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RUN-LENGTH FUNCTION OF THE BOLYAI-RÉNYI EXPANSION OF REAL NUMBERS

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Abstract. By iterating the Bolyai-Rényi transformation $T(x) = (x+1)^2 \pmod 1$, almost every real number $x \in [0,1)$ can be expanded as a continued radical expression

$$x = -1 + \sqrt{x_1 + \sqrt{x_2 + \ldots + \sqrt{x_n + \ldots}}}$$

with digits $x_n \in \{0,1,2\}$ for all $n \in \mathbb{N}$. For any real number $x \in [0,1)$ and digit $i \in \{0,1,2\}$, let $r_n(x,i)$ be the maximal length of consecutive i's in the first n digits of the Bolyai-Rényi expansion of x. We study the asymptotic behavior of the run-length function $r_n(x,i)$. We prove that for any digit $i \in \{0,1,2\}$, the Lebesgue measure of the set

$$D(i) = \Big\{x \in [0,1)\colon \lim_{n\to\infty} \frac{r_n(x,i)}{\log n} = \frac{1}{\log \theta_i}\Big\}$$

is 1, where $\theta_i = 1 + \sqrt{4i+1}$. We also obtain that the level set

$$E_\alpha(i) = \Big\{x \in [0,1)\colon \lim_{n\to\infty} \frac{r_n(x,i)}{\log n} = \alpha\Big\}$$

is of full Hausdorff dimension for any $0 \leqslant \alpha \leqslant \infty$.

Keywords: run-length function; Bolyai-Rényi expansion; Lebesgue measure; Hausdorff dimension

MSC 2020: 11K55, 28A80

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1. Introduction

The Bolyai-Rényi transformation $T\colon\ [0,1) \to [0,1)$ is defined by

$$T(x) = (x+1)^2 \pmod 1 = \begin{cases} x^2+2x & \text{if } x \in I_0 := [0, \sqrt{2}-1), \\ x^2+2x-1 & \text{if } x \in I_1 := [\sqrt{2}-1, \sqrt{3}-1), \\ x^2+2x-2 & \text{if } x \in I_2 := [\sqrt{3}-1, 1). \end{cases}$$

It is well known that by iterating the transformation T, almost every real number $x \in [0,1)$ can be expanded as a continued radical expression. For any $x \in [0,1)$, define $x_1 := x_1(x) = i$ if $x \in I_i$, or equivalently,

$$x_1 = \lfloor (x+1)^2 \rfloor - 1,$$

where $\lfloor \cdot \rfloor$ denotes the integer part of a real number. For any integer $n \geqslant 2$, define $x_n = x_1(T^{n-1}(x))$ with T^{n-1} the $(n-1)$th iteration of T. It is easy to check that

$$(T^{n-1}(x)+1)^2 = x_n + T^n(x) + 1. \tag{1.1}$$

Thus,

$$\begin{aligned} x &= -1 + \sqrt{x_1 + T(x) + 1} = -1 + \sqrt{x_1 + \sqrt{x_2 + T^2(x) + 1}} = \ldots \\ &= -1 + \sqrt{x_1 + \sqrt{x_2 + \ldots + \sqrt{x_n + T^n(x) + 1}}}. \end{aligned} \tag{1.2}$$

If $x_n \neq 0$ for infinitely many $n \in \mathbb{N}$, or equivalently, $T^n(x) \neq 0$ for all $n \in \mathbb{N}$ (see Remark 2.1), then we have

$$x = -1 + \lim_{n\to\infty} \sqrt{x_1 + \sqrt{x_2 + \ldots + \sqrt{x_n}}} =: -1 + \sqrt{x_1 + \sqrt{x_2 + \ldots + \sqrt{x_n + \ldots}}}.$$

This kind of expansion was originally introduced by Bolyai in 1832 to approximate roots of certain equations, see [3]. In his influential paper in 1957, Rényi in [6] studied the ergodic properties of various 'f-expansions' of real numbers, including the continued fraction expansion, the β-expansion and the Bolyai-Rényi expansion. Unlike the well known Gauss transformation and the β-transformation, an exact formula for the density function of the unique absolutely continuous invariant probability measure of the Bolyai-Rényi transformation is still not known. This fact hinders the study of the statistical properties of the Bolyai-Rényi expansion, for example, the frequency of the digits. In [3], Jenkinson and Pollicott computed the frequency $\mathcal{F}(i)$ of the digit

$i \in \{0,1,2\}$ occurring in almost every Bolyai-Rényi expansion by numerical method. They obtained that

$$\mathcal{F}(0) = 0.46407962943 \pm 9.6 \times 10^{-7},$$
$$\mathcal{F}(1) = 0.3044190449 \pm 5 \times 10^{-7},$$
$$\mathcal{F}(2) = 0.2315013256 \pm 3.6 \times 10^{-7}.$$

In this paper, we study the consecutive appearance of the same digit $i \in \{0,1,2\}$ in the Bolyai-Rényi expansion of almost all real numbers $x \in [0,1)$. For any $n \in \mathbb{N}$, the run-length function $r_n(x,i)$ is defined as the maximal length of consecutive i's in the first n digits of the Bolyai-Rényi expansion of x (we will use the same symbol for continued fraction expansion and β-expansion, where it causes no confusion), i.e.,

$$r_n(x,i) = \max\{k \geqslant 0\colon\ x_{j+1} = \ldots = x_{j+k} = i \text{ for some } 0 \leqslant j \leqslant n-k\}.$$

Following Erdös and Rényi's (see [1]) interesting result that the maximal average gain of a player over $\lfloor \log_2 n \rfloor$ consecutive repetitions of a fair game tends to 1 almost surely, the run-length functions of the continued fraction expansion and the β-expansion of real numbers have recently been well studied. For continued fraction expansion, Song and Zhou in [7] proved that for any $i \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{r_n(x,i)}{\log_{(i+\sqrt{i^2+4})/2} n} = \frac{1}{2}, \quad \mathcal{L}\text{-a.e. } x \in [0,1),$$

where $\mathcal{L}$ denotes the Lebesgue measure. For β-expansion, Tong, Yu and Zhao in [9] proved that for any $\beta > 1$,

$$\lim_{n\to\infty} \frac{r_n(x,0)}{\log_\beta n} = 1, \quad \mathcal{L}\text{-a.e. } x \in [0,1).$$

Hausdorff dimensions of the corresponding level sets are also determined in both of their papers. (For more metric results about the run-length function in various expansions of real numbers, the readers are referred to [4], [8], [10], [11] and references therein.) It should be noted that in both of the above two papers, Philipp's metric result (see [5]) about the mixing of Gauss transformation or β-transformation (see Lemma 2.3 in [7] and Lemma 2.2 in [9]) plays an important role in the estimate of the Lebesgue measure. However, to the best knowledge of the authors, there is no such strong result for the Bolyai-Rényi transformation. To overcome this disadvantage, we study the properties of Bolyai-Rényi transformation and obtain the following results.

