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Czechoslovak Mathematical Journal, Vol. 74 (2024), No. 1, 273–282

Persistent URL: http://dml.cz/dmlcz/152279

Czechoslovak Mathematical Journal, 74 (149) (2024), 273–282

A CONJECTURE ON MINIMUM PERMANENTS

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Received April 19, 2023. Published online December 11, 2023.

Abstract. We consider the permanent function on the faces of the polytope of certain doubly stochastic matrices, whose nonzero entries coincide with those of fully indecomposable square $(0,1)$-matrices containing the identity submatrix. We show that a conjecture in K. Pula, S. Z. Song, I. M. Wanless (2011), is true for some cases by determining the minimum permanent on some faces of the polytope of doubly stochastic matrices.

Keywords: permanent; doubly stochastic; barycentric; cohesive matrix

MSC 2020: 15A15

1. Introduction and preliminaries

Let Ω_n be the polytope of $n\times n$ doubly stochastic matrices, that is, the $n\times n$ nonnegative matrices whose row and column sums are all equal to 1. The *permanent* of an $n\times n$ matrix $A=[a_{ij}]$ is defined by

$$\operatorname{per}(A)=\sum_{\sigma} a_{1\sigma(1)}\dots a_{n\sigma(n)},$$

where σ runs over all permutations of $\{1,2,\dots,n\}$.

Let $D=[d_{ij}]$ be an n-square nonnegative matrix, and let

$$\Omega(D)=\{X=[x_{ij}]\in\Omega_n\colon\ x_{ij}=0 \text{ whenever } d_{ij}=0\}.$$

Then $\Omega(D)$ is a face of Ω_n, and since it is compact, $\Omega(D)$ contains a *minimizing matrix* A such that $\operatorname{per}(A)\leqslant\operatorname{per}(X)$ for all $X\in\Omega(D)$.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2016R1D1A1B02006812). The first author (Gi-Sang Cheon) was partially supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIP) (2016R1A5A1008055).

DOI: 10.21136/CMJ.2023.0186-23

Brualdi in [1] defined an n-square (0,1) matrix D to be *cohesive* if there is a matrix Z in the interior of $\Omega(D)$ for which

$$\operatorname{per}(Z) = \min\{\operatorname{per}(X)\colon\ X \in \Omega(D)\}.$$

And he defined an n-square (0,1) matrix D to be *barycentric* if

$$\operatorname{per}(b(D)) = \min\{\operatorname{per}(X)\colon\ X \in \Omega(D)\},$$

where the *barycenter* $b(D)$ of $\Omega(D)$ is given by

$$b(D) = \frac{1}{\operatorname{per}(D)} \sum_{P \leqslant D} P,$$

where the summation extends over the set of all permutation matrices P with $P \leqslant D$ and $\operatorname{per}(D)$ is their number.

Let I_n denote the identity matrix of order n and 0_k the $k \times k$ zero matrix, and $J_{r,s}$ denote the $r \times s$ matrix all of whose entries are 1.

In [3], the authors considered two matrices $U_{m,n} = \begin{bmatrix} I_n & J_{n,m} \\ J_{m,n} & 0_m \end{bmatrix}$ and $V_{m,n} = \begin{bmatrix} I_n & J_{n,m} \\ J_{m,n} & J_{m,m} \end{bmatrix}$ and suggested determining the minimum permanents and minimizing matrices on $\Omega(U_{m,n})$ and $\Omega(V_{m,n})$, respectively, for $m \geqslant 2$ and $n \geqslant 3$. This face $\Omega(V_{m,n})$ is an extended one of $\Omega(W_n)$ in Theorem 5 in [1]. Song determined the minimum permanents on $\Omega(V_{2,n})$ in [4] and on $\Omega(V_{m,3})$ in [6], respectively.

