \operatorname{coz}(\alpha_m)$. □

Also, it is easy to prove that if for every two c-completely separated sublocales A and B in a frame L there exists a clopen sublocale $U \in \mathcal{S}(L)$ such that $A \subseteq U \subseteq L \backslash B$, then for every $\{\alpha_1, \ldots, \alpha_k\} \subseteq \mathcal{C}_c(L)$ with $\top = \bigvee_{i=1}^{k} \operatorname{coz}(\alpha_i)$ there exists $\{a_i\}_{i=1}^{k} \subseteq L$ satisfying the conditions of Lemma 4.4.

Corollary 4.5. *A frame L is strongly zero-dimensional if and only if for every two completely separated sublocales A and B there exists a clopen sublocale $U \in \mathcal{S}(L)$ such that $A \subseteq U \subseteq L \setminus B$.*

Proof. According to the previous lemma, it is obvious. □

Proposition 4.6. *The following statements are equivalent:*

(1) *$\mathcal{R}(L)$ is a clean ring.*
(2) *$\mathcal{R}^*(L)$ is a clean ring.*
(3) *The set of clean elements in $\mathcal{R}(L)$ is a subring of $\mathcal{R}(L)$.*
(4) *L is strongly zero-dimensional.*
(5) *Every zero-divisor in $\mathcal{R}(L)$ is clean.*
(6) *$\mathcal{R}(L)$ has a clean prime ideal.*

P r o o f. (1) $\Rightarrow$ (2) Let $\alpha \in \mathcal{R}^*(L)$. Then $k\colon \mathbb{R} \to \mathbb{R}$ by

$$k(x) = \begin{cases} 1 & \text{if } x \geqslant \dfrac{2}{3}, \\ 3x - 1 & \text{if } \dfrac{1}{3} < x < \dfrac{2}{3}, \\ 0 & \text{if } x \leqslant \dfrac{1}{3}. \end{cases}$$

Moreover, $\mathcal{O}(k)(G \mapsto k^{-1}(G))\colon \mathcal{O}(\mathbb{R}) \to \mathcal{O}(\mathbb{R})$ is a homomorphism frame. Put $\beta = \alpha \circ \mathcal{O}(k)$. We show that for $A_{(\alpha,2/3)} = \mathfrak{c}_L(\mathrm{coz}((\alpha - \frac{2}{3}) \wedge \mathbf{0}))$ and $A_{(\alpha,1/3)} = \mathfrak{c}_L(\mathrm{coz}((\alpha - \frac{1}{3}) \wedge \mathbf{0}))$, there exists an idempotent e such that $A_{(\alpha,2/3)} \leqslant \mathfrak{c}_L(\mathrm{coz}(e)) \leqslant A_{(\alpha,1/3)}$. Since $\beta \in \mathcal{R}(L)$ is clean, there exists an idempotent $e \in \mathcal{R}(L)$ such that

$$\mathfrak{c}_L(\mathrm{coz}(\beta - \mathbf{1})) \subseteq \mathfrak{c}_L(\mathrm{coz}(e)) \subseteq \mathfrak{o}_L(\mathrm{coz}(\beta)).$$

Since

$$\begin{aligned} &A_{(\alpha,2/3)} \subseteq \mathfrak{c}_L(\mathrm{coz}(\beta - \mathbf{1})) \\ &\quad\Leftrightarrow \mathfrak{c}_L\Big(\mathrm{coz}\Big(\Big(\alpha - \frac{\mathbf{2}}{\mathbf{3}}\Big) \wedge \mathbf{0}\Big)\Big) \subseteq \mathfrak{c}_L(\mathrm{coz}(\beta - \mathbf{1})) \Leftrightarrow \mathrm{coz}(\beta - \mathbf{1}) \subseteq \mathrm{coz}\Big(\Big(\alpha - \frac{\mathbf{2}}{\mathbf{3}}\Big) \wedge \mathbf{0}\Big) \\ &\quad\Leftrightarrow \beta(-,-1) \vee \beta(1,-) \leqslant \alpha\Big(-,\frac{2}{3}\Big) \Leftrightarrow \bot \leqslant \alpha\Big(-,\frac{2}{3}\Big), \end{aligned}$$