Theorem 1.1. *For any* $i \in \{0,1,2\}$, *let*

$$D(i) = \Big\{x \in [0,1)\colon \lim_{n\to\infty} \frac{r_n(x,i)}{\log n} = \frac{1}{\log \theta_i}\Big\},$$

where $\theta_i = 1 + \sqrt{4i+1}$. *Then* $\mathcal{L}(D(i)) = 1$.

Theorem 1.2. *For any* $i \in \{0,1,2\}$ *and* $0 \leqslant \alpha \leqslant \infty$, *let*

$$E_\alpha(i) = \Big\{x \in [0,1)\colon \lim_{n\to\infty} \frac{r_n(x,i)}{\log n} = \alpha\Big\}.$$

Then $\dim_H E_\alpha(i) = 1$, *where* $\dim_H$ *denotes the Hausdorff dimension.*

The rest of this paper is organized as follows: we first introduce some notions as well as the properties of the Bolyai-Rényi expansion in the next section. Then we prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4.

2. PRELIMINARIES

In this section, we introduce some notions and study the properties of the Bolyai-Rényi expansion.

Suppose that $u = u_1u_2\dots u_m$ and $v = v_1v_2\dots v_n$ are two words (i.e., finite sequences) of integers, $\xi = \xi_1\xi_2\dots$ is an infinite sequence of integers. Denote by $|u| := m$ the length of u. Write $uv = u_1\dots u_mv_1\dots v_n$ for the concatenation of u and v. For any integer $k \geqslant 0$, the symbol u^k denotes the k times self-concatenation of u. For any $1 \leqslant j \leqslant |u|$ and $l \geqslant 1$, $u|_j = u_1u_2\dots u_j$ is a truncation of u and $\xi|_l = \xi_1\xi_2\dots\xi_l$ is a truncation of ξ.

Define the lexicographical order '$\prec$' between two distinct infinite sequences ξ, η of integers as follows: $\xi \prec \eta$ if $\xi_1 < \eta_1$ or there exists an integer $j > 1$ such that $\xi_j < \eta_j$ and $\xi_l = \eta_l$ for all $1 \leqslant l < j$.

Let $\Omega = \{0,1,2\}$. For any $n \in \mathbb{N}$, let

$$\Omega^n = \{u_1\dots u_n\colon\ u_j \in \Omega \text{ for all } 1 \leqslant j \leqslant n\}$$

and $\Omega^* = \bigcup\limits_{n=1}^{\infty} \Omega^n$. For any $x \in [0,1)$, denote the infinite sequence $x_1x_2\dots$ by $\varepsilon(x)$, where

$$x_n = x_1(T^{n-1}(x)) = \lfloor (T^{n-1}(x)+1)^2 \rfloor - 1, \quad n \in \mathbb{N}.$$

Remark 2.1. For any $x \in [0,1)$ we have $x_n \neq 0$ for infinitely many $n \in \mathbb{N}$ if and only if $T^n(x) \neq 0$ for all $n \in \mathbb{N}$. In fact, if $T^{n_0}(x) = 0$ for some $n_0 \in \mathbb{N}$, then for any $n \geqslant n_0$ we have $T^n(x) = 0$, which implies that $x_{n+1} = 0$. On the other hand, if there exists some $n_1 \in \mathbb{N}$ such that $x_n = 0$ for all $n > n_1$, then by (1.1), we have $1 > T^n(x) \geqslant 2T^{n-1}(x) \geqslant \ldots \geqslant 2^{n-n_1}T^{n_1}(x) \geqslant 0$, which implies that $T^{n_1}(x) = 0$.

By (1.2) and Remark 2.1, except for a countable set, every real number $x \in [0,1)$ has a Bolyai-Rényi expansion

$$x = -1 + \lim_{n\to\infty} \sqrt{x_1 + \sqrt{x_2 + \ldots + \sqrt{x_n}}} =: -1 + \sqrt{x_1 + \sqrt{x_2 + \ldots + \sqrt{x_n + \ldots}}}.$$

Lemma 2.2. *For any $x, \widetilde{x} \in [0,1)$ we have $\varepsilon(x) \prec \varepsilon(\widetilde{x})$ if and only if $x < \widetilde{x}$.*

Proof. We first prove by induction that if $x < \widetilde{x}$, then $\varepsilon(x) = \varepsilon(\widetilde{x})$ or $\varepsilon(x) \prec \varepsilon(\widetilde{x})$. Since $x < \widetilde{x}$, it is clear that $x_1 = \lfloor (x+1)^2 \rfloor - 1 \leqslant \widetilde{x}_1$. If $x_1 < \widetilde{x}_1$, then we have $\varepsilon(x) \prec \varepsilon(\widetilde{x})$. Now, assume that for some $n \in \mathbb{N}$ we have $x_j = \widetilde{x}_j$ for all $1 \leqslant j \leqslant n$. Since

$$\begin{aligned} x &= -1 + \sqrt{x_1 + \sqrt{x_2 + \ldots + \sqrt{x_n + T^n(x) + 1}}} \\ &< \widetilde{x} = -1 + \sqrt{x_1 + \sqrt{x_2 + \ldots + \sqrt{x_n + T^n(\widetilde{x}) + 1}}}, \end{aligned}$$

it follows that $T^n(x) < T^n(\widetilde{x})$, which implies $x_{n+1} = \lfloor (T^n(x)+1)^2 \rfloor - 1 \leqslant \widetilde{x}_{n+1}$. Therefore, we have $\varepsilon(x) = \varepsilon(\widetilde{x})$ or $\varepsilon(x) \prec \varepsilon(\widetilde{x})$.