In [3], the authors determined the forms of minimizing matrices on $\Omega(V_{m,n})$, and gave a conjecture as follows:

Conjecture 1.1 ([3], Conjecture 2.1). *Let $V_{m,n}$ be cohesive but not barycentric for $1 < n < m + \sqrt{m}$, while for $n \geqslant m + \sqrt{m}$, $V_{m,n}$ is not cohesive and $b(U_{m,n})$ is a minimizing matrix on $\Omega(V_{m,n})$.*

In this paper, we show the first part of this Conjecture 1.1 is true for a case $V_{4,5}$ by determining the minimum permanent on $\Omega(V_{4,5})$ and by calculating the barycenter $b(V_{4,5})$ on $\Omega(V_{4,5})$ in Section 2. We also show the second part of this Conjecture 1.1 is true for a case in Section 3.

Recall that an n-square nonnegative matrix is said to be *fully indecomposable* if it contains no $k \times (n-k)$ zero submatrix for $k = 1, \ldots, n-1$.

For a matrix A, let $A(i, j, \ldots, k \mid l, m, \ldots, n)$ denote the submatrix obtained from A by deleting rows $i, j, \ldots, k$ and columns $l, m, \ldots, n$. In particular, we simplify the notation $A(i, j, \ldots, k \mid i, j, \ldots, k)$ to $A(i, j, \ldots, k)$.

We use the following well-known lemma of [2].

Lemma 1.2 ([2]). *Let $D = [d_{ij}]$ be an n-square fully indecomposable $(0,1)$ matrix, and $A = [a_{ij}]$ be a minimizing matrix on $\Omega(D)$. Then A is fully indecomposable, and for (i,j) such that $d_{ij} = 1$,*

$$\operatorname{per}(A(i \mid j)) = \operatorname{per}(A) \quad \text{if } a_{ij} > 0, \quad \operatorname{per}(A(i \mid j)) \geqslant \operatorname{per}(A) \quad \text{if } a_{ij} = 0.$$

For our purpose, we write the following three theorems that were obtained in [3].

Theorem 1.3 ([3], Theorem 2.1). *For $m \geqslant 2$, $n \geqslant 2$, the minimizing matrix on $\Omega(V_{m,n})$ is of the form*

$$\begin{bmatrix} dI_n & aJ_{n,m} \\ aJ_{m,n} & xJ_{m,m} \end{bmatrix}$$

with $ma + d = 1 = na + mx$.

Theorem 1.4 ([3], Corollary 2.5). *For $m \geqslant 2$, $n \geqslant 2$, $V_{m,n}$ is cohesive for $1 < n < m + \sqrt{m}$, while it is not cohesive for $n \geqslant 2m$. But for $m + \sqrt{m} \leqslant n < 2m$, it remains to be determined whether it is cohesive or not.*

Theorem 1.5 ([3], Corollary 3.3). *The minimum permanent on $\Omega(V_{4,n})$ is*

$$\operatorname{per} b(U_{4,n}) = 4! \cdot \frac{(n-1)(n-2)(n-3)(n-4)^{n-4}}{n^{n+3}}$$

for $n \geqslant 6$.

2. Cohesiveness of $V_{4,5}$

In this section, we determine the minimum permanent on $\Omega(V_{4,5})$, which was not determined in Theorem 1.5. Therefore, we show that $V_{4,5}$ is a cohesive matrix, but not barycentric matrix, which shows that the first part of Conjecture 1.1 is true for this case.

Theorem 2.1. *We have that $V_{4,5}$ is cohesive but not barycentric. And the minimum permanent on $\Omega(V_{4,5})$ is*

$$\frac{3}{8}a - \frac{93}{8}a^2 + \frac{1377}{8}a^3 - \frac{12231}{8}a^4 + 8676a^5 - 31140a^6 + 64992a^7 - 60384a^8, \tag{2.1}$$

where a is the unique real root of

$$\begin{aligned} &\frac{3}{32} - \frac{117}{32}a + \frac{2157}{32}a^2 - \frac{24183}{32}a^3 + \frac{44631}{8}a^4 \\ &\qquad - 27567a^5 + 88500a^6 - 167832a^7 + 142944a^8 = 0 \end{aligned} \tag{2.2}$$

with $\frac{1}{8} < a \leqslant \frac{1}{5}$.