and

$$\begin{aligned} &\mathfrak{o}_L(\mathrm{coz}(\beta)) \subseteq A_{(\alpha,1/3)} \\ &\quad\Leftrightarrow \mathfrak{o}_L(\mathrm{coz}(\beta)) \subseteq \mathfrak{c}_L\Big(\mathrm{coz}\Big(\Big(\alpha - \frac{\mathbf{1}}{\mathbf{3}}\Big) \wedge \mathbf{0}\Big)\Big) \Leftrightarrow \mathfrak{o}_L(\mathrm{coz}(\beta)) \subseteq \mathfrak{c}_L\Big(\alpha\Big(-,\frac{1}{3}\Big)\Big) \\ &\quad\Leftrightarrow \bot = \mathfrak{o}_L(\mathrm{coz}(\beta)) \wedge \mathfrak{o}_L\Big(\alpha\Big(-,\frac{1}{3}\Big)\Big) \Leftrightarrow \bot = \mathfrak{o}_L\Big(\mathrm{coz}(\beta) \wedge \alpha\Big(-,\frac{1}{3}\Big)\Big) \\ &\quad\Leftrightarrow \bot = \mathrm{coz}(\beta) \wedge \alpha\Big(-,\frac{1}{3}\Big), \end{aligned}$$

we conclude that

$$A_{(\alpha,2/3)} \subseteq \mathfrak{c}_L(\mathrm{coz}(\beta - \mathbf{1})) \subseteq \mathfrak{c}_L(\mathrm{coz}(e)) \subseteq \mathfrak{o}_L(\mathrm{coz}(\beta)) \subseteq A_{(\alpha,1/3)},$$

which follows from Lemma 3.8 that α is clean in $\mathcal{R}^*(L)$.

(2) $\Rightarrow$ (3) We show that every element in $\mathcal{R}(L)$ is clean. Let $\alpha \in \mathcal{R}(L)$. Then $\beta = (-\mathbf{1} \vee \alpha) \wedge \mathbf{1} \in \mathcal{R}^*(L)$. By the assumption, there exists an idempotent $e \in \mathcal{R}(L)$ such that

$$\mathfrak{c}_L(\mathrm{coz}(\beta - \mathbf{1})) \subseteq \mathfrak{c}_L(\mathrm{coz}(e)) \subseteq \mathfrak{o}_L(\mathrm{coz}(\beta)).$$

Since $\mathfrak{o}_L(\mathrm{coz}(\beta)) = \mathfrak{o}_L(\mathrm{coz}(\alpha))$ and $\mathrm{coz}(\alpha - \mathbf{1}) = \alpha(-,1) \vee \alpha(1,-)$, we conclude that

$$\mathrm{coz}(((-\mathbf{1} \vee \alpha) \wedge \mathbf{1}) - \mathbf{1})) = ((-\mathbf{1} \vee \alpha) \wedge \mathbf{1})(-,1) \vee ((-\mathbf{1} \vee \alpha) \wedge \mathbf{1})(1,-).$$

Also, we have $[(-\mathbf{1}\vee\alpha)\wedge\mathbf{1}](-,1)=\alpha(-,1)$ and $[(-\mathbf{1}\vee\alpha)\wedge\mathbf{1}](1,-)=\bot$. Then

$$\operatorname{coz}(\beta-\mathbf{1})=\alpha(-,1)\vee\bot=\alpha(-,1)\leqslant\alpha(-,1)\vee\alpha(1,-)=\operatorname{coz}(\alpha-\mathbf{1}).$$

Therefore, we can conclude that

$$\mathfrak{c}_L(\operatorname{coz}(\alpha-\mathbf{1}))\subseteq\mathfrak{c}_L(\operatorname{coz}(\beta-\mathbf{1}))\subseteq\mathfrak{c}_L(\operatorname{coz}(e))\subseteq\mathfrak{o}_L(\operatorname{coz}(\beta))=\mathfrak{o}_L(\operatorname{coz}(\alpha)),$$

and so, by Proposition 3.2, α is clean.