Next, we show that if $x < \widetilde{x}$, then the two infinite sequences $\varepsilon(x)$ and $\varepsilon(\widetilde{x})$ can not be the same. Otherwise, for any $n \in \mathbb{N}$ we have

$$0 < \widetilde{x} - x \leqslant \sqrt{x_1 + \sqrt{x_2 + \ldots + \sqrt{x_n + 2}}} - \sqrt{x_1 + \sqrt{x_2 + \ldots + \sqrt{x_n + 1}}}.$$

Since the right-hand side of the above inequality tends to 0 as $n \to \infty$, we come to a contradiction. This concludes the proof. □

For any $n \in \mathbb{N}$ and $w = w_1 w_2 \ldots w_n \in \Omega^n$, let

$$l_w = -1 + \sqrt{w_1 + \sqrt{w_2 + \ldots + \sqrt{w_n + 1}}}, \; r_w = -1 + \sqrt{w_1 + \sqrt{w_2 + \ldots + \sqrt{w_n + 2}}}$$

and

$$I(w) = \{x \in [0,1)\colon\ x_1 = w_1, \ldots, x_n = w_n\},$$

which is called a *cylinder* of order n. It is clear that $I(w) \cap I(w') = \emptyset$ for distinct $w, w' \in \Omega^n$ and

$$[0,1) = \bigcup_{w \in \Omega^n} I(w).$$

Proposition 2.3. *For any $n \in \mathbb{N}$ and $w \in \Omega^n$, the set $I(w)$ is a left-closed and right-open interval $[l_w, r_w)$, the length $|I(w)|$ of which satisfies*

$$4^{-n} \leqslant |I(w)| = r_w - l_w \leqslant 2^{-n}.$$

Proof. By Lemma 2.2, it is easy to observe that the set $I(w)$ is an interval. Recall that for any $x \in I(w)$ we have

$$x = -1 + \sqrt{w_1 + \sqrt{w_2 + \ldots + \sqrt{w_n + T^n(x) + 1}}}. \tag{2.1}$$

Since $T^n(x) \in [0,1)$, it follows that $l_w \leqslant x < r_w$. On the other hand, assume that $l_w \leqslant x < r_w$. Then we have

$$x = -1 + \sqrt{w_1 + \sqrt{w_2 + \ldots + \sqrt{w_n + t + 1}}}$$

for some $0 \leqslant t < 1$. Since $w_j \in \Omega = \{0,1,2\}$ for all $1 \leqslant j \leqslant n$, it follows that for any $1 \leqslant k \leqslant n$,

$$1 \leqslant \sqrt{w_k + \sqrt{w_{k+1} + \ldots + \sqrt{w_n + t + 1}}} < 2.$$

Thus, by induction, we can obtain that

$$T^{k-1}(x) = -1 + \sqrt{w_k + \sqrt{w_{k+1} + \ldots + \sqrt{w_n + t + 1}}},$$

which implies that $x_k = \lfloor (T^{k-1}(x)+1)^2 \rfloor - 1 = w_k$. Therefore, $x \in I(w)$.

Since

$$\begin{aligned}|I(w)| = r_w - l_w &= \sqrt{w_1 + \ldots + \sqrt{w_n + 2}} - \sqrt{w_1 + \ldots + \sqrt{w_n + 1}} \\ &= \frac{1}{\sqrt{w_1 + \ldots + \sqrt{w_n + 2}} + \sqrt{w_1 + \ldots + \sqrt{w_n + 1}}} \cdot \ldots \cdot \frac{1}{\sqrt{w_n + 2} + \sqrt{w_n + 1}},\end{aligned}$$

we have

$$\begin{aligned}2^{-n} &\geqslant 2^{1/2^n} - 1 = r_{0^n} - l_{0^n} = |I(0^n)| \geqslant |I(w)| \geqslant |I(2^n)| \\ &= \frac{1}{\sqrt{2 + \ldots + \sqrt{2+2}} + \sqrt{2 + \ldots + \sqrt{2+1}}} \cdot \ldots \cdot \frac{1}{\sqrt{2+2} + \sqrt{2+1}} \geqslant 4^{-n},\end{aligned} \tag{2.2}$$

where the first inequality follows from the fact that $2^s - 1 \leqslant s$ for any $s \in [0,1]$. □

Lemma 2.4. *For any $n \in \mathbb{N}$ and $w \in \Omega^n$, define the function $f_w \colon \mathbb{R} \to \mathbb{R}$ by*

$$f_w(x) = (\ldots(((x+1)^2 - w_1)^2 - w_2)^2 \ldots - w_{n-1})^2 - w_n - 1.$$

On the interval $I(w)$, we have $f_w(x) = T^n(x)$ and for any $x, \widetilde{x} \in I(w)$

$$\frac{1}{2} \leqslant \frac{f_w'(x)}{f_w'(\widetilde{x})} \leqslant 2.$$

Remark 2.5. It is clear that $f_w(x) = T_{w_n} \circ \ldots \circ T_{w_2} \circ T_{w_1}(x)$ if we let

$$T_i(x) = (x+1)^2 - i - 1 = x^2 + 2x - i, \quad i \in \{0,1,2\}$$

be the ith branch of the transformation T.

Proof of Lemma 2.4. By (2.1), it is clear that $f_w(x) = T^n(x)$ for any $x \in I(w)$. Assume that $x, \widetilde{x} \in I(w)$. By repeatedly using (2.1), we can obtain that

$$f'_w(x) = 2^n (T^{n-1}(x)+1)(T^{n-2}(x)+1)\ldots(T(x)+1)(x+1). \tag{2.3}$$

Then

$$\frac{f'_w(x)}{f'_w(\widetilde{x})} = \frac{T^{n-1}(x)+1}{T^{n-1}(\widetilde{x})+1} \cdot \frac{T^{n-2}(x)+1}{T^{n-2}(\widetilde{x})+1} \cdot \ldots \cdot \frac{T(x)+1}{T(\widetilde{x})+1} \cdot \frac{x+1}{\widetilde{x}+1}.$$

For any $0 \leqslant k \leqslant n-1$, since

$$x_j(T^k(x)) = x_1(T^{j-1}(T^k(x))) = x_{j+k}(x) \quad \text{for all } 1 \leqslant j \leqslant n-k,$$

it follows that $T^k(x) \in I(w_{k+1}\ldots w_n)$. Then by (2.2), we have

$$\frac{T^k(x)+1}{T^k(\widetilde{x})+1} \leqslant 1 + \frac{|I(w_{k+1}\ldots w_n)|}{T^k(\widetilde{x})+1} \leqslant 1 + |I(w_{k+1}\ldots w_n)| \leqslant 2^{1/2^{(n-k)}}.$$

Therefore,

$$\frac{f'_w(x)}{f'_w(\widetilde{x})} \leqslant 2^{1/2} \cdot 2^{1/2^2} \cdot \ldots \cdot 2^{1/2^n} \leqslant 2.$$

□

Lemma 2.6. *For any $w, v \in \Omega^*$ we have*

$$\frac{1}{2} \leqslant \frac{|I(wv)|}{|I(w)| \cdot |I(v)|} \leqslant 2.$$

Proof. Let the function f_w be defined as in Lemma 2.4. Then it is clear that $f_w(l_w) = 0$ and $f_w(r_w) = 1$. By the mean value theorem, there exists an $x \in I(w)$ such that

$$|I(w)| = r_w - l_w = \frac{f_w(r_w) - f_w(l_w)}{f'_w(x)} = \frac{1}{f'_w(x)}. \tag{2.4}$$

