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## REPRESENTATION FUNCTIONS FOR BINARY LINEAR FORMS

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*Abstract.* Let  $\mathbb{Z}$  be the set of integers,  $\mathbb{N}_0$  the set of nonnegative integers and  $F(x_1, x_2) = u_1x_1 + u_2x_2$  be a binary linear form whose coefficients  $u_1, u_2$  are nonzero, relatively prime integers such that  $u_1u_2 \neq \pm 1$  and  $u_1u_2 \neq -2$ . Let  $f: \mathbb{Z} \rightarrow \mathbb{N}_0 \cup \{\infty\}$  be any function such that the set  $f^{-1}(0)$  has asymptotic density zero. In 2007, M. B. Nathanson (2007) proved that there exists a set  $A$  of integers such that  $r_{A,F}(n) = f(n)$  for all integers  $n$ , where  $r_{A,F}(n) = |\{(a, a') : n = u_1a + u_2a' : a, a' \in A\}|$ . We add the structure of difference for the binary linear form  $F(x_1, x_2)$ .

*Keywords:* representation function; binary linear form; density

*MSC 2020:* 11B13, 11B34

## 1. INTRODUCTION

Let  $\mathbb{Z}$  be the set of integers and  $\mathbb{N}_0$  the set of nonnegative integers. The research on representation functions attracts much interest. In 2007, Nathanson in [2] considered the representation functions for binary linear forms and obtained the following nice result:

**Theorem A** ([2], Theorem 1). *Let  $F(x_1, x_2) = u_1x_1 + u_2x_2$  be a binary linear form whose coefficients  $u_1, u_2$  are nonzero, relatively prime integers such that  $u_1u_2 \neq \pm 1$  and  $u_1u_2 \neq -2$ . Let  $f: \mathbb{Z} \rightarrow \mathbb{N}_0 \cup \{\infty\}$  be any function such that the set  $f^{-1}(0)$  has asymptotic density zero. There exists a set  $A$  of integers such that  $r_{A,F}(n) = f(n)$  for all integers  $n$ , where*

$$r_{A,F}(n) = |\{(a, a') : n = u_1a + u_2a' : a, a' \in A\}|.$$

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For a nonempty set  $A$  of integers and an integer  $n$ , denote by  $d_A(n)$  the number of  $(a, a')$  with  $a, a' \in A$  such that  $n = a - a'$ . It is clear that  $d_A(-n) = d_A(n)$  for any positive integer  $n$ . In this note, we consider adding the structure of difference  $d_A(n)$ . The following problem is posed naturally:

**Problem 1.1.** *For the above  $F(x_1, x_2)$ , does there exist a set  $A$  of integers such that  $r_{A,F}(n) = 1$  for all integers  $n$  and  $d_A(n) = 1$  for all positive integers  $n$ ?*

Assume there exists such set  $A$ . Then for any nonzero integer  $b$  we could choose  $a_1, a_2, a_3, a_4$  with  $a_1 \neq a_2$  from  $A$  such that  $a_1 - a_2 = u_2b$  and  $a_3 - a_4 = u_1b$ . Then  $u_1(a_1 - a_2) = u_2(a_3 - a_4)$ , namely,  $u_1a_1 + u_2a_4 = u_1a_2 + u_2a_3$ , a contradiction with  $r_{A,F}(n) = 1$  for all integers  $n$ . Thus, the answer to the above problem is negative. Based on this, we obtain the following result:

**Theorem 1.1.** *Let  $F(x_1, x_2) = u_1x_1 + u_2x_2$  be a binary linear form with coefficients  $u_1, u_2$  are nonzero, relatively prime integers such that  $u_1u_2 \neq \pm 1$  and  $u_1u_2 \neq -2$ . Let  $f: \mathbb{Z} \rightarrow \mathbb{N}_0 \cup \{\infty\}$  be any function such that the set  $f^{-1}(0)$  (write it as  $T$ ) has asymptotic density zero and  $f(n) \geq 2$  for all integers  $n \notin T$ . Then there exists a set  $A$  of integers such that  $r_{A,F}(n) = f(n)$  for all integers  $n$  and  $d_A(n) = 1$  for all positive integers  $n$ .*

For related results, one may refer to [1] and [3] for details.