P r o o f. Let $A_{4,5}$ be a minimizing matrix on $\Omega(V_{4,5})$. Then we have $A_{4,5} = \begin{bmatrix} dI_5 & aJ_{5,4} \\ aJ_{4,5} & xJ_{4,4} \end{bmatrix}$ with $4a + d = 1 = 5a + 4x$ from Theorem 1.3. Since $5a \leqslant 1$ and $4a + d = 1$, we have

$$d = 1 - 4a \geqslant 1 - 4 \cdot \frac{1}{5} \geqslant a. \tag{2.3}$$

If x is not zero, then we have $\operatorname{per} A_{4,5}(9 \mid 9) = \operatorname{per} A_{4,5}(1 \mid 9)$ from Lemma 1.2. Now consider

$$\begin{aligned} \operatorname{per} A_{4,5}(9 \mid 9) &= 9a^2 \operatorname{per} A_{4,5}(9,8,1) + d \operatorname{per} , A_{4,5}(9,1) \\ &= 9a^2(2x^2d^4 + 16xa^2d^3 + 24a^4d^2) \\ &\quad + d(6x^3d^4 + 72x^2a^2d^3 + 216xa^4d^2 + 144a^6d) \end{aligned} \tag{2.4}$$

and

$$\operatorname{per} A_{4,5}(1 \mid 9) = 4a \operatorname{per} A_{4,5}(9,1) = 4a(6x^3d^4 + 72x^2a^2d^3 + 216xa^4d^2 + 144a^6d). \tag{2.5}$$

If $d \geqslant 4a$, then (2.4) and (2.5) show that

$$\operatorname{per} A_{4,5}(9 \mid 9) \geqslant 9a^2(2x^2d^4 + 16xa^2d^3 + 24a^4d^2) + \operatorname{per} A_{4,5}(1 \mid 9) > \operatorname{per} A_{4,5}(1 \mid 9),$$

which is impossible. Thus, we have $a \leqslant d < 4a$ from (2.3). Hence, $\frac{1}{8} < a \leqslant \frac{1}{5}$. From the doubly stochastic property of $A_{4,5}$, we may substitute $d = 1 - 4a$, $x = \frac{1}{4}(1 - 5a)$ to equations (2.4) and (2.5). Then we get an equation from the equality of equations (2.4) and (2.5) as

$$\begin{aligned} p(a) &= \frac{3}{32} - \frac{117}{32}a + \frac{2157}{32}a^2 - \frac{24183}{32}a^3 + \frac{44631}{8}a^4 \\ &\quad - 27567a^5 + 88500a^6 - 167832a^7 + 142944a^8 \\ &= 0, \end{aligned}$$

which is the equation in (2.2). This equation $p(a) = 0$ has a unique zero for a with $\frac{1}{8} < a \leqslant \frac{1}{5}$. By the use of computer graphic package, we have the graph of $p(a)$ in Figure 1 with $\frac{1}{8} < a \leqslant \frac{1}{5}$.

By the use of the computer system Mathematica, we get approximate values $a = 0.1702614\ldots$, $x = 0.0371732\ldots$, and $d = 0.3189542\ldots$ And hence, $V_{4,5}$ is a cohesive matrix.

Moreover, the minimum permanent is

$$\frac{3}{8}a - \frac{93}{8}a^2 + \frac{1377}{8}a^3 - \frac{12231}{8}a^4 + 8676a^5 - 31140a^6 + 64992a^7 - 60384a^8,$$

which is the value in (2.1) from $\operatorname{per} A_{4,5} = \operatorname{per} A_{4,5}(1 \mid 9)$. By the use of the computer system Mathematica, we get that the approximate minimum permanent value in (2.1) is $0.0012954\ldots$

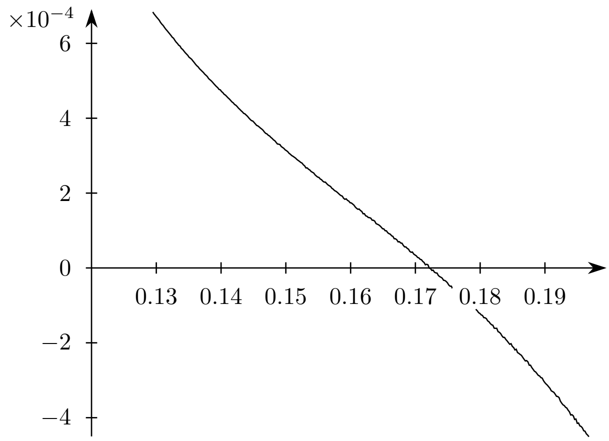

Figure 1. Graph of $p(a)$.