Assume that $|v| = n$. By the definition of f_w, we have

$$f_w(l_{wv}) = -1 + \sqrt{v_1 + \ldots + \sqrt{v_n + 1}} = l_v,$$
$$f_w(r_{wv}) = -1 + \sqrt{v_1 + \ldots + \sqrt{v_n + 2}} = r_v.$$

Then by the mean value theorem again, there exists $\widetilde{x} \in I(wv) \subset I(w)$ such that

$$|I(wv)| = r_{wv} - l_{wv} = \frac{f_w(r_{wv}) - f_w(l_{wv})}{f_w'(\widetilde{x})} = \frac{|I(v)|}{f_w'(\widetilde{x})}. \tag{2.5}$$

Therefore, by (2.4)–(2.5) and Lemma 2.4, it follows that

$$\frac{1}{2} \leqslant \frac{|I(wv)|}{|I(w)| \cdot |I(v)|} = \frac{f_w'(x)}{f_w'(\widetilde{x})} \leqslant 2.$$

□

Lemma 2.7. *Given $w \in \Omega^*$ and $n \in \mathbb{N}$ for any $i \in \Omega = \{0,1,2\}$ we have*

$$\frac{1}{\theta_i^{n+2}} \leqslant \frac{|I(wi^n)|}{|I(w)|} \leqslant \frac{1}{\theta_i^{n-2}},$$

where $\theta_i = 1 + \sqrt{4i+1}$.

Proof. Since $\theta_i \geqslant 2$, by Lemma 2.6, it suffices to show

$$\frac{1}{\theta_i^{n+1}} \leqslant |I(i^n)| \leqslant \frac{1}{\theta_i^{n-1}}.$$

If $i = 0$, since $\frac{1}{2}s \leqslant 2^s - 1 \leqslant s$ for any $s \in [0,1]$, it follows that

$$\frac{1}{\theta_0^{n+1}} = \frac{1}{2^{n+1}} \leqslant |I(0^n)| = r_{0^n} - l_{0^n} = 2^{1/2^n} - 1 \leqslant \frac{1}{2^n} = \frac{1}{\theta_0^n}.$$

If $i = 1$, let $x = \frac{1}{2}(\sqrt{5} - 1) \in I_1 = [\sqrt{2} - 1, \sqrt{3} - 1)$. It is easy to check that $T(x) = x$, which implies that $x \in I(1^n)$. By (2.3), we have $f_{1^n}'(x) = 2^n(x+1)^n = \theta_1^n$. As in (2.4), there exists an $\widetilde{x} \in I(1^n)$ such that $|I(1^n)| = 1/f_{1^n}'(\widetilde{x})$. So by Lemma 2.4,

$$\frac{1}{\theta_1^{n+1}} \leqslant \frac{1}{2\theta_1^n} \leqslant |I(1^n)| \leqslant \frac{2}{\theta_1^n} \leqslant \frac{1}{\theta_1^{n-1}}.$$

If $i = 2$, by Proposition 2.3, we have $|I(2^n)| \geqslant 4^{-n}$. As in (2.4), there exists an $\widetilde{x} \in I(2^n)$ such that $|I(2^n)| = 1/f_{2^n}'(\widetilde{x})$. Letting x tend to $r_{2^n} = 1$ on $I(2^n)$, by Lemma 2.4 we have $T^k(x) \to 1$ for all $1 \leqslant k \leqslant n-1$, which together with (2.3) implies $f_{2^n}'(x) \to 4^n$. Thus, by Lemma 2.4 again, it follows that $|I(2^n)| \leqslant 2/f_{2^n}'(x) \to 2/4^n$. Therefore,

$$\frac{1}{\theta_2^n} = \frac{1}{4^n} \leqslant |I(2^n)| \leqslant \frac{2}{4^n} \leqslant \frac{1}{\theta_2^{n-1}}.$$

□

Lemma 2.8. *For any* $w \in \Omega^*$ *we have* $|I(w0)| \geqslant |I(w1)|$ *and*

$$|I(w01)| \leqslant |I(w02)| + |I(w10)| + |I(w11)|.$$

Proof. Assume that $w \in \Omega^n$ with $n \in \mathbb{N}$. By Proposition 2.3, it is clear that

$$\begin{aligned}|I(w0)| &= r_{w0} - l_{w0} = \sqrt{w_1 + \ldots + \sqrt{w_n + \sqrt{0+2}}} - \sqrt{w_1 + \ldots + \sqrt{w_n + \sqrt{0+1}}}\\
&= \frac{1}{\sqrt{w_1 + \ldots + \sqrt{w_n + \sqrt{2}}} + \sqrt{w_1 + \ldots + \sqrt{w_n + 1}}} \cdot \ldots \cdot \frac{1}{\sqrt{2}+1}\\
&\geqslant \frac{1}{\sqrt{w_1 + \ldots + \sqrt{w_n + \sqrt{3}}} + \sqrt{w_1 + \ldots + \sqrt{w_n + \sqrt{2}}}} \cdot \ldots \cdot \frac{1}{\sqrt{3}+\sqrt{2}}\\
&= \sqrt{w_1 + \ldots + \sqrt{w_n + \sqrt{1+2}}} - \sqrt{w_1 + \ldots + \sqrt{w_n + \sqrt{1+1}}} = |I(w1)|.\end{aligned}$$

By Lemma 2.2 and Proposition 2.3, we know that $I(w01)$, $I(w02)$, $I(w10)$ and $I(w11)$ are consecutive intervals from left to right. Let the function f_w be defined as in Lemma 2.4. Then we have

$$\begin{aligned}f_w(l_{w01}) &= -1 + \sqrt{0 + \sqrt{1+1}} = -1 + \sqrt[4]{2},\\
f_w(r_{w01}) = f_w(l_{w02}) &= -1 + \sqrt{0 + \sqrt{1+2}} = -1 + \sqrt[4]{3},\\
f_w(r_{w11}) &= -1 + \sqrt{1 + \sqrt{1+2}} = -1 + \sqrt{1+\sqrt{3}}.\end{aligned}$$

Thus, by the mean value theorem, there are $x, \widetilde{x} \in I(w)$ such that

$$\frac{|I(w01)|}{|J|} = \frac{\sqrt[4]{3} - \sqrt[4]{2}}{\sqrt{1+\sqrt{3}} - \sqrt[4]{3}} \cdot \frac{f'_w(x)}{f'_w(\widetilde{x})} \leqslant \frac{2(\sqrt[4]{3} - \sqrt[4]{2})}{\sqrt{1+\sqrt{3}} - \sqrt[4]{3}} \approx 0.7533,$$

where $J = I(w02) \cup I(w10) \cup I(w11)$ and the inequality follows from Lemma 2.4. □

3. LEBESGUE MEASURE

In this section, we prove Theorem 1.1 with the help of Lemma 2.7 and the Borel-Cantelli lemma. The proof is split into two propositions (3.1 to get the lim sup and 3.2 to get the lim inf).

