

Ali Abkar

Polyanalytic Besov spaces and approximation by dilatations

Czechoslovak Mathematical Journal, Vol. 74 (2024), No. 1, 305–317

Persistent URL: <http://dml.cz/dmlcz/152282>

Terms of use:

© Institute of Mathematics AS CR, 2024

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

POLYANALYTIC BESOV SPACES AND APPROXIMATION
BY DILATATIONS

ALI ABKAR, Qazvin

Received July 31, 2023. Published online November 29, 2023.

Abstract. Using partial derivatives $\partial f/\partial z$ and $\partial f/\partial \bar{z}$, we introduce Besov spaces of polyanalytic functions in the open unit disk, as well as in the upper half-plane. We then prove that the dilatations of functions in certain weighted polyanalytic Besov spaces converge to the same functions in norm. When restricted to the open unit disk, we prove that each polyanalytic function of degree q can be approximated in norm by polyanalytic polynomials of degree at most q .

Keywords: mean approximation; polyanalytic Besov space; polyanalytic Bergman space; dilatation; non-radial weight; angular weight

MSC 2020: 30E10, 30H20, 30H25, 46E15

1. INTRODUCTION

1.1. What this paper reports. In the definition of classical Besov spaces of analytic functions, one uses the notion of the derivative of a given analytic function f . In the case when f is not analytic, for instance in the wider class of polyanalytic functions, this definition does not work; we recall that a polyanalytic function of degree q is a complex-valued function f that satisfies the partial differential equation

$$\left(\frac{\partial}{\partial \bar{z}}\right)^q f = \frac{1}{2^q} \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^q f = 0, \quad z = x + iy,$$

where q is a positive integer. If $q = 1$, the class of polyanalytic functions coincide with that of analytic ones. In this paper we use the partial derivatives $\partial f/\partial z$ and $\partial f/\partial \bar{z}$ to define the space which we call *polyanalytic Besov space*. This is a Banach space and deserves investigation from various perspectives, that is, the function theoretic perspective and the operator theoretic perspective. In the current paper, we define

the space both on the unit disk, and on the upper half-plane. We study the problem of approximating in norm of elements of these spaces by well-behaved functions like dilatations. We consider both radial and non-radial weight functions; we also investigate the same problem in angular weighted spaces.

1.2. Preliminaries and notations. Let \mathbb{D} denote the open unit disk in the complex plane, and let q denote a fixed natural number. A function $f: \mathbb{D} \rightarrow \mathbb{C}$ is called a *polyanalytic function* of degree q ; or simply a *polyanalytic function* if f is q times continuously differentiable (with respect to x, y) on the unit disk and satisfies

$$\left(\frac{\partial}{\partial \bar{z}}\right)^q f = \frac{1}{2^q} \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^q f = 0, \quad z = x + iy.$$

These functions are also known as q -analytic functions, and they enjoy the following representation in terms of analytic functions (see [5]):

$$(1) \quad f(z) = h_0(z) + \bar{z}h_1(z) + \dots + \bar{z}^{q-1}h_{q-1}(z),$$

where each $h_k, 0 \leq k \leq q - 1$, is an analytic function on the unit disk. The set of polyanalytic functions of degree q on a given domain Ω of the complex plane is denoted by $\text{Pol}_q(\Omega)$. It is worth mentioning that polyanalytic functions are not as flexible as the analytic functions; for instance, they may vanish on a curve lying in the unit disk without vanishing identically in the whole domain, see [4].

A *weight* on a given domain Ω is a positive integrable function $w: \Omega \rightarrow [0, \infty)$. In the following discussions, the domain Ω is either the unit disk or the upper half-plane. We will be clear when we shift to discuss the upper half-plane case.

For a positive number p , we say that a function $f \in \text{Pol}_q(\mathbb{D})$ belongs to the weighted polyanalytic Bergman space $\mathcal{A}_{q,w}^p(\mathbb{D})$ if

$$\|f\|_{\mathcal{A}_{q,w}^p(\mathbb{D})} = \left(\int_{\mathbb{D}} |f(z)|^p w(z) dA(z)\right)^{1/p} < \infty,$$

where $dA(z) = dx dy$.