(3) $\Rightarrow$ (4) Let A and B be two completely separated sublocales of L. Hence, there exists $\alpha\in\mathcal{R}(L)$ such that $A\subseteq\mathfrak{c}_L(\operatorname{coz}(\alpha))$ and $B\subseteq\mathfrak{c}_L(\operatorname{coz}(\alpha-\frac{\mathbf{1}}{\mathbf{2}}))$. Define $f\colon\mathbb{R}\to\mathbb{R}$ by

$$f(x)=\begin{cases}\dfrac{1}{2} & \text{if } x\geqslant\dfrac{1}{2},\\[2ex] x & \text{if } -\dfrac{1}{2}<x<\dfrac{1}{2},\\[2ex] -\dfrac{1}{2} & \text{if } x\leqslant-\dfrac{1}{2},\end{cases}$$

and put $\beta:=\alpha\circ\tau^{-1}\circ\mathcal{O}f\circ\tau$. Then $\operatorname{coz}(\beta)=\operatorname{coz}(\alpha)$.

Suppose $p\in\mathbb{R}$. If $\frac{1}{2}<p$, then

$$\beta(p,-)=\alpha\circ\tau^{-1}\circ\mathcal{O}f\circ\tau(p,-)=\alpha(\bot)=\bot=\frac{1}{2}(p,-),$$

and if $p<\frac{1}{2}$, then $\frac{1}{2}(p,-)=\top\geqslant\beta(p,-)$. Therefore, $\beta\leqslant\frac{1}{2}$. Now if we assume $-\frac{1}{2}\leqslant p$, then $-\frac{1}{2}(p,-)=\bot\leqslant\beta(p,-)$, and if $p<-\frac{1}{2}$, then

$$\beta(p,-)=\alpha\circ\tau^{-1}\circ\mathcal{O}f\circ\tau(p,-)=\alpha(\top)=\top=-\frac{1}{2}(p,-).$$

Therefore, we conclude that $|\beta|\leqslant\frac{1}{2}<1$ and by Example 3.4, β is clean. Since

$$A\subseteq\mathfrak{c}_L(\operatorname{coz}(\beta))=\mathfrak{c}_L(\operatorname{coz}(\beta))=\mathfrak{c}_L(\operatorname{coz}(2\beta)),$$

then $\mathfrak{o}_L(\operatorname{coz}(2\beta))\subseteq L\setminus A$. Also, since

$$\operatorname{coz}(2\beta-\mathbf{1})=\operatorname{coz}\Big(\beta-\frac{\mathbf{1}}{\mathbf{2}}\Big)=\beta\Big(-,\frac{1}{2}\Big)\vee\beta\Big(\frac{1}{2},-\Big)=\alpha\Big(-,\frac{1}{2}\Big)\leqslant\operatorname{coz}\Big(\alpha-\frac{\mathbf{1}}{\mathbf{2}}\Big),$$

we conclude that

$$B\subseteq\mathfrak{c}_L\Big(\operatorname{coz}\Big(\alpha-\frac{\mathbf{1}}{\mathbf{2}}\Big)\Big)\subseteq\mathfrak{c}_L\Big(\operatorname{coz}\Big(2\Big(\beta-\frac{\mathbf{1}}{\mathbf{2}}\Big)\Big)\Big).$$

Since 2β is clean, there exists an idempotent $e\in\mathcal{R}(L)$ such that

$$B\subseteq\mathfrak{c}_L(\operatorname{coz}(2\beta-\mathbf{1}))\subseteq\mathfrak{c}_L(\operatorname{coz}(e))\subseteq\mathfrak{o}_L(\operatorname{coz}(2\beta))\subseteq L\setminus A.$$

Now by Corollary 4.5, L is a strongly zero-dimensional frame.

(4) $\Rightarrow$ (5) Let $\alpha \in \mathcal{R}(L)$ be given. Put

$$A := L \setminus \mathfrak{o}_L\Big(\operatorname{coz}\Big(|\alpha| - \frac{\mathbf{1}}{\mathbf{2}}\Big) \wedge \mathbf{0}\Big) = \mathfrak{c}_L\Big(\operatorname{coz}\Big(|\alpha| - \frac{\mathbf{1}}{\mathbf{2}}\Big) \wedge \mathbf{0}\Big).$$

Then

$$A \cap \mathfrak{c}_L(\operatorname{coz}(\alpha)) = \mathfrak{c}_L\Big(\operatorname{coz}(\alpha) \vee |\alpha|\Big(-, \frac{1}{2}\Big)\Big) = \mathfrak{c}_L(\top) = \mathsf{O}.$$

By Lemma 4.3, A and $\mathfrak{c}_L(\operatorname{coz}(\alpha))$ are completely separated, and by our hypothesis, there exists an idempotent e such that

$$A \subseteq \mathfrak{c}_L(\operatorname{coz}(e)) \subseteq L \setminus \mathfrak{c}_L(\operatorname{coz}(\alpha)) = \mathfrak{o}_L(\operatorname{coz}(\alpha)).$$