Proposition 3.1. *For any* $i \in \Omega = \{0, 1, 2\}$ *and* $\varepsilon > 0$ *we have*

$$\limsup_{n\to\infty} \frac{r_n(x,i)}{\log_{\theta_i} n} \leqslant 1 + \varepsilon, \quad \mathcal{L}\text{-a.e. } x \in [0,1).$$

Proof. It suffices to show that

$$\mathcal{L}\{x \in [0,1) \colon\ r_n(x,i) > (1+\varepsilon)\log_{\theta_i} n \text{ for infinitely many } n \in \mathbb{N}\} = 0.$$

For any $x \in [0,1)$ and $n \in \mathbb{N}$, let $l_n(x,i)$ denote the maximal length of consecutive i's following the digit x_n in the infinite sequence $\varepsilon(x) = x_1x_2\ldots$, i.e.,

$$l_n(x,i) = \sup\{k \geqslant 0 \colon\ x_{n+j} = i \text{ for all } 1 \leqslant j \leqslant k\}.$$

It is easy to check that

$$\begin{aligned}&\{x \in [0,1) \colon\ r_n(x,i) > (1+\varepsilon)\log_{\theta_i} n \text{ for infinitely many } n \in \mathbb{N}\}\\ &\quad\subset \{x \in [0,1) \colon\ l_n(x,i) > (1+\varepsilon)\log_{\theta_i} n \text{ for infinitely many } n \in \mathbb{N}\}\\ &\quad\subset \bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\bigcup_{w\in\Omega^n} I(wi^{\lceil(1+\varepsilon)\log_{\theta_i} n\rceil}),\end{aligned}$$

where $\lceil y\rceil$ denotes the smallest integer not less than y. Then by Lemma 2.7,

$$\begin{aligned}&\mathcal{L}\{x \in [0,1) \colon\ r_n(x,i) > (1+\varepsilon)\log_{\theta_i} n \text{ for infinitely many } n \in \mathbb{N}\}\\ &\quad\leqslant \liminf_{m\to\infty}\sum_{n=m}^{\infty}\sum_{w\in\Omega^n} |I(wi^{\lceil(1+\varepsilon)\log_{\theta_i} n\rceil})|\\ &\quad\leqslant \liminf_{m\to\infty}\sum_{n=m}^{\infty}\sum_{w\in\Omega^n} \frac{|I(w)|}{{\theta_i}^{(1+\varepsilon)\log_{\theta_i} n-2}} = \liminf_{m\to\infty}\sum_{n=m}^{\infty}\frac{{\theta_i}^2}{n^{1+\varepsilon}} = 0.\end{aligned}$$

□

Proposition 3.2. *For any* $i \in \Omega = \{0,1,2\}$ *and* $0 < \varepsilon < 1$, *we have*

$$\liminf_{n\to\infty}\frac{r_n(x,i)}{\log_{\theta_i} n} \geqslant 1-\varepsilon, \quad \mathcal{L}\text{-a.e. } x \in [0,1).$$

Proof. For any $n \in \mathbb{N}$, let $\varrho_n = \lceil(1-\varepsilon)\log_{\theta_i} n\rceil$ and

$$A_n = \{x \in [0,1) \colon\ r_n(x,i) < \varrho_n\}.$$

It suffices to prove that $\mathcal{L}\Bigl(\bigcap\limits_{m=1}^{\infty}\bigcup\limits_{n=m}^{\infty} A_n\Bigr) = 0$. Suppose that $m \in \mathbb{N}$ is large. For any $n \geqslant m$, let $k_n = \lfloor n/\varrho_n\rfloor$. For any $1 \leqslant k \leqslant k_n$, let

$$\mathbb{A}_{n,k} = \{w \in \Omega^{k\varrho_n} \colon\ w_{(j-1)\varrho_n+1}w_{(j-1)\varrho_n+2}\ldots w_{j\varrho_n} \neq i^{\varrho_n} \text{ for all } 1 \leqslant j \leqslant k\}$$

and $A_{n,k} = \bigcup\limits_{w\in\mathbb{A}_{n,k}} I(w)$. Let $\mathbb{A}_{n,0} = \{\emptyset\}$ with $\emptyset$ the empty word and $A_{n,0} = [0,1)$. It is easy to check that $A_n \subset A_{n,k_n} \subset A_{n,k_n-1} \subset \ldots \subset A_{n,1} \subset A_{n,0}$. Since

$$A_{n,k} = \bigcup_{u\in\mathbb{A}_{n,k}} I(u) = \bigcup_{w\in\mathbb{A}_{n,k-1}} \bigcup_{\substack{v\in\Omega^{\varrho_n}\\ v\neq i^{\varrho_n}}} I(wv) = \bigcup_{w\in\mathbb{A}_{n,k-1}} (I(w) - I(wi^{\varrho_n})),$$

by Lemma 2.7 we have

$$\begin{aligned}\mathcal{L}(A_{n,k}) &= \sum_{w\in\mathbb{A}_{n,k-1}} (|I(w)| - |I(wi^{\varrho_n})|)\\ &\leqslant \Bigl(1 - \frac{1}{\theta_i^{\varrho_n+2}}\Bigr) \sum_{w\in\mathbb{A}_{n,k-1}} |I(w)| = \Bigl(1 - \frac{1}{\theta_i^{\varrho_n+2}}\Bigr)\mathcal{L}(A_{n,k-1})\end{aligned}$$

for all $1 \leqslant k \leqslant k_n$. Then

$$\mathcal{L}(A_n) \leqslant \mathcal{L}(A_{n,k_n}) \leqslant (1-\theta_i^{-2-\varrho_n})^{k_n}\mathcal{L}(A_{n,0}) = (1-\theta_i^{-2-\varrho_n})^{k_n},$$

which implies that

$$\sum_{n=m}^{\infty} \mathcal{L}(A_n) \leqslant \sum_{n=m}^{\infty} (1-\theta_i^{-2-\varrho_n})^{k_n} \leqslant \sum_{n=m}^{\infty} \mathrm{e}^{-k_n/\theta_i^{\varrho_n+2}} \leqslant \sum_{n=m}^{\infty} \mathrm{e}^{-n^{\varepsilon}/(4(1-\varepsilon)\theta_i^3 \log_{\theta_i} n)} < \infty.$$

Therefore, by the Borel-Cantelli lemma, we have $\mathcal{L}\Bigl(\bigcap\limits_{m=1}^{\infty}\bigcup\limits_{n=m}^{\infty} A_n\Bigr) = 0$. □

Proof of Theorem 1.1. It is a direct corollary of Propositions 3.1 and 3.2. □

4. HAUSDORFF DIMENSION

In this section, the lower bound of the Hausdorff dimension of the set

$$E_\alpha(i) = \Bigl\{x \in [0,1)\colon \lim_{n\to\infty} \frac{r_n(x,i)}{\log n} = \alpha\Bigr\}$$

is given by the Hausdorff dimension of a Cantor subset F.