In the following, we present a definition for the polyanalytic Besov space by using partial differential operators ∂_z and $\partial_{\bar{z}}$ instead of the differentiation operator $f \mapsto f'$ which is used for the definition of analytic Besov spaces. Let us start by recalling the differential operators ($z = x + iy$):

$$\partial_z = \frac{1}{2} \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \partial_{\bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Given a weight w on the unit disk, we say that $f \in \text{Pol}_q(\mathbb{D})$ belongs to the weighted polyanalytic Dirichlet space $\mathcal{D}_{q,w}^p$ if

$$\int_{\mathbb{D}} [|\partial_z f(z)|^p + |\partial_{\bar{z}} f(z)|^p] w(z) \, dA(z) < \infty.$$

We define the norm of $f \in \mathcal{D}_{q,w}^p$ by

$$\|f\|_{\mathcal{D}_{q,w}^p} = \left(|f(0)|^p + \int_{\mathbb{D}} [|\partial_z f(z)|^p + |\partial_{\bar{z}} f(z)|^p] w(z) \, dA(z) \right)^{1/p}.$$

When $p = 2$, $\mathcal{D}_{q,w}^2$ is a Hilbert space whose inner product is

$$\langle f, g \rangle_{\mathcal{D}_{q,w}^2} = f(0)\overline{g(0)} + \int_{\mathbb{D}} [\partial_z f(z)\overline{\partial_z g(z)} + \partial_{\bar{z}} f(z)\overline{\partial_{\bar{z}} g(z)}] w(z) \, dA(z).$$

Similarly, we define the weighted polyanalytic Besov space $\mathcal{B}_{q,w}^p$ as the space of functions $f \in \text{Pol}_q(\mathbb{D})$ for which

$$\int_{\mathbb{D}} (1 - |z|^2)^{p-2} [|\partial_z f(z)|^p + |\partial_{\bar{z}} f(z)|^p] w(z) \, dA(z) < \infty.$$

The norm of a function $f \in \mathcal{B}_{q,w}^p$ is given by

$$\|f\|_{\mathcal{B}_{q,w}^p} = \left(|f(0)|^p + \int_{\mathbb{D}} (1 - |z|^2)^{p-2} [|\partial_z f(z)|^p + |\partial_{\bar{z}} f(z)|^p] w(z) \, dA(z) \right)^{1/p}.$$

Note that when $q = 1$ (which means that f is analytic), we have $\partial_z f(z) = f'(z)$, and $\partial_{\bar{z}} f(z) = 0$, so that the polyanalytic Besov (or Dirichlet) space reduces to the analytic Besov (or Dirichlet) space. Therefore, the spaces defined above are natural generalizations of the classical Dirichlet and Besov spaces.

In the case when the underlying set is the upper half-plane \mathbb{C}^+ , the polyanalytic Besov space on the upper half-plane is defined in a similar fashion – we replace $f(0)$ by $f(i)$ and $dA(z)$ by a suitable Gaussian weight function: a function $f \in \text{Pol}_q(\mathbb{C}^+)$ belongs to the Besov space if the following expression is finite (for details see Section 3):

$$\|f\|_{\mathcal{B}_{\mathbb{C}^+,w}^p}^p = |f(i)|^p + \int_{\mathbb{C}^+} [|\partial_z f(z)|^p + |\partial_{\bar{z}} f(z)|^p] \text{Im}(z)^{\alpha+p-2} e^{-\beta|z|^2} \, dA(z).$$

It is a natural question in operator theory of function spaces to decide whether the polynomials are dense in the space or not. To answer this question, the most important strategy is to see if the dilatations

$$f_r(z) = f(rz), \quad z \in \Omega, \quad 0 < r < 1,$$

are convergent to f in norm.

In this paper we consider non-radial weight functions w that satisfy the following condition:

There is a constant $C > 0$, a nonnegative integer k , and an $r_0 \in (0, 1)$ such that

$$(2) \quad r^k w\left(\frac{z}{r}\right) \leq Cw(z), \quad |z| < r, \quad r_0 \leq r < 1.$$

We prove that the dilatations f_r converge to f in norm as $r \rightarrow 1^-$. If, moreover, w is chosen in such a way that the polyanalytic polynomials of degree at most q are included in the weighted Besov space, and if each f_r can be approximated by the polyanalytic polynomials of degree at most q , then we can approximate each f by polyanalytic polynomials of degree at most q .

We should remark that the current author in his recent paper (see [3]) (of course as a by product of his arguments on $\mathcal{A}_{q,w}^p(\mathbb{D})$) suggested that polyanalytic Besov spaces be defined as functions in $\text{Pol}_q(\mathbb{D})$ that satisfy

$$\int_{\mathbb{D}} (1 - |z|^2)^{p-2} |\partial_z f(z)|^p dA(z) < \infty.$$

This definition is clearly true but has some disadvantages. The best practice is to consider both partial derivatives of f as we did in the definition presented in the current paper.

The theory of polyanalytic functions and their applications can be found in [4], [5], [9]. Similar approximation problems for analytic functions were already studied in [1] and [2]. As for the Bergman and Fock spaces of polyanalytic functions we refer the reader to [7], [8], [10], [11], and the references therein.

2. POLYANALYTIC BESOV SPACES IN THE UNIT DISK

We start by investigating the function spaces defined on the unit disk.