Now consider the barycenter

$$b(V_{4,5}) = \begin{bmatrix} \frac{209}{501} I_5 & \frac{73}{501} J_{5,4} \\ \frac{73}{501} J_{4,5} & \frac{34}{501} JI_{4,4} \end{bmatrix}.$$

In this case, using the computer system Mathematica, we have

$$\operatorname{per} b(V_{4,5})(9 \mid 9) = \frac{2234989336077244912}{1323063084794002334667},$$

but

$$\operatorname{per} b(V_{4,5})(1 \mid 9) = \frac{1767936530142554816}{1323063084794002334667},$$

and hence, $\operatorname{per} b(V_{4,5})(9 \mid 9) > \operatorname{per} b(V_{4,5})(1 \mid 9)$. Hence, the barycenter $b(V_{4,5})$ is not a minimizing matrix from Lemma 1.2. Therefore, $V_{4,5}$ is not barycentric as is conjectured. □

Hence, we have shown that the first part of Conjecture 1.1 is true for the case $V_{4,5}$.

3. Special matrices $V_{m,n}$ that are not cohesive

In this section, we investigate a special case of $V_{m,n}$. Let us consider the special case $m = 4k$ and $n = 7k$ with $k \geqslant 2$ for $V_{m,n}$. This class of matrices is a case, where $m + \sqrt{m} < n < 2m$, which is contained in the second part of Conjecture 1.1. That is, it is not cohesive, which shows that the second part of Conjecture 1.1 is true in this case. In this special case, we will show that $A_0 = b(U_{4k,7k})$ is the unique minimizing matrix on $\Omega(V_{4k,7k})$.

From Theorem 1.3, the minimizing matrix on $\Omega(V_{m,n})$ is of the form:

$$A_x = \begin{bmatrix} dI_n & aJ_{n,m} \\ aJ_{m,n} & xJ_{m,m} \end{bmatrix} = \begin{bmatrix} \Big(\dfrac{n-m+m^2x}{n}\Big)I_n & \Big(\dfrac{1-mx}{n}\Big)J_{n,m} \\ \Big(\dfrac{1-mx}{n}\Big)J_{m,n} & xJ_{m,m} \end{bmatrix}.$$

Then we have

(3.1)

$$\begin{aligned} \operatorname{per} A_x &= \sum_{i=0}^{m} \binom{m}{i}^2 \frac{i!\,n!\,(m-i)!}{(n-m+i)!} x^i \Big(\frac{1}{n}(1-mx)\Big)^{2m-2i} \Big(\frac{1}{n}(n-m+m^2x)\Big)^{n-m+i} \\ &= \sum_{i=0}^{m} \binom{m}{i} \frac{n!\,m!}{(n-m+i)!\,n^{n+m-i}} x^i (1-mx)^{2m-2i}(n-m+m^2x)^{n-m+i}. \end{aligned}$$

In particular,

$$\operatorname{per} A_0 = \frac{n!\,m!\,(n-m)^{n-m}}{(n-m)!\,n^{n+m}}, \tag{3.2}$$

and hence,

(3.3)

$$\frac{\operatorname{per} A_x}{\operatorname{per} A_0} = \sum_{i=0}^{m} \binom{m}{i} \frac{n^i(n-m)!}{(n-m+i)!} x^i (1-mx)^{2m-2i} \Big(1+\frac{m^2x}{n-m}\Big)^{n-m} (n-m+m^2x)^i.$$

Now, let $y = mx$ and express m and n in terms of k. Then we get

$$\frac{\operatorname{per} A_x}{\operatorname{per} A_0} = \Big(1+\frac{4}{3}y\Big)^{3k} \sum_{i=0}^{4k} \binom{4k}{i} \frac{(7k)^i(3k)!}{(3k+i)!} (1-y)^{8k-2i} \Big(\frac{3}{4}y+y^2\Big)^i \tag{3.4}$$

for $0 \leqslant y \leqslant 1$.