4.1. Cantor subset. Fix $i \in \{0,1,2\}$ and an integer $N \geqslant 5$. For any $k \in \mathbb{N}$, let

$$\delta_k = \begin{cases} \alpha & \text{if } 0 \leqslant \alpha < \infty,\\ k & \text{if } \alpha = \infty.\end{cases}$$

Let n_0 be a positive integer large enough such that for all $n \geqslant n_0$,

$$n \geqslant \delta_1 \log n + N.$$

For any $k \in \mathbb{N}$, let $n_k = 2^{k^2} n_0$. Since $\delta_k \leqslant k\delta_1$ and $n_k \geqslant 2kn_{k-1}$ we have

$$n_k - n_{k-1} - \delta_k \log n_{k-1} \geqslant k(n_{k-1} - \delta_1 \log n_{k-1}) \geqslant N. \tag{4.1}$$

It is easy to check that

$$\lim_{k\to\infty} \frac{\log n_{k+1}}{\log n_k} = 1, \quad \lim_{k\to\infty} \delta_k = \alpha \quad \text{and} \quad \lim_{k\to\infty} \frac{\delta_{k+1} \log n_k}{n_k - n_{k-1}} = 0. \tag{4.2}$$

Now, we construct a Cantor subset F of $E_\alpha(i)$. Let $\mathbb{F}_0 = \{0^{n_0}\}$. Inductively, assume that for an integer $k \geqslant 1$, the set $\mathbb{F}_{k-1}$ has already been defined such that $|\tau| = n_{k-1}$ and $\tau|_{n_0} = 0^{n_0}$ for any $\tau \in \mathbb{F}_{k-1}$. Let

$$t_k = \left\lfloor \frac{n_k - n_{k-1} - \delta_k \log n_{k-1}}{N} \right\rfloor \quad \text{and} \quad s_k = n_k - n_{k-1} - Nt_k. \tag{4.3}$$

By (4.1), we have $t_k \geqslant 1$ and

$$\delta_k \log n_{k-1} \leqslant s_k < \delta_k \log n_{k-1} + N. \tag{4.4}$$

Define

$$\mathbb{F}_k = \{\tau i^{s_k} w^{(1)} \dots w^{(t_k)} \colon \tau \in \mathbb{F}_{k-1},\ w^{(j)} \in \mathbb{V}_N \text{ for all } 1 \leqslant j \leqslant t_k\},$$

where

$$\mathbb{V}_N = \{10w01\colon w \in \Omega^{N-4}\}.$$

It is easy to check that $|v| = n_k$ and $v|_{n_0} = 0^{n_0}$ for all $v \in \mathbb{F}_k$. Let

$$F = \bigcap_{k=0}^{\infty} \bigcup_{v \in \mathbb{F}_k} I(v) \subset [0,1).$$

Lemma 4.1. *We have $F \subset E_\alpha(i)$.*

Proof. Let x be an arbitrary element in F. For any $n \geqslant n_1$, there exists a unique integer $k \geqslant 2$ such that $n_{k-1} \leqslant n < n_k$ and $x \in \bigcup_{v\in\mathbb{F}_k} I(v)$. So $x_1x_2\dots x_{n_k} \in \mathbb{F}_k$. By the definition of $\mathbb{F}_k$, it is easy to check that

$$s_{k-1} \leqslant r_n(x,i) \leqslant \max_{1\leqslant j\leqslant k} s_j + n_0.$$

Then

$$\frac{s_{k-1}}{\log n_k} \leqslant \frac{r_n(x,i)}{\log n} \leqslant \frac{\max_{1\leqslant j\leqslant k} s_j + n_0}{\log n_{k-1}}.$$

By (4.2) and (4.4), it follows that

$$\lim_{n\to\infty}\frac{r_n(x,i)}{\log n}=\alpha,$$

which implies that $x\in E_\alpha(i)$. This concludes the proof. $\square$

4.2. Mass distribution. In this subsection, we distribute a Borel probability measure μ on the set F so that we can provide a lower bound for the Hausdorff dimension of F by the mass distribution principle, see Lemma 4.2.

Recall that $\mathbb{V}_N=\{10w01\colon\ w\in\Omega^{N-4}\}$ with $N\geqslant 5$. Let d be the unique positive solution of the equation

$$\sum_{u\in\mathbb{V}_N}|I(u)|^d=1.$$

It is clear that $0<d<1$. Let $\Delta_0=0$ and $\Delta_k=\sum\limits_{j=1}^{k}(t_j+2s_j+1)$ for any $k\in\mathbb{N}$.

Now, we first define $\mu(I(u))$ for any cylinder $I(u)$ with $|u|\geqslant n_0$ such that Kolmogorov's consistency condition is satisfied. Then the set function μ can be extended to a Borel probability measure on $[0,1]$.

(i) For any $v\in\Omega^{n_0}$, let $\mu(I(v))=1$ if $v\in\mathbb{F}_0$ (i.e., $v=0^{n_0}$); otherwise, let $\mu(I(v))=0$. Let $C=4^{n_0d}$. By Proposition 2.3, it is clear that

$$\mu(I(v))\leqslant 1=|I(v)|^{-d}\cdot|I(v)|^d\leqslant C|I(v)|^d=C2^{\Delta_0d}\cdot|I(v)|^d.$$