Theorem 2.1. *Let $0 < p < \infty$ and let w satisfy condition (2). Then the polyanalytic polynomials of degree at most q are dense in the weighted polyanalytic Dirichlet space $\mathcal{D}_{q,w}^p$.*

P r o o f. We first prove that $\|f_r - f\|_{\mathcal{D}_{q,w}^p} \rightarrow 0$ as $r \rightarrow 1^-$. To this end, we work with

$$\|f\|_{\mathcal{D}_{q,w}^p}^p = \int_{\mathbb{D}} [|\partial_z f(z)|^p + |\partial_{\bar{z}} f(z)|^p] w(z) dA(z).$$

Let us recall the equations

$$(3) \quad \|f_r - f\|_{\mathcal{D}_{q,w}^p}^p = \int_{\mathbb{D}} |\partial_z f_r(z) - \partial_z f(z)|^p w(z) dA(z) + \int_{\mathbb{D}} |\partial_{\bar{z}} f_r(z) - \partial_{\bar{z}} f(z)|^p w(z) dA(z),$$

and

$$\partial_z f_r(z) = r\partial_z f(rz), \quad \partial_{\bar{z}} f_r(z) = r\partial_{\bar{z}} f(rz).$$

Replacing z by z/r , we obtain

$$\begin{aligned} \int_{\mathbb{D}} |\partial_z f_r(z)|^p w(z) \, dA(z) &= r^{p-k-2} \int_{r\mathbb{D}} |\partial_z f(z)|^p r^k w\left(\frac{z}{r}\right) \, dA(z) \\ &\leq Cr^{p-k-2} \int_{r\mathbb{D}} |\partial_z f(z)|^p w(z) \, dA(z). \end{aligned}$$

Now apply the dominated convergence theorem to get

$$\limsup_{r \rightarrow 1^-} \int_{\mathbb{D}} |\partial_z f_r(z)|^p w(z) \, dA(z) \leq \int_{\mathbb{D}} |\partial_z f(z)|^p w(z) \, dA(z),$$

which in turn leads to

$$\lim_{r \rightarrow 1^-} \int_{\mathbb{D}} |\partial_z f_r(z) - \partial_z f(z)|^p w(z) \, dA(z) = 0.$$

This means that the first term in (3) tends to zero. Similarly, we show that

$$\lim_{r \rightarrow 1^-} \int_{\mathbb{D}} |\partial_{\bar{z}} f_r(z) - \partial_{\bar{z}} f(z)|^p w(z) \, dA(z) = 0,$$

from which it follows that $f_r \rightarrow f$ in norm. As for the second part, assume that $\varepsilon > 0$ is given. By the above argument, there is an $0 < r < 1$ such that $\|f - f_r\|_{\mathcal{D}_{q,w}^p}^p < \varepsilon$. We now use (1) to write

$$f_r(z) = f(rz) = h_0(rz) + r\bar{z}h_1(rz) + \dots + r^{n-1}\bar{z}^{q-1}h_{q-1}(rz).$$

Due to the fact that the dilatations $(h_k)_r$, $0 \leq k \leq q-1$, are analytic in a closed disk containing \mathbb{D} , we conclude that f_r is q -analytic in a disk bigger than the unit disk. By Mergelyan's theorem each analytic component $(h_k)_r$ can be uniformly approximated by analytic polynomials. Therefore, we can approximate f_r uniformly on \mathbb{D} by a q -analytic polynomial, say Q . Since $d\mu(z) = w(z) \, dA(z)$ is a finite measure on the unit disk, we obtain

$$\|f_r - Q\|_{\mathcal{D}_{q,w}^p}^p = \int_{\mathbb{D}} |f_r - Q|^p \, d\mu \leq \varepsilon^p \int_{\mathbb{D}} d\mu(z) = \varepsilon^p C,$$

where $C = \mu(\mathbb{D})$. Finally, we have

$$\|f - Q\|_{\mathcal{D}_{q,w}^p}^p \leq \|f - f_r\|_{\mathcal{D}_{q,w}^p}^p + \|f_r - Q\|_{\mathcal{D}_{q,w}^p}^p \leq \varepsilon(1 + C^{1/p}).$$

This proves the theorem. □

In the second theorem below we address the polyanalytic Besov spaces on the unit disk.