Then let us begin with some arithmetic lemmas, which were used in [5] but we include them for our purpose.

Lemma 3.1. *We have*

$$\binom{4k}{i} \geqslant 2^i \binom{2k}{i}$$

for all $0 \leqslant i \leqslant 2k$.

P r o o f. For $i = 0$, it is holds with equality. For $1 \leqslant i \leqslant 2k$,

$$\binom{4k}{i} = \frac{(4k)(4k-1)\dots(4k-i+1)}{i!} = \frac{2^i(2k)(2k-\frac{1}{2})\dots(2k-\frac{1}{2}i+\frac{1}{2})}{i!} \geqslant 2^i\binom{2k}{i}$$

for all $1 \leqslant i \leqslant 2k$. □

For our purpose, let us denote

$$P_i = \frac{(7k)^i}{(3k+1)(3k+2)\dots(3k+i)} = \frac{(7k)^i(3k)!}{(3k+i)!}$$

for $i = 1, 2, \dots, 4k$.

Lemma 3.2. *We have*

(i) $P_i \geqslant (\frac{7}{4})^i$ *for all* $1 \leqslant i \leqslant k$,

(ii) $P_i = P_k \times (7k)^{i-k}/((4k+1)(4k+2)\dots(3k+i)) \geqslant (\frac{7}{4})^k(\frac{7}{5})^{i-k}$ *for all* $k+1 \leqslant i \leqslant 2k$,

(iii) $P_i = P_{2k} \times (7k)^{i-2k}/((5k+1)(5k+2)\dots(3k+i)) \geqslant (\frac{49}{20})^k$ *for all* $2k+1 \leqslant i \leqslant 4k$.

Proof. It is clear from the definition of P_i. □

Lemma 3.3. *We have*

(i) $P_i \geqslant (14k)^i/(6k+i+1)^i$ *for all* $1 \leqslant i \leqslant 2k$,

(ii) $P_i \geqslant (14k)^i/(6k+i+1)^i \geqslant 1.86^i$ *for all* $1 \leqslant i \leqslant k$,

(iii) $P_i \geqslant (14k)^i/(6k+i+1)^i \geqslant 1.64^i$ *for all* $1 \leqslant i \leqslant 2k$.

Proof. (i) We have

$$P_i = \frac{(7k)^i}{(3k+1)(3k+2)\dots(3k+i)} = \frac{(\frac{7}{3})^i}{(1+1/3k)(1+2/3k)\dots(1+i/3k)}. \tag{3.5}$$

Using the arithmetic mean and geometric mean inequality, we have

(3.6)
$$\Big(1+\frac{1}{3k}\Big)\Big(1+\frac{2}{3k}\Big)\dots\Big(1+\frac{i}{3k}\Big) \leqslant \Big[\frac{1}{i}\Big(i+\frac{i(i+1)}{6k}\Big)\Big]^i = \Big(1+\frac{i+1}{6k}\Big)^i = \Big(\frac{6k+i+1}{6k}\Big)^i.$$

Then the result follows from (3.5) and (3.6).

(ii) For $1 \leqslant i \leqslant k$, we have $6k+i+1 \leqslant 7k+1$. Then $14k - 1.86(6k+i+1) \geqslant 14k - 1.86(7k+1) = 0.98k - 1.86 > 0$ for $k \geqslant 2$.

(iii) For $1 \leqslant i \leqslant 2k$, we have $6k+i+1 \leqslant 8k+1$. Then $14k - 1.64(6k+i+1) \geqslant 14k - 1.64(8k+1) = 0.88k - 1.64 > 0$ for $k \geqslant 2$. □

Theorem 3.4. *For* $m = 4k$ *and* $n = 7k$ *with* $k \geqslant 2$, *the unique minimizing matrix in* $\Omega(V_{m,n})$ *is* A_0.

P r o o f. From equation (3.4) we consider 3 cases for the interval $0 \leqslant y \leqslant 1$.