## 2. PROOF OF THEOREM 1.1

In the following text, the function  $F(x_1, x_2)$  is defined as above. Then there exist integers  $v_1, v_2$  such that  $F(v_1, v_2) = u_1v_1 + u_2v_2 = 1$ . We may assume that  $u_1 + u_2 > 0$ . For a nonempty finite set  $A$  of integers and an integer  $k$ , let  $|k|$  be the absolute value of  $k$ ,  $\max|A| = \max\{|a|: a \in A\}$ ,  $kA = \{ka: a \in A\}$ ,  $k + A = \{k + a: a \in A\}$  and  $A - A = \{a - a': a, a' \in A\}$ .

**Lemma 2.1.** *Let  $f: \mathbb{Z} \rightarrow \mathbb{N}_0 \cup \{\infty\}$  be any function such that the set  $f^{-1}(0)$  (write it as  $T$ ) has asymptotic density zero and  $f(n) \geq 2$  for all integers  $n \notin T$ . Let  $A$  be a finite set of integers such that  $r_{A,F}(n) \leq f(n)$  for all integers  $n$  and  $d_A(n) \leq 1$  for all positive integers  $n$ . If  $k$  is an integer with  $k \notin T$  and  $r_{A,F}(k) < f(k)$ , then there exists a finite subset  $A'$  with  $A \subseteq A' \subseteq \mathbb{Z}$  such that:*

- (i)  $r_{A',F}(k) = r_{A,F}(k) + 1$  and  $r_{A',F}(n) \leq f(n)$  for all integers  $n$ ;
- (ii)  $d_{A'}(n) \leq 1$  for all positive integers  $n$ .

*P r o o f.* Since  $T$  is a set of integers with density zero, we know that there exists a positive integer  $c$  with  $c > 3(|u_1| + |u_2|) \max |A| + (|u_1| + |u_2|)(|v_1| + |v_2|)|k|$  such that the set

$$\begin{aligned} \mathbb{A} = \{ & u_1(v_1k + u_2c) + u_2A, u_1(v_2k - u_1c) + u_2A, u_1A + u_2(v_1k + u_2c), \\ & u_1A + u_2(v_2k - u_1c), (u_1 + u_2)(v_1k + u_2c), (u_1 + u_2)(v_2k - u_1c), \\ & u_1(v_2k - u_1c) + u_2(v_1k + u_2c)\} \end{aligned}$$

satisfies  $\mathbb{A} \cap T = \emptyset$ . Let  $A' = A \cup \{v_1k + u_2c, v_2k - u_1c\}$ . Obviously,  $r_{A',F}(k) = r_{A,F}(k) + 1$ . Furthermore, by  $c > 3(|u_1| + |u_2|) \max |A| + (|u_1| + |u_2|)(|v_1| + |v_2|)|k|$  and the definition of  $u_1, u_2$ , we know that  $r_{A',F}(n) \leq 2$  for any  $n \in \mathbb{A}$ ,  $r_{A',F}(n) = r_{A,F}(n)$  for any  $n \in \mathbb{Z} \setminus (\{k\} \cup \mathbb{A})$ . Thus, (i) holds. We could deduce from  $c > 3(|u_1| + |u_2|) \times \max |A| + (|u_1| + |u_2|)(|v_1| + |v_2|)|k|$  and the precondition that  $d_{A'}(n) \leq 1$  for all positive integers  $n$ , (ii) holds. This completes the proof of Lemma 2.1.  $\square$

**Lemma 2.2.** *Let  $f: \mathbb{Z} \rightarrow \mathbb{N}_0 \cup \{\infty\}$  be any function such that the set  $f^{-1}(0)$  (write it as  $T$ ) has asymptotic density zero and  $f(n) \geq 2$  for all integers  $n \notin T$ . Let  $A$  be a finite set of integers such that  $d_A(n) \leq 1$  for all positive integers  $n$  and  $r_{A,F}(n) \leq f(n)$  for all integers  $n$ . If  $k$  is a positive integer with  $d_A(k) = 0$ , then there exists a finite subset  $A'$  with  $A \subseteq A' \subseteq \mathbb{Z}$  such that:*

- (i)  $d_{A'}(k) = 1$  and  $d_{A'}(n) \leq 1$  for all positive integers  $n$ ;
- (ii)  $r_{A',F}(n) \leq f(n)$  for all integers  $n$ .