Theorem 2.2. *Let $2 \leq p < \infty$ and let w satisfy condition (2). Then the polyanalytic polynomials of degree at most q are dense in the weighted polyanalytic Besov space $\mathcal{B}_{q,w}^p$.*

Proof. It suffices to verify that for each $f \in \mathcal{B}_{q,w}^p$,

$$(4) \quad \limsup_{r \rightarrow 1^-} \int_{\mathbb{D}} (1 - |z|^2)^{p-2} |\partial_z f_r(z)|^p w(z) \, dA(z) \leq \int_{\mathbb{D}} (1 - |z|^2)^{p-2} |\partial_z f(z)|^p w(z) \, dA(z),$$

and

$$(5) \quad \limsup_{r \rightarrow 1^-} \int_{\mathbb{D}} (1 - |z|^2)^{p-2} |\partial_{\bar{z}} f_r(z)|^p w(z) \, dA(z) \leq \int_{\mathbb{D}} (1 - |z|^2)^{p-2} |\partial_{\bar{z}} f(z)|^p w(z) \, dA(z).$$

Again, we work with the semi-norm

$$\|f\|_{\mathcal{B}_{q,w}^p}^p = \int_{\mathbb{D}} (1 - |z|^2)^{p-2} [|\partial_z f(z)|^p + |\partial_{\bar{z}} f(z)|^p] w(z) \, dA(z).$$

To establish (4), we implement a change of variables to get

$$\begin{aligned} \|\partial_z f_r\|_{\mathcal{B}_{q,w}^p}^p &= \int_{\mathbb{D}} (1 - |z|^2)^{p-2} |\partial_z f_r(z)|^p w(z) \, dA(z) \\ &= r^{p-k-2} \int_{r\mathbb{D}} \left(\frac{r^2 - |z|^2}{r^2} \right)^{p-2} |\partial_z f(z)|^p r^k w\left(\frac{z}{r}\right) \, dA(z). \end{aligned}$$

Since

$$r \mapsto \left(\frac{r^2 - |z|^2}{r^2} \right)^{p-2}$$

is increasing in r when $p \geq 2$, the monotone convergence theorem can be applied to the last integral. This in turn implies that

$$\limsup_{r \rightarrow 1^-} \int_{\mathbb{D}} (1 - |z|^2)^{p-2} |\partial_z f_r(z)|^p w(z) \, dA(z) \leq \int_{\mathbb{D}} (1 - |z|^2)^{p-2} |\partial_z f(z)|^p w(z) \, dA(z),$$

which proves (4). The proof of (5) is similar. In sum, the foregoing arguments show that

$$\|f - f_r\|_{\mathcal{B}_{q,w}^p} \rightarrow 0, \quad r \rightarrow 1^-.$$

Again, we use an argument similar to the one used in the proof of Theorem 2.1 to conclude that the polyanalytic polynomials of degree at most q are dense in $\mathcal{B}_{q,w}^p$. \square

3. POLYANALYTIC BESOV SPACES IN THE UPPER HALF-PLANE

Recall that an analytic function f on the upper half-plane $\mathbb{C}^+ = \{z = x + iy \in \mathbb{C} : \text{Im}(z) = y > 0\}$ belongs to the Bergman space of the upper half-plane if

$$\int_{\mathbb{C}^+} |f(z)|^p \text{Im}(z)^\alpha e^{-\beta|z|^2} dA(z) < \infty,$$

where α and β are some nonnegative numbers. In the literature, one usually encounters this definition when $\beta = 0$, see [6]. Therefore, it is natural to define the weighted Bergman space of polyanalytic functions on the upper half-plane, $\mathcal{A}_{q,w}^p(\mathbb{C}^+)$, as the space of polyanalytic functions of degree q on \mathbb{C}^+ for which

$$\|f\|_{\mathcal{A}_{q,w}^p(\mathbb{C}^+)}^p = \int_{\mathbb{C}^+} |f(z)|^p w(z) \text{Im}(z)^\alpha e^{-\beta|z|^2} dA(z) < \infty.$$

This suggests that the weighted Dirichlet space and the weighted Besov space of analytic functions should be normed, respectively, by

$$\|f\|_{D_{c^+,w}^p}^p = |f(i)|^p + \int_{\mathbb{C}^+} |f'(z)|^p w(z) \text{Im}(z)^\alpha e^{-\beta|z|^2} dA(z),$$

and

$$\|f\|_{B_{c^+,w}^p}^p = |f(i)|^p + \int_{\mathbb{C}^+} |f'(z)|^p w(z) \text{Im}(z)^{\alpha+p-2} e^{-\beta|z|^2} dA(z).$$

The advantage of this definition is that, as in the unit disk case, when $p = 2$, the Besov space reduces to the Dirichlet space. In analogy with the above definitions, we now declare the weighted Besov space of polyanalytic functions as the space of functions $f \in \text{Pol}_q(\mathbb{C}^+)$ such that

$$\|f\|_{B_{c^+,w}^p}^p = |f(i)|^p + \int_{\mathbb{C}^+} [|\partial_z f(z)|^p + |\partial_{\bar{z}} f(z)|^p] w(z) \text{Im}(z)^{\alpha+p-2} e^{-\beta|z|^2} dA(z)$$

is finite. By our definition, when $p = 2$, we get to the Dirichlet space of polyanalytic functions on the upper half-plane, i.e., polyanalytic functions on the upper half-plane for which

$$\|f\|_{D_{c^+,w}^p}^p = |f(i)|^p + \int_{\mathbb{C}^+} [|\partial_z f(z)|^p + |\partial_{\bar{z}} f(z)|^p] w(z) \text{Im}(z)^\alpha e^{-\beta|z|^2} dA(z) < \infty.$$

We now turn to the approximation of functions in these spaces by dilatations.