(ii) Inductively, assume that for some $k\geqslant 1$ the set function $\mu(I(\tau))$ has already been defined for every $\tau\in\Omega^{n_{k-1}}$ so that $\mu(I(\tau))\leqslant C2^{\Delta_{k-1}d}\cdot|I(\tau)|^d$. For any $v\in\Omega^{n_k}$, if $v\notin\mathbb{F}_k$, let $\mu(I(v))=0$. Otherwise, if $v\in\mathbb{F}_k$ with $v=\tau i^{s_k}w^{(1)}\dots w^{(t_k)}$, where $\tau\in\mathbb{F}_{k-1}$ and $w^{(j)}\in\mathbb{V}_N$ for all $1\leqslant j\leqslant t_k$, let

$$\mu(I(v))=\mu(I(\tau))\cdot\prod_{j=1}^{t_k}|I(w^{(j)})|^d. \tag{4.5}$$

Since $\tau\in\mathbb{F}_{k-1}\subset\Omega^{n_{k-1}}$, by Lemma 2.6 and Proposition 2.3, we have

$$\begin{aligned}\mu(I(v))&\leqslant C2^{\Delta_{k-1}d}\cdot|I(\tau)|^d\cdot\prod_{j=1}^{t_k}|I(w^{(j)})|^d\leqslant C2^{\Delta_{k-1}d}\cdot 2^{(t_k+1)d}\cdot\frac{|I(v)|^d}{|I(i^{s_k})|^d}\\&\leqslant C2^{\Delta_{k-1}d}\cdot 2^{(t_k+1)d}\cdot 4^{s_kd}\cdot|I(v)|^d=C2^{\Delta_kd}\cdot|I(v)|^d.\end{aligned} \tag{4.6}$$

(iii) For any $n > n_0$ there exists a unique integer $k \geqslant 1$ such that $n_{k-1} < n \leqslant n_k$. For any $u \in \Omega^n$, let

$$\mu(I(u)) = \sum_{\substack{v \in \Omega^{n_k} \\ v|_n = u}} \mu(I(v)).$$

We claim that for any $k \geqslant 1$ and $\tau \in \Omega^{n_{k-1}}$,

$$\mu(I(\tau)) = \sum_{\substack{v \in \Omega^{n_k} \\ v|_{n_{k-1}} = \tau}} \mu(I(v)).$$

This implies that the set function μ satisfies Kolmogorov's consistency condition and so can be uniquely extended to a Borel probability measure on $[0,1]$. In fact, if $\tau \notin \mathbb{F}_{k-1}$, then $\mu(I(\tau)) = 0$ and $\mu(I(v)) = 0$ for any $v \in \Omega^{n_k}$ with $v|_{n_{k-1}} = \tau$ since $v \notin \mathbb{F}_k$. If $\tau \in \mathbb{F}_{k-1}$, then for any $v \in \Omega^{n_k}$ with $v|_{n_{k-1}} = \tau$, we have $\mu(I(v)) \neq 0$ if and only if $v \in \mathbb{F}_k$, i.e., $v = \tau i^{s_k} w^{(1)} \dots w^{(t_k)}$ with $w^{(j)} \in \mathbb{V}_N$ for all $1 \leqslant j \leqslant t_k$. So, by (4.5) and the definition of d, it follows that

$$\begin{aligned} \sum_{\substack{v \in \Omega^{n_k} \\ v|_{n_{k-1}} = \tau}} \mu(I(v)) &= \sum_{\tau i^{s_k} w^{(1)} \dots w^{(t_k)} \in \mathbb{F}_k} \mu(I(\tau i^{s_k} w^{(1)} \dots w^{(t_k)})) \\ &= \sum_{w^{(1)} \in \mathbb{V}_N} \dots \sum_{w^{(t_k)} \in \mathbb{V}_N} \mu(I(\tau)) \cdot \prod_{j=1}^{t_k} |I(w^{(j)})|^d = \mu(I(\tau)). \end{aligned}$$

4.3. Hausdorff dimension. We first cite a well known lemma that will be used in the estimate of the Hausdorff dimension of F.

Lemma 4.2 ([2]). *Suppose that $F \subset \mathbb{R}$ is a Borel set and μ is a Borel probability measure on $\mathbb{R}$ with $\mu(F) > 0$. If for any $x \in F$,*

$$\liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \geqslant s,$$

where $B(x,r)$ denotes the ball centered at x with radius r, then $\dim_H F \geqslant s$.

Lemma 4.3. *We have $\dim_H F \geqslant (1 - N^{-1})d$.*

Proof. Let x be an arbitrary element in F. Write $\varepsilon := \varepsilon(x)$ for simplicity. For any $0 < r < |I(\varepsilon|_{n_1})|$, there exists a unique integer $k \geqslant 2$ such that

$$|I(\varepsilon|_{n_k})| \leqslant r < |I(\varepsilon|_{n_{k-1}})|.$$

By the definition of F, we have $\varepsilon|_{n_{k-1}} \in \mathbb{F}_{k-1}$ and $\varepsilon|_{n_k} \in \mathbb{F}_k$.

We split the estimate of the measure of $B(x,r)$ into two cases.

Case I: $|I(\varepsilon|_{n_{k-1}+s_k+N})| \leqslant r < |I(\varepsilon|_{n_{k-1}})|$. Since $\varepsilon|_{n_{k-1}} \in \mathbb{F}_{k-1}$ with $k \geqslant 2$, by the definition of $\mathbb{F}_{k-1}$, it follows that the word $\varepsilon|_{n_{k-1}}$ ends with a word $w^{(t_{k-1})} \in \mathbb{V}_N$, i.e., $\varepsilon|_{n_{k-1}} = \varepsilon|_{n_{k-1}-N} w^{(t_{k-1})}$. So we can write $w01 := \varepsilon|_{n_{k-1}}$ for simplicity. Note that by Lemma 2.2, the interval $I(w00)$ lies next to the left-hand side of $I(w01)$ and the interval $J := I(w02) \cup I(w10) \cap I(w11)$ lies next to the right-hand side of $I(w01)$. Then, by Lemma 2.8, it follows that

$$B(x,r) = (x-r, x+r) \subset I(w00) \cup I(\varepsilon|_{n_{k-1}}) \cup J,$$

because $x \in I(\varepsilon|_{n_{k-1}})$ and $r < |I(\varepsilon|_{n_{k-1}})|$. By the definition of $\mathbb{F}_{k-1}$, it is easy to see that $w00, w02, w10, w11 \notin \mathbb{F}_{k-1}$, which implies that $\mu(B(x,r)) \leqslant \mu(I(\varepsilon|_{n_{k-1}}))$ by the definition of μ. On the other hand, by Lemma 2.6 and Proposition 2.3, we have

$$r \geqslant |I(\varepsilon|_{n_{k-1}+s_k+N})| \geqslant 2^{-1}|I(\varepsilon|_{n_{k-1}})| \cdot 4^{-s_k-N}.$$

Hence, by (4.6),

$$\mu(B(x,r)) \leqslant \mu(I(\varepsilon|_{n_{k-1}})) \leqslant C2^{\Delta_{k-1}d} \cdot |I(\varepsilon|_{n_{k-1}})|^d \leqslant C2^{\Delta_{k-1}d} \cdot 2^{(2s_k+2N+1)d} r^d.$$

Therefore, by Proposition 2.3 again, it follows that

$$\begin{aligned} \frac{\log \mu(B(x,r))}{\log r} &\geqslant d + \frac{\log C + (\Delta_{k-1} + 2s_k + 2N + 1)d \log 2}{\log r} \\ &\geqslant d - \frac{\log C + (\Delta_{k-1} + 2s_k + 2N + 1)d \log 2}{-\log |I(\varepsilon|_{n_{k-1}})|} \\ &\geqslant d - \frac{\log C + (\Delta_{k-1} + 2s_k + 2N + 1)d \log 2}{n_{k-1} \log 2}. \end{aligned} \tag{4.7}$$

Case II: $|I(\varepsilon|_{n_{k-1}+s_k+(t+1)N})| \leqslant r < |I(\varepsilon|_{n_{k-1}+s_k+tN})|$ for some $1 \leqslant t < t_k$.