Case 1: $\frac{1}{2} \leqslant y \leqslant 1$. By equation (3.4),

$$\frac{\operatorname{per} A_x}{\operatorname{per} A_0} \geqslant \Big(1+\frac{4}{3}y\Big)^{3k} \frac{(7k)^{4k}(3k)!}{(3k+4k)!}(1-y)^{8k-8k}\Big(\frac{3}{4}y+y^2\Big)^{4k}.$$

Applying Lemma 3.2, we have for all $\frac{1}{2} \leqslant y \leqslant 1$,

$$\frac{\operatorname{per} A_x}{\operatorname{per} A_0} \geqslant \Big(1+\frac{4}{3}y\Big)^{3k}\Big(\frac{49}{20}\Big)^{k}\Big(\frac{3}{4}y+y^2\Big)^{4k} = [f(y)]^k > 1,$$

since $f(y) = (1+\frac{4}{3}y)^3(\frac{49}{20})(\frac{3}{4}y+y^2)^4$ is increasing between $\frac{1}{2} \leqslant y \leqslant 1$, and the least value of $f(y)$ is $f(\frac{1}{2}) = (1+\frac{2}{3})^3(\frac{49}{20})(\frac{3}{8}+\frac{1}{4})^4 = \frac{3828125}{2211840} > 1$. That is, $\operatorname{per} A_x > \operatorname{per} A_0$, which is what we want.

Case 2: $\frac{3}{8} \leqslant y \leqslant \frac{1}{2}$. Applying Lemma 3.2 to equation (3.4), we have

$$\begin{aligned}
\frac{\operatorname{per} A_x}{\operatorname{per} A_0} &= \Big(1+\frac{4}{3}y\Big)^{3k}\sum_{i=0}^{4k}\binom{4k}{i}\frac{(7k)^i(3k)!}{(3k+i)!}(1-y)^{8k-2i}\Big(\frac{3}{4}y+y^2\Big)^{i}\\
&\geqslant \Big(1+\frac{4}{3}y\Big)^{3k}\sum_{j=0}^{2k}\binom{4k}{j+2k}\Big(\frac{49}{20}\Big)^{k}(1-y)^{8k-2(j+2k)}\Big(\frac{3}{4}y+y^2\Big)^{j+2k}\\
&= \Big(1+\frac{4}{3}y\Big)^{3k}\Big(\frac{49}{20}\Big)^{k}\Big(\frac{3}{4}y+y^2\Big)^{2k}\sum_{j=0}^{2k}\binom{4k}{2k-j}(1-y)^{4k-2i}\Big(\frac{3}{4}y+y^2\Big)^{j}\\
&= \Big(\frac{49}{20}\Big)^{k}\Big(1+\frac{4}{3}y\Big)^{3k}\Big(\frac{3}{4}y+y^2\Big)^{2k}\sum_{j=0}^{2k}\binom{4k}{2k-j}(1-2y+y^2)^{2k-j}\Big(\frac{3}{4}y+y^2\Big)^{j}\\
&\geqslant \Big(\frac{49}{20}\Big)^{k}\Big(1+\frac{4}{3}y\Big)^{3k}\Big(\frac{3}{4}y+y^2\Big)^{2k}\\
&\quad\times\sum_{j=0}^{2k}\binom{2k}{2k-j}2^{2p-j}(1-2y+y^2)^{2k-j}\Big(\frac{3}{4}y+y^2\Big)^{j}\\
&= \Big(\frac{49}{20}\Big)^{k}\Big(1+\frac{4}{3}y\Big)^{3k}\Big(\frac{3}{4}y+y^2\Big)^{2k}\Big(2-4y+2y^2+\frac{3}{4}y+y^2\Big)^{2k}\\
&= \Big[\Big(\frac{49}{20}\Big)\Big(1+\frac{4}{3}y\Big)^{3}\Big(\frac{3}{4}y+y^2\Big)^{2}\Big(2-\frac{13}{4}y+3y^2\Big)^{2}\Big]^{k} = g(y)^k,
\end{aligned}$$

where

$$g(y) = \Big[\Big(\frac{49}{20}\Big)\Big(1+\frac{4}{3}y\Big)^{3}\Big(\frac{3}{4}y+y^2\Big)^{2}\Big]\Big[\Big(2-\frac{13}{4}y+3y^2\Big)^{2}\Big].$$