Also,

$$\operatorname{coz}\Big(|\alpha| - \frac{\mathbf{1}}{\mathbf{2}}\Big) = |\alpha|\Big(-, \frac{1}{2}\Big) = \alpha\Big(-\frac{1}{2}, \frac{1}{2}\Big) \leqslant \operatorname{coz}(\alpha - \mathbf{1}).$$

That implies that $\mathfrak{c}_L(\operatorname{coz}(\alpha - \mathbf{1})) \subseteq A$, and by Proposition 3.2, α is clean.

(5) $\Rightarrow$ (6) Clearly, each minimal prime ideal would be clean.

(6) $\Rightarrow$ (1) Let P be a clean prime ideal in $\mathcal{R}(L)$ and $\alpha \in \mathcal{R}(L)$. Put $A = \mathfrak{c}_L(\operatorname{coz}(\alpha))$ and $B = \mathfrak{c}_L(\operatorname{coz}(\alpha - \mathbf{1}))$. If $A = \mathsf{O}$, then $\operatorname{coz}(\alpha) = \top$. Therefore, α is a unit, and $\alpha = \alpha + \mathbf{0}$ is clean. If $B = \mathsf{O}$, then $\operatorname{coz}(\alpha - \mathbf{1}) = \top$. Therefore, α is a unit, and $\alpha = (\alpha - \mathbf{1}) + \mathbf{1}$ is clean. Let $A \neq \mathsf{O} \neq B$. Put $O = \mathfrak{c}_L(\operatorname{coz}((\alpha - \frac{\mathbf{1}}{\mathbf{3}}) \vee \mathbf{0})) = \mathfrak{c}_L(\alpha(\frac{1}{3}, -))$ and $C = \mathfrak{c}_L(\operatorname{coz}((\alpha - \frac{\mathbf{1}}{\mathbf{3}}) \wedge \mathbf{0})) = \mathfrak{c}_L(\alpha(-, \frac{1}{3}))$. Indeed A and C are disjoint. Put

$$\gamma = \frac{((\alpha - \frac{\mathbf{1}}{\mathbf{3}}) \wedge \mathbf{0})^2}{\alpha^2 + ((\alpha - \frac{\mathbf{1}}{\mathbf{3}}) \wedge \mathbf{0})^2} \quad \text{and} \quad \delta = \frac{((\alpha - \frac{\mathbf{1}}{\mathbf{3}}) \vee \mathbf{0})^2}{\alpha^2 + ((\alpha - \frac{\mathbf{1}}{\mathbf{3}}) \vee \mathbf{0})^2}.$$

Thus,

$$\operatorname{coz}\Big(\alpha^2 + \Big(\Big(\alpha - \frac{\mathbf{1}}{\mathbf{3}}\Big) \wedge \mathbf{0}\Big)^2\Big) = \operatorname{coz}(\alpha) \vee \operatorname{coz}\Big(\Big(\alpha - \frac{\mathbf{1}}{\mathbf{3}}\Big) \wedge \mathbf{0}\Big) = \operatorname{coz}(\alpha) \vee \operatorname{coz}\Big(\alpha\Big(-, \frac{1}{3}\Big)\Big) = \top,$$

and

$$\operatorname{coz}(\gamma) = \operatorname{coz}\Big(\Big(\alpha - \frac{\mathbf{1}}{\mathbf{3}}\Big) \wedge \mathbf{0}\Big) = \alpha\Big(-, \frac{1}{3}\Big).$$

Therefore, $\mathfrak{c}_L(\operatorname{coz}(\gamma)) = \mathfrak{c}_L\Big(\alpha\Big(-, \frac{1}{3}\Big)\Big) = C$. Also,

$$\operatorname{coz}(\gamma - \mathbf{1}) = \operatorname{coz}(\alpha) \quad \text{and} \quad \mathfrak{c}_L(\operatorname{coz}(\gamma - \mathbf{1})) = \mathfrak{c}_L(\operatorname{coz}(\alpha)) = A.$$