As in Case I, write $w01 := \varepsilon|_{n_{k-1}+s_k+tN}$ for simplicity. We can also obtain that

$$B(x,r) \subset I(w00) \cup I(\varepsilon|_{n_{k-1}+s_k+tN}) \cup I(w02) \cup I(w10) \cup I(w11).$$

By the definition of μ, for any $u \in \Omega^{n_{k-1}+s_k+tN}$ we have $\mu(I(u)) \neq 0$ only if there exists some $v \in \mathbb{F}_k$ such that $v|_{n_{k-1}+s_k+tN} = u$, which means that the word u must end with 01 by the definition of $\mathbb{F}_k$. This implies $\mu(B(x,r)) \leqslant \mu(I(\varepsilon|_{n_{k-1}+s_k+tN}))$. Assume that

$$\varepsilon|_{n_{k-1}+s_k+tN} = \varepsilon|_{n_{k-1}} i^{s_k} w^{(1)} \dots w^{(t)} =: u$$

with $w^{(j)} \in \mathbb{V}_N$ for all $1 \leqslant j \leqslant t$. By (4.5)–(4.6) and the definition of d,

$$\begin{aligned}\mu(I(u)) &= \sum_{\substack{v\in\mathbb{F}_k \\ v|_{n_{k-1}+s_k+tN}=u}} \mu(I(v)) = \sum_{uw^{(t+1)}\dots w^{(t_k)}\in\mathbb{F}_k} \mu(I(uw^{(t+1)}\dots w^{(t_k)})) \\ &= \sum_{w^{(t+1)}\in\mathbb{V}_N} \cdots \sum_{w^{(t_k)}\in\mathbb{V}_N} \mu(I(\varepsilon|_{n_{k-1}})) \cdot \prod_{j=1}^{t_k} |I(w^{(j)})|^d \\ &= \mu(I(\varepsilon|_{n_{k-1}})) \cdot \prod_{j=1}^{t} |I(w^{(j)})|^d \leqslant C2^{\Delta_{k-1}d} \cdot |I(\varepsilon|_{n_{k-1}})|^d \cdot \prod_{j=1}^{t} |I(w^{(j)})|^d.\end{aligned}$$

On the other hand, by Lemma 2.6 and Proposition 2.3, we have

$$r \geqslant |I(\varepsilon|_{n_{k-1}+s_k+(t+1)N})| \geqslant 2^{-(t+2)}|I(\varepsilon|_{n_{k-1}})| \cdot 4^{-s_k-N} \cdot \prod_{j=1}^{t} |I(w^{(j)})|.$$

Hence,

$$\mu(B(x,r)) \leqslant \mu(I(\varepsilon|_{n_{k-1}+s_k+tN})) = \mu(I(u)) \leqslant C2^{\Delta_{k-1}d} \cdot 2^{(t+2s_k+2N+2)d} r^d.$$

Therefore, by Proposition 2.3 again, it follows that

$$\begin{aligned}\frac{\log\mu(B(x,r))}{\log r} &\geqslant d + \frac{\log C + (\Delta_{k-1}+t+2s_k+2N+2)d\log 2}{\log r} \\ &\geqslant d - \frac{\log C + (\Delta_{k-1}+t+2s_k+2N+2)d\log 2}{-\log|I(\varepsilon|_{n_{k-1}+s_k+tN})|} \\ &\geqslant d - \frac{\log C + (\Delta_{k-1}+t+2s_k+2N+2)d\log 2}{(n_{k-1}+s_k+tN)\log 2}.\end{aligned} \tag{4.8}$$

Recall that $\Delta_k = \sum_{j=1}^{k}(t_j + 2s_j + 1)$ for any $k \in \mathbb{N}$. By (4.2)–(4.4) and the Stolz-Cesàro Theorem, we have

$$\lim_{k\to\infty} \frac{\Delta_{k-1}+2s_k}{n_{k-1}+s_k} = \lim_{k\to\infty} \frac{\Delta_{k-1}+2s_k}{n_{k-1}} = \lim_{k\to\infty} \frac{t_k+2s_{k+1}+1}{n_k-n_{k-1}} = \frac{1}{N}.$$

Letting $r \to 0$, which implies $k \to \infty$, it follows that the right hand sides of both (4.7) and (4.8) tend to $(1-N^{-1})d$. Therefore, by Lemma 4.2, we can obtain that $\dim_H F \geqslant (1-N^{-1})d$. □

Proof of Theorem 1.2. Fix $i \in \Omega = \{0,1,2\}$. For any integer $N \geqslant 5$, let $0 < d < 1$ be the unique positive solution of the equation

$$\sum_{u \in \mathbb{V}_N} |I(u)|^d = 1,$$

where $\mathbb{V}_N = \{10w01\colon\ w \in \Omega^{N-4}\}$. By Lemmas 4.1 and 4.3, we have

$$\dim_H E_\alpha(i) \geqslant \Bigl(1 - \frac{1}{N}\Bigr) d.$$

Note that by Lemma 2.6 and Proposition 2.3, we can obtain that

$$\begin{aligned} 1 &= \sum_{u \in \mathbb{V}_N} |I(u)|^d = \sum_{w \in \Omega^{N-4}} |I(10w01)|^d \geqslant \sum_{w \in \Omega^{N-4}} 4^{-5d} |I(w)|^d \\ &= \frac{1}{4^{5d}} \sum_{w \in \Omega^{N-4}} |I(w)| \cdot |I(w)|^{d-1} \geqslant \frac{1}{4^{5d}} \sum_{w \in \Omega^{N-4}} |I(w)| \cdot 2^{(1-d)(N-4)} = \frac{2^{(1-d)(N-4)}}{2^{10d}}. \end{aligned}$$

It follows that $d \to 1$ as $N \to \infty$. Therefore, $\dim_H E_\alpha(i) \geqslant 1$. ☐

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