Theorem 3.1. *Let $0 < p < \infty$ and let w satisfy condition (2). Then for each $f \in \mathcal{D}_{q,w}^p(\mathbb{C}^+)$, the dilatations f_r converge to f in norm.*

P r o o f. It is enough to work with the semi-norm

$$\|f\|_{\mathcal{D}_{q,w}^p(\mathbb{C}^+)}^p = \int_{\mathbb{C}^+} [|\partial_z f(z)|^p + |\partial_{\bar{z}} f(z)|^p] w(z) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2} dA(z).$$

By the definition,

$$(6) \quad \|f_r - f\|_{\mathcal{D}_{q,w}^p(\mathbb{C}^+)}^p = \int_{\mathbb{C}^+} |\partial_z f_r(z) - \partial_z f(z)|^p w(z) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2} dA(z) \\ + \int_{\mathbb{C}^+} |\partial_{\bar{z}} f_r(z) - \partial_{\bar{z}} f(z)|^p w(z) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2} dA(z).$$

We implement a change of variables and use the equality $\partial_z f_r(z) = r\partial_z f(rz)$ to write

$$\int_{\mathbb{C}^+} |\partial_z f_r(z)|^p w(z) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2} dA(z) \\ = r^{p-(k+\alpha+2)} \int_{\mathbb{C}^+} |\partial_z f(z)|^p r^k w\left(\frac{z}{r}\right) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2/r^2} dA(z) \\ \leq Cr^{p-(k+\alpha+2)} \int_{\mathbb{C}^+} |\partial_z f(z)|^p w(z) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2/r^2} dA(z).$$

Now, the dominated convergence theorem leads to

$$\limsup_{r \rightarrow 1^-} \int_{\mathbb{C}^+} |\partial_z f_r(z)|^p w(z) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2} dA(z) \\ \leq \int_{\mathbb{C}^+} |\partial_z f(z)|^p w(z) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2} dA(z).$$

Thus, the first term on the right-hand side of (6) tends to zero. The same argument is valid for the second term on the right-hand side of (6), and finally

$$\lim_{r \rightarrow 1^-} \|f_r - f\|_{\mathcal{D}_{q,w}^p(\mathbb{C}^+)}^p = 0.$$

□

In the Bergman space case, the problem can be solved more easily.

Corollary 3.2. *Let $0 < p < \infty$ and let w satisfy condition (2). Then for each function $f \in \mathcal{A}_{q,w}^p(\mathbb{C}^+)$, the dilatations f_r converge to f in norm.*

P r o o f. We first note that

$$\|f_r\|_{\mathcal{A}_{q,w}^p(\mathbb{C}^+)}^p = \int_{\mathbb{C}^+} |f(rz)|^p w(z) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2} dA(z) \\ = \frac{1}{r^{\alpha+k+2}} \int_{\mathbb{C}^+} |f(z)|^p r^k w\left(\frac{z}{r}\right) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2/r^2} dA(z) \\ = \frac{1}{r^{\alpha+k+2}} \int_{\mathbb{C}^+} |f(z)|^p r^k w\left(\frac{z}{r}\right) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2} e^{\beta|z|^2(1-r^{-2})} dA(z).$$

It is easy to see that for $0 < r_0 < r < 1$,

$$e^{\beta|z|^2(1-r^{-2})} \leq 1,$$

and hence

$$\begin{aligned} \|f_r\|_{\mathcal{A}_{q,w}^p(\mathbb{C}^+)}^p &= \frac{1}{r^{\alpha+k+2}} \int_{\mathbb{C}^+} |f(z)|^p r^k w\left(\frac{z}{r}\right) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2} e^{\beta|z|^2(1-r^{-2})} dA(z) \\ &\leq \frac{C}{r^{\alpha+k+2}} \int_{\mathbb{C}^+} |f(z)|^p w(z) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2} dA(z) < \infty. \end{aligned}$$

We now apply the dominated convergence theorem to obtain

$$\lim_{r \rightarrow 1^-} \|f_r\|_{\mathcal{A}_{q,w}^p(\mathbb{C}^+)}^p = \int_{\mathbb{C}^+} |f(z)|^p w(z) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2} dA(z) = \|f\|_{\mathcal{A}_{q,w}^p(\mathbb{C}^+)}^p,$$

from which the result follows. \square

It is now time to state the similar result for the polyanalytic Besov spaces on \mathbb{C}^+ .