Similarly, $D = \mathfrak{c}_L(\operatorname{coz}(\delta))$ and $B = \mathfrak{c}_L(\operatorname{coz}(\delta - \mathbf{1}))$. Since

$$\operatorname{coz}(\gamma\delta) = \operatorname{coz}\Big(\Big(\alpha - \frac{\mathbf{1}}{\mathbf{3}}\Big) \wedge \mathbf{0}\Big) \wedge \operatorname{coz}\Big(\Big(\alpha - \frac{\mathbf{1}}{\mathbf{3}}\Big) \vee \mathbf{0}\Big) = \alpha\Big(-, \frac{1}{3}\Big) \wedge \alpha\Big(\frac{1}{3}, -\Big) = \bot,$$

we conclude that $\gamma\delta \in P$ implies that either $\gamma \in P$ or $\delta \in P$. Therefore, γ or δ is clean.

Let γ be clean. Then there exists an idempotent element $e \in \mathcal{R}(L)$ such that

$$\mathfrak{c}_L(\operatorname{coz}(\gamma - \mathbf{1})) \subseteq \mathfrak{c}_L(\operatorname{coz}(e)) \subseteq \mathfrak{o}_L(\operatorname{coz}(\gamma)).$$

Also,

$$L \setminus B = L \setminus \mathfrak{c}_L(\operatorname{coz}(\delta - \mathbf{1})) = \mathfrak{o}_L(\operatorname{coz}(\delta - \mathbf{1})),$$

and

$$\begin{aligned}\mathfrak{o}_L(\operatorname{coz}(\gamma)) \vee \mathfrak{o}_L(\operatorname{coz}(\delta - \mathbf{1})) &= \mathfrak{o}_L(\operatorname{coz}(\gamma) \vee \operatorname{coz}(\delta - \mathbf{1})) = \mathfrak{o}_L\Big(\alpha\Big(-, \frac{1}{3}\Big) \vee \operatorname{coz}(\alpha - \mathbf{1})\Big) \\ &= \mathfrak{o}_L(\operatorname{coz}(\alpha - \mathbf{1})) = \mathfrak{o}_L(\operatorname{coz}(\delta - \mathbf{1})).\end{aligned}$$

Then

$$\mathfrak{c}_L(\operatorname{coz}(\alpha)) = \mathfrak{c}_L(\operatorname{coz}(\gamma - \mathbf{1})) \subseteq \mathfrak{c}_L(\operatorname{coz}(e)) \subseteq \mathfrak{o}_L(\operatorname{coz}(\gamma)) \subseteq L \setminus \mathfrak{c}_L(\operatorname{coz}(\alpha - \mathbf{1})).$$

Similarly, if δ is clean, then there exists an idempotent element $e \in \mathcal{R}(L)$ such that

$$\mathfrak{c}_L(\operatorname{coz}(\alpha - \mathbf{1})) = \mathfrak{c}_L(\operatorname{coz}(\delta - \mathbf{1})) \subseteq \mathfrak{c}_L(\operatorname{coz}(e)) \subseteq \mathfrak{o}_L(\operatorname{coz}(\delta)) = L \setminus \mathfrak{c}_L(\operatorname{coz}(\alpha)).$$

Therefore, α is clean. □

Note 4.7. *Given that a frame L is c-strongly zero-dimensional if and only if for every two c-completely separated sublocales A and B there exists a clopen sublocale $U \in \mathcal{S}(L)$ such that $A \subseteq U \subseteq L \setminus B$, the preceding proposition yields immediately a similar result for c-strongly zero-dimensional frames L, by replacing $\mathcal{R}(L)$ by $\mathcal{C}_c(L)$ and $\mathcal{R}^*(L)$ by $\mathcal{C}_c^*(L)$ in all statements.*

Acknowledgements. We would like to express our deep gratitude to the referee for improving the article with useful comments.

References

[1] *A. R. Aliabad, M. Mahmoudi*: Pre-image of functions in $C(L)$. Categ. Gen. Algebr. Struct. Appl. *15* (2021), 35–58. zbl MR doi

[2] *F. Azarpanah*: When is $C(X)$ a clean ring? Acta. Math. Hung. *94* (2002), 53–58. zbl MR doi

[3] *R. N. Ball, A. W. Hager*: On the localic Yosida representation of an archimedean lattice ordered group with weak order unit. J. Pure Appl. Algebra *70* (1991), 17–43. zbl MR doi

[4] *R. N. Ball, J. Walters-Wayland*: C- and C^*-quotients in pointfree topology. Diss. Math. *412* (2002), 1–62. zbl MR doi