Then we have $g(y) > 1$ since the value of the first square brackets of $g(y)$ is increasing for $y \in [\frac{3}{8}, \frac{1}{2}]$ and the least value of the first square brackets of $g(y)$ is $\frac{964467}{655360}$ at $y = \frac{3}{8}$, and the value of the second square brackets of $g(y)$ is greater than $(\frac{645}{576})^2$ from $(2-\frac{13}{4}y+3y^2)^2 = (3(y-\frac{13}{24})^2+\frac{645}{576})^2$ for all $y \in [\frac{3}{8}, \frac{1}{2}]$. Thus, we have that $\operatorname{per} A_x/\operatorname{per} A_0 > 1$ for this case. That is, $\operatorname{per} A_x > \operatorname{per} A_0$, which is what we want.

Case 3: $0 \leqslant y \leqslant \frac{3}{8}$. Applying Lemmas 3.1 and 3.3 to equation (3.4), we have

$$\begin{aligned}
\frac{\operatorname{per} A_x}{\operatorname{per} A_0} &\geqslant \Big(1+\frac{4}{3}y\Big)^{3k}(1-y)^{4k}\sum_{i=0}^{2k} 2^i \binom{2k}{i}\frac{(7k)^i(3k)!}{(3k+i)!}(1-y)^{4k-2i}\Big(\frac{3}{4}y+y^2\Big)^i \\
&\geqslant \Big(1+\frac{4}{3}y\Big)^{3k}(1-y)^{4k}\sum_{i=0}^{2k}\binom{2k}{i}(2\times 1.64)^i(1-2y+y^2)^{2k-i}\Big(\frac{3}{4}y+y^2\Big)^i \\
&= \Big(1+\frac{4}{3}y\Big)^{3k}(1-y)^{4k}(1-2y+y^2+2.46y+3.28y^2)^{2k} = h(y)^k,
\end{aligned}$$

where

$$h(y) = \Big(1+\frac{4}{3}y\Big)^3(1-y)^4(1+0.46y+4.28y^2)^2.$$

By the use of computer graphic package, we have the graph of $h(y)$ in Figure 2 with $0 \leqslant y \leqslant \frac{3}{8}$.

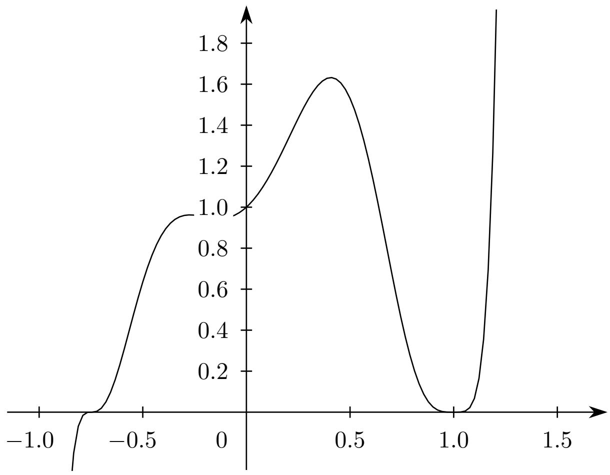

Figure 2. Graph of $h(y)$.

Thus, $h(0) = 1$, $h(\frac{1}{4}) = 1.433\ldots > 1$ and $h(\frac{3}{8}) = 1.621\ldots > 1$. That is, $h(y) \geqslant 1$ for all $y \in [0, \frac{3}{8}]$. That is, $\operatorname{per} A_x \geqslant \operatorname{per} A_0$, which is what we want.

By Cases 1, 2 and 3, we conclude that $A_0 = b(U_{m,n})$ is the unique minimizing matrix in $\Omega(V_{m,n})$ for this special case. □

As a concluding remark, we determined the minimum permanent on the faces of $\Omega(V_{4,5})$ and $\Omega(V_{4k,7k})$ with $k = 1, 2, 3, \ldots$, which shows that Conjecture 1.1 is true for these cases.

Acknowledgements. The authors would like to thank the referee for valuable comments and helpful suggestions.

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