Theorem 3.3. *Let $2 \leq p < \infty$ and let w satisfy condition (2). Then for each $f \in \mathcal{B}_{q,w}^p(\mathbb{C}^+)$, the dilatations f_r converge to f in norm.*

Proof. We fix a function $f \in \mathcal{B}_{q,w}^p(\mathbb{C}^+)$, and for simplicity write

$$d\mu(z) = w(z) \operatorname{Im}(z)^{\alpha+p-2} e^{-\beta|z|^2} dA(z)$$

and

$$\|f\|_{\mathcal{B}_{q,w}^p(\mathbb{C}^+)}^p = \int_{\mathbb{C}^+} [|\partial_z f(z)|^p + |\partial_{\bar{z}} f(z)|^p] d\mu(z).$$

It then follows that

$$\begin{aligned} \int_{\mathbb{C}^+} |\partial_z f_r(z)|^p d\mu(z) &= \frac{r^p}{r^{k+\alpha+p}} \int_{\mathbb{C}^+} |\partial_z f(z)|^p r^k w\left(\frac{z}{r}\right) \operatorname{Im}(z)^{\alpha+p-2} e^{-\beta|z|^2/r^2} dA(z) \\ &\leq \frac{C}{r^{k+\alpha}} \int_{\mathbb{C}^+} |\partial_z f(z)|^p d\mu(z), \end{aligned}$$

from which we get

$$\limsup_{r \rightarrow 1^-} \int_{\mathbb{C}^+} |\partial_z f_r(z)|^p d\mu(z) \leq \int_{\mathbb{C}^+} |\partial_z f(z)|^p d\mu(z).$$

This in turn shows that

$$\lim_{r \rightarrow 1^-} \int_{\mathbb{C}^+} |\partial_z f_r(z) - \partial_z f(z)|^p w(z) \operatorname{Im}(z)^{\alpha+p-2} e^{-\beta|z|^2} dA(z) = 0.$$

The proof that

$$\lim_{r \rightarrow 1^-} \int_{\mathbb{C}^+} |\partial_{\bar{z}} f_r(z) - \partial_{\bar{z}} f(z)|^p w(z) \operatorname{Im}(z)^{\alpha+p-2} e^{-\beta|z|^2} dA(z) = 0$$

is similar. Putting these pieces together we obtain

$$\lim_{r \rightarrow 1^-} \|f_r - f\|_{\mathcal{B}_{q,w}^p(\mathbb{C}^+)}^p = 0.$$

□

Example 3.4.

- (a) Let $\beta > 0$ and $n \in \mathbb{N}$. Consider the weight $w(z) = e^{-\beta|z|^n}$, which satisfies (2), since

$$w\left(\frac{z}{r}\right) = e^{-\beta|z|^n/r^n} \leq e^{-\beta|z|^n} = w(z), \quad 0 < r < 1.$$

- (b) Let

$$w(z) = e^{-\beta|\operatorname{Re}(z)|^n} = e^{-\beta|x|^n}.$$

Again, we see that $w(z/r) \leq w(z)$.

- (c) Let $w(z) = \exp(|z|)$, then

$$\frac{d}{dr} w\left(\frac{z}{r}\right) = -\frac{|z|}{r^2} \exp\left(\frac{|z|}{r}\right) < 0$$

while

$$\frac{d}{dr} \left[r w\left(\frac{z}{r}\right) \right] = \left(1 - \frac{|z|}{r}\right) \exp\left(\frac{|z|}{r}\right) > 0, \quad |z| < r.$$

Note also that $r^2 e^{|z|/r}$ is increasing for $r > \frac{1}{2}$ since

$$\frac{d}{dr} (r^2 e^{|z|/r}) = 2r e^{|z|/r} + r^2 \left(\frac{-1}{r^2} e^{|z|/r}\right) = (2r - 1) e^{|z|/r} > 0.$$

4. ANGULAR WEIGHTS

We now consider the *angular weights*. These are weights that depend only on the argument of z , that is, $w(re^{i\theta}) = w(\theta)$.

Theorem 4.1. *Let $w: \mathbb{D} \rightarrow (0, \infty)$ be an angular weight satisfying*

$$\int_0^{2\pi} w(\theta) d\theta < \infty.$$

Then the polyanalytic polynomials of degree at most q are dense in $\mathcal{B}_{q,w}^p$, $2 \leq p < \infty$.