[5] *B. Banaschewski*: Pointfree topology and the spectra of f-rings. Ordered Algebraic Structures. Kluwer Academic, Dordrecht, 1997, pp. 123–148. zbl MR doi

[6] *B. Banaschewski*: The real numbers in pointfree topology. Textos de Mathemática. Series B 12. Universidade de Coimbra, Coimbra, 1997. zbl MR

[7] *B. Banaschewski*: Gelfand and exchange rings: Their spectra in pointfree topology. Arab. J. Sci. Eng., Sect. C, Theme Issues *25* (2000), 3–22. zbl MR
[8] *B. Banaschewski*: On the pointfree counterpart of the local definition of classical continuous maps. Categ. Gen. Algebr. Struct. Appl. *8* (2018), 1–8. zbl MR
[9] *B. Banaschewski, C. Gilmour*: Pseudocompactness and the cozero part of a frame. Commentat. Math. Univ. Carol. *37* (1996), 577–587. zbl MR
[10] *T. Dube*: Concerning P-frames, essential P-frames, and strongly zero-dimensional frames. Algebra Univers. *61* (2009), 115–138. zbl MR doi
[11] *M. Elyasi, A. A. Estaji, M. Robat Sarpoushi*: Locally functionally countable subalgebra of $\mathcal{R}(L)$. Arch. Math., Brno *56* (2020), 127–140. zbl MR doi
[12] *A. A. Estaji, A. Karimi Feizabadi, M. Robat Sarpoushi*: $\ddagger_c$-ideals and prime ideals in ring $\mathcal{R}_c(L)$. Filomat *32* (2018), 6741–6752. zbl MR doi
[13] *A. A. Estaji, M. Robat Sarpoushi, M. Elyasi*: Further thoughts on the ring $\mathcal{R}_c(L)$ in frames. Algebra Univers. *80* (2019), Article ID 43, 14 pages. zbl MR doi
[14] *M. J. Ferreira, J. Picado, S. M. Pinto*: Remainders in pointfree topology. Topology Appl. *245* (2018), 21–45. zbl MR doi
[15] *P. T. Johnstone*: Stone Spaces. Cambridge Studies in Advanced Mathematics 3. Cambridge University Press, Cambridge, 1982. zbl MR
[16] *P. T. Johnstone*: Cartesian monads on toposes. J. Pure Appl. Algebra *116* (1997), 199–220. zbl MR doi
[17] *P. T. Johnstone*: Topos Theory. London Mathematical Society Monographs 10. Academic Press, New York, 1997. zbl MR
[18] *O. A. S. Karamzadeh, M. Namdari, S. Soltanpour*: On the locally functionally countable subalgebra of $C(X)$. Appl. Gen. Topol. *16* (2015), 183–207. zbl MR doi
[19] *A. Karimi Feizabadi, A. A. Estaji, M. Robat Sarpoushi*: Pointfree topology version of image of real-valued continuous functions. Categ. Gen. Algebr. Struct. Appl. *9* (2018), 59–75. zbl MR doi
[20] *H. Kou, M. K. Luo*: Strongly zero-dimensional locales. Acta Math. Sin., Engl. Ser. *18* (2002), 47–54. zbl MR doi
[21] *R. Mehri, R. Mohamadian*: On the locally countable subalgebra of $C(X)$ whose local domain is cocountable. Hacet. J. Math. Stat. *46* (2017), 1053–1068. zbl MR doi
[22] *J. Picado, A. Pultr*: Frames and Locales: Topology Without Points. Frontiers in Mathematics. Springer, Berlin, 2012. zbl MR doi
[23] *M. Robat Sarpoushi*: Pointfree Topology Version of Continuous Functions with Countable Image: Ph.D. Thesis. Hakim Sabzevari University, Sabzevar, 2017.
[24] *M. Taha, A. A. Estaji, M. Robat Sarpoushi*: On the regularity of $\mathcal{C}_c$(L). 53^{nd} Annual Iranian Mathematics Confrence, University of Science & Technology of Mazandaran, September 5–8, 2022. pp. 1323–1326.

Authors' address: Ali Akbar Estaji (corresponding author), Maryam Taha, Faculty of Mathematics and Computer Sciences, Hakim Sabzevari University, Tovhid Town, Sabzevar, Iran, E-mail: aaestaji@hsu.ac.ir, tahamaryam872@gmail.com.