*P r o o f.* Since  $T$  is a set of integers with density zero, we know that there exists a positive integer  $c$  with  $c > 3(|u_1| + |u_2|) \max |A| + (|u_1| + |u_2|)|k|$  such that the set

$$\begin{aligned} \mathbb{A} = \{ & u_1c + u_2A, u_1(c + k) + u_2A, u_1A + u_2c, u_1A + u_2(c + k), \\ & (u_1 + u_2)c, (u_1 + u_2)(c + k), (u_1 + u_2)c + u_1k, (u_1 + u_2)c + u_2k\}. \end{aligned}$$

satisfies  $\mathbb{A} \cap T = \emptyset$ . Let  $A' = A \cup \{c, c + k\}$ . Obviously,  $d_{A'}(k) = 1$ . Furthermore, by  $c > 3(|u_1| + |u_2|) \max |A| + (|u_1| + |u_2|)|k|$  and the definition of  $u_1, u_2$ , we know that  $r_{A',F}(n) \leq 2$  for any  $n \in \mathbb{A}$  and  $r_{A',F}(n) = r_{A,F}(n)$  for any  $n \in \mathbb{Z} \setminus \mathbb{A}$ . Thus, (i) holds. We could deduce from  $c > 3(|u_1| + |u_2|) \max |A| + (|u_1| + |u_2|)|k|$  and the precondition that  $d_{A'}(n) \leq 1$  for all positive integers  $n$ , (ii) holds. This completes the proof of Lemma 2.2.  $\square$

Now we return to the proof of Theorem 1.1. The idea is from Theorem 1.1 of [1]. We also give the details for the sake of completeness.

*P r o o f* of Theorem 1.1. We firstly rewrite the set of integers  $T$  in the form

$$T_{l_1} \cup T_{l_2} \cup \dots \cup T_{l_k} \cup \dots,$$

where  $l_1 < l_2 < \dots$  are nonnegative integers (if  $0 \notin T$ , then  $l_1 > 0$ ) and  $T_{l_i} = T \cap \{-l_i, l_i\}$ . Let  $t_1$  be the element  $b$  with  $b \notin T$  and  $|b|$  be minimal. Since  $T$  has density zero, then we can find a positive integer  $c$  with  $c > (|u_1| + |u_2|)(|v_1| + |v_2|)|t_1|$  such that

$$\mathbb{S} = \{(u_1 + u_2)(v_1 t_1 + u_2 c), (u_1 + u_2)(v_2 t_1 - u_1 c), u_1(v_2 t_1 - u_1 c) + u_2(v_1 t_1 + u_2 c)\} \cap T = \emptyset.$$

Let  $B_0 = \{t_1 v_1 + u_2 c, t_1 v_2 - u_1 c\}$ . Obviously,  $r_{B_0, F}(t_1) = 1$ . Furthermore, by  $c > (|u_1| + |u_2|)(|v_1| + |v_2|)|t_1|$  and the definition of  $u_1, u_2$ , we know that  $r_{B_0, F}(n) \leq 2$  for any  $n \in \mathbb{S}$ ,  $r_{B_0, F}(n) = 0$  for any  $n \in \mathbb{Z} \setminus \mathbb{S}$  and  $d_{B_0}(n) \leq 1$  for all positive integers  $n$ . It follows from Lemmas 2.1 and 2.2 that there exists a finite subset  $A_1$  with  $B_0 \subseteq A_1 \subseteq \mathbb{Z}$  such that:

- (i)  $r_{A_1, F}(n) = f(n)$  for all integers  $n$  with  $|n| \leq l_1$ , and  $r_{A_1, F}(n) \leq f(n)$  for all integers  $n$ ;
- (ii)  $d_{A_1}(n) = 1$  for all positive integers  $n$  with  $n \leq l_1$ .

We proceed with the above procedure. Then by Lemmas 2.1 and 2.2, we could construct  $A_1 \subseteq A_2 \subseteq \dots \subseteq A_k \subseteq \dots$  of finite sets of integers such that for any positive integer  $k$ , we have:

- (i)  $r_{A_k, F}(n) = f(n)$  for all integers  $n$  with  $|n| \leq l_k$ , and  $r_{A_k, F}(n) \leq f(n)$  for all integers  $n$ ;
- (ii)  $d_{A_k}(n) = 1$  for all positive integers  $n$  with  $n \leq l_k$ .

Let

$$A = \bigcup_{k=1}^{\infty} A_k.$$

Then we have  $r_{A, F}(n) = f(n)$  for all integers  $n$  and  $d_A(n) = 1$  for all positive integers  $n$ .

This completes the proof of Theorem 1.1. □

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