P r o o f. It is easy to see that $w(z) = w(z/r) = w(\theta)$, so that the semi-norm of f_r is

$$\begin{aligned} \|f_r\|_{\mathcal{B}_{q,w}^p}^p &= \int_{\mathbb{D}} (1 - |z|^2)^{p-2} [|\partial_z f_r(z)|^p + |\partial_{\bar{z}} f_r(z)|^p] w(z) \, dA(z) \\ &= r^{p-2} \int_{r\mathbb{D}} \left(\frac{r^2 - |z|^2}{r^2} \right)^{p-2} [|\partial_z f(z)|^p + |\partial_{\bar{z}} f(z)|^p] w(z) \, dA(z). \end{aligned}$$

This implies that

$$\limsup_{r \rightarrow 1^-} \|f_r\|_{\mathcal{B}_{q,w}^p}^p = \|f\|_{\mathcal{B}_{q,w}^p}^p,$$

and finally

$$\lim_{r \rightarrow 1^-} \|f_r - f\|_{\mathcal{B}_{q,w}^p}^p = 0.$$

But in the unit disk, each f_r can be approximated by polyanalytic polynomials of degree at most q (see the argument at the end of the proof of Theorem 2.1). In this way, the assertion follows. \square

Theorem 4.2. *Let $w(se^{i\theta}) = \omega(s)v(\theta)$ be a weight function on \mathbb{D} , where ω and v satisfy*

$$\int_0^1 s\omega(s) \, ds < \infty, \quad \int_0^{2\pi} v(\theta) \, d\theta < \infty,$$

and $r^k \omega(s/r) \leq C\omega(s)$ for an integer $k \geq 0$. Then the polyanalytic polynomials of degree at most q are dense in $\mathcal{B}_{q,w}^p$, $2 \leq p < \infty$.

P r o o f. We start by writing

$$\begin{aligned} \|f_r\|_{\mathcal{B}_{q,w}^p}^p &= \int_{\mathbb{D}} (1 - |z|^2)^{p-2} [|\partial_z f_r(z)|^p + |\partial_{\bar{z}} f_r(z)|^p] w(s)v(\theta) \, dA(z) \\ &= r^{p-2-k} \int_{r\mathbb{D}} \left(\frac{r^2 - |z|^2}{r^2} \right)^{p-2} [|\partial_z f(z)|^p + |\partial_{\bar{z}} f(z)|^p] r^k w\left(\frac{s}{r}\right) v(\theta) \, dA(z) \\ &\leq Cr^{p-2-k} \int_{r\mathbb{D}} \left(\frac{r^2 - |z|^2}{r^2} \right)^{p-2} [|\partial_z f(z)|^p + |\partial_{\bar{z}} f(z)|^p] w(s)v(\theta) \, dA(z). \end{aligned}$$

Therefore, the dominated convergence theorem applies

$$\limsup_{r \rightarrow 1^-} \|f_r\|_{\mathcal{B}_{q,w}^p}^p = \|f\|_{\mathcal{B}_{q,w}^p}^p.$$

As in the preceding theorem, the rest is routine. \square

We now state the upper half-plane analogs of these theorems.

Theorem 4.3. *Let $0 < p < \infty$ and let w be an angular weight on \mathbb{C}^+ satisfying*

$$\int_0^\pi w(\theta) \, d\theta < \infty.$$

Then for each $f \in \mathcal{D}_{q,w}^p(\mathbb{C}^+)$, the dilatations f_r converge to f in norm.

Proof. Let us write

$$(7) \quad \|f_r - f\|_{\mathcal{D}_{q,w}^p(\mathbb{C}^+)}^p = \int_{\mathbb{C}^+} |\partial_z f_r(z) - \partial_z f(z)|^p w(z) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2} dA(z) \\ + \int_{\mathbb{C}^+} |\partial_{\bar{z}} f_r(z) - \partial_{\bar{z}} f(z)|^p w(z) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2} dA(z).$$

We replace z by z/r , and use $\partial_z f_r(z) = r\partial_z f(rz)$ and $w(z) = w(z/r)$ to get

$$\int_{\mathbb{C}^+} |\partial_z f_r(z)|^p w(z) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2} dA(z) \\ = r^{p-(\alpha+2)} \int_{\mathbb{C}^+} |\partial_z f(z)|^p w\left(\frac{z}{r}\right) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2/r^2} dA(z) \\ \leq r^{p-(\alpha+2)} \int_{\mathbb{C}^+} |\partial_z f(z)|^p w(z) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2} dA(z),$$

and finally,

$$\limsup_{r \rightarrow 1^-} \int_{\mathbb{C}^+} |\partial_z f_r(z)|^p w(z) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2} dA(z) \\ = \int_{\mathbb{C}^+} |\partial_z f(z)|^p w(z) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2} dA(z).$$

This implies that

$$\lim_{r \rightarrow 1^-} \int_{\mathbb{C}^+} |\partial_z f_r(z) - \partial_z f(z)|^p w(z) \operatorname{Im}(z)^\alpha e^{-\beta|z|^2} dA(z) = 0.$$

Similarly, one shows that the second term on the right-hand side of (7) tends to zero, from which we obtain

$$\lim_{r \rightarrow 1^-} \|f_r - f\|_{\mathcal{D}_{q,w}^p(\mathbb{C}^+)}^p = 0.$$

□

Theorem 4.4. Let $w(se^{i\theta}) = \omega(s)v(\theta)$ be an angular weight on \mathbb{C}^+ , where ω and v satisfy

$$\int_0^1 s\omega(s) ds < \infty, \quad \int_0^\pi v(\theta) d\theta < \infty,$$

and $r^k \omega(s/r) \leq C\omega(s)$ for an integer $k \geq 0$. Then for each $f \in \mathcal{B}_{q,w}^p(\mathbb{C}^+)$, $2 \leq p < \infty$, the dilatations f_r converge to f in norm.

Proof. Again, we see that

$$\begin{aligned} \|f_r\|_{\mathcal{B}_{q,w}^p(\mathbb{C}^+)}^p &= \int_{\mathbb{D}} (1 - |z|^2)^{p-2} [|\partial_z f_r(z)|^p + |\partial_{\bar{z}} f_r(z)|^p] w(s) v(\theta) \, dA(z) \\ &= r^{p-2-k} \int_{r\mathbb{D}} \left(\frac{r^2 - |z|^2}{r^2} \right)^{p-2} [|\partial_z f(z)|^p + |\partial_{\bar{z}} f(z)|^p] r^k w\left(\frac{s}{r}\right) v(\theta) \, dA(z) \\ &\leq Cr^{p-2-k} \int_{r\mathbb{D}} \left(\frac{r^2 - |z|^2}{r^2} \right)^{p-2} [|\partial_z f(z)|^p + |\partial_{\bar{z}} f(z)|^p] w(s) v(\theta) \, dA(z). \end{aligned}$$

Therefore, application of the dominated convergence theorem yields

$$\limsup_{r \rightarrow 1^-} \|f_r\|_{\mathcal{B}_{q,w}^p(\mathbb{C}^+)}^p \leq \|f\|_{\mathcal{B}_{q,w}^p(\mathbb{C}^+)}^p,$$

from which the result follows. \square

References

- [1] *A. Abkar*: Norm approximation by polynomials in some weighted Bergman spaces. *J. Funct. Anal.* *191* (2002), 224–240. [zbl](#) [MR](#) [doi](#)
- [2] *A. Abkar*: Approximation in weighted analytic Besov spaces and in generalized Fock spaces. *Complex Anal. Oper. Theory* *16* (2022), Article ID 11, 19 pages. [zbl](#) [MR](#) [doi](#)
- [3] *A. Abkar*: Mean approximation in Bergman spaces of polyanalytic functions. *Anal. Math. Phys.* *12* (2022), Article ID 52, 16 pages. [zbl](#) [MR](#) [doi](#)
- [4] *L. D. Abreu, H. G. Feichtinger*: Function spaces of polyanalytic functions. *Harmonic and Complex Analysis and its Applications. Trends in Mathematics.* Springer, Cham, 2014, pp. 1–38. [zbl](#) [MR](#) [doi](#)
- [5] *M. B. Balk*: Polyanalytic Functions. *Mathematical Research* 63. Akademie, Berlin, 1991. [zbl](#) [MR](#)
- [6] *P. Duren, E. A. Gallardo-Gutiérrez, A. Montes-Rodríguez*: A Paley-Wiener theorem for Bergman spaces with application to invariant subspaces. *J. Lond. Math. Soc.* *39* (2007), 459–466. [zbl](#) [MR](#) [doi](#)
- [7] *A. Haimi, H. Hedenmalm*: Asymptotic expansion of polyanalytic Bergman kernels. *J. Funct. Anal.* *267* (2014), 4667–4731. [zbl](#) [MR](#) [doi](#)
- [8] *A. D. Koševlev*: On the kernel function of the Hilbert space of functions polyanalytic in a disk. *Dokl. Akad. Nauk SSSR* *232* (1977), 277–279. (In Russian.) [zbl](#) [MR](#)
- [9] *N. I. Muskhelishvili*: Some Basic Problems of the Mathematical Theory of Elasticity. Nauka, Moscow, 1966. (In Russian.) [zbl](#) [MR](#)
- [10] *A. K. Ramazanov*: On the structure of spaces of polyanalytic functions. *Math. Notes* *72* (2002), 692–704. [zbl](#) [MR](#) [doi](#)
- [11] *N. L. Vasilevski*: On the structure of Bergman and poly-Bergman spaces. *Integral Equations Oper. Theory* *33* (1999), 471–488. [zbl](#) [MR](#) [doi](#)

Author's address: Ali Abkar, Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin 34149, Iran, e-mail: abkar@sci.ikiu.ac.ir.