Zhen-Hang Yang; Jing-Feng Tian

Complete monotonicity of the remainder in an asymptotic series related to the psi function

Czechoslovak Mathematical Journal, Vol. 74 (2024), No. 1, 337–351

Persistent URL: `http://dml.cz/dmlcz/152284`

Czechoslovak Mathematical Journal, 74 (149) (2024), 337–351

COMPLETE MONOTONICITY OF THE REMAINDER IN AN ASYMPTOTIC SERIES RELATED TO THE PSI FUNCTION

Zhen-Hang Yang, Hangzhou, Jing-Feng Tian, Baoding

Received August 2, 2023. Published online February 12, 2024.

Abstract. Let $p, q \in \mathbb{R}$ with $p-q \geqslant 0$, $\sigma = \frac{1}{2}(p+q-1)$ and $s = \frac{1}{2}(1-p+q)$, and let

$$\mathcal{D}_m(x;p,q) = \mathcal{D}_0(x;p,q) + \sum_{k=1}^{m} \frac{B_{2k}(s)}{2k(x+\sigma)^{2k}},$$

where

$$\mathcal{D}_0(x;p,q) = \frac{\psi(x+p)+\psi(x+q)}{2} - \ln(x+\sigma).$$

We establish the asymptotic expansion

$$\mathcal{D}_0(x;p,q) \sim -\sum_{n=1}^{\infty} \frac{B_{2n}(s)}{2n(x+\sigma)^{2n}} \quad \text{as } x \to \infty,$$

where $B_{2n}(s)$ stands for the Bernoulli polynomials. Further, we prove that the functions $(-1)^m \mathcal{D}_m(x;p,q)$ and $(-1)^{m+1}\mathcal{D}_m(x;p,q)$ are completely monotonic in x on $(-\sigma,\infty)$ for every $m \in \mathbb{N}_0$ if and only if $p-q \in [0,\frac{1}{2}]$ and $p-q=1$, respectively. This not only unifies the two known results but also yields some new results.

Keywords: psi function; asymptotic expansion; complete monotonicity

MSC 2020: 41A60, 33B15, 26A48

1. Introduction

We begin with recalling several necessary knowledge. Classical Euler's gamma function Γ and psi (digamma) function ψ are defined by [1]

$$\Gamma(z) = \int_0^{\infty} t^{z-1} \mathrm{e}^{-t} \, \mathrm{d}t \quad \text{and} \quad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad (\mathrm{Re}(z) > 0), \tag{1.1}$$

respectively.

DOI: 10.21136/CMJ.2024.0354-23

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and satisfies

$$(-1)^k f^{(k)}(x) \geqslant 0 \quad \text{for } k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \text{ and } x \in I,$$

see [10], [12]. A positive function f is called *logarithmically completely monotonic on an interval* I if f has derivatives of all orders on I and satisfies

$$(-1)^k [\ln f(x)]^{(k)} \geqslant 0 \quad \text{for } k \in \mathbb{N} \text{ and } x \in I,$$

see [3], [9]. It was pointed out in [9] that if f is logarithmically completely monotonic on I then f is completely monotonic on I, and not vice versa. The famous Bernstein Theorem (see Theorem 12b in [12]) tells us that the function $f(x)$ is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty \mathrm{e}^{-xt}\, \mathrm{d}\mu(t),$$

where $\mu(t)$ is nondecreasing and the integral converges for $0 < x < \infty$.

A function $f\colon (0, \infty) \to \mathbb{R}$ is a Bernstein function if f is of class C^∞, $f(x) \geqslant 0$ for all $x > 0$ and derivatives of all orders on I and satisfies

$$(-1)^{k-1} f^{(k)}(x) \geqslant 0 \quad \text{for all } k \in \mathbb{N} \text{ and } x > 0,$$

see Definition 3.1 in [10]. Evidently, a nonnegative C^∞-function $f\colon (0, \infty) \to \mathbb{R}$ is a Bernstein function if and only if f' is a completely monotone function.

The Bernoulli polynomials $B_k(x)$ are defined by

$$\frac{t\mathrm{e}^{xt}}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n, \quad |t| < 2\pi, \tag{1.2}$$

respectively, which satisfy the following properties, see equations (23.1.8), (23.1.6), (23.2.5), (23.1.14), (23.1.21) in [1]:

(P1) $B_n(1-x) = (-1)^n B_n(x)$;

(P2) $B_n(x+1) - B_n(x) = nx^{n-1}$, $n = 0, 1, 2, \ldots$;

(P3) $B_n'(x) = nB_{n-1}(x)$ and $n \int_a^x B_{n-1}(t)\, \mathrm{d}t = B_n(x) - B_n(a)$, $n = 1, 2, \ldots$;

(P4) $(-1)^{n+1} B_{2n+1}(x) > 0$, $x \in (0, \frac{1}{2})$, $n = 1, 2, \ldots$;

(P5) $B_n(\frac{1}{2}) = -(1 - 2^{1-n}) B_n$, $n = 0, 1, 2, \ldots$.

The motivation of this paper is caused by several known results, which suggest that the remainders in those asymptotic series involving gamma function have complete monotonicity property. The earliest result was proved in Theorem 8 of [2] by Alzer.

Theorem A. *Let*

$$f_0(x) = \ln\Gamma(x) - \Big(x - \frac{1}{2}\Big)\ln x + x - \frac{1}{2}\ln(2\pi),$$
$$f_n(x) = f_0(x) - \sum_{k=1}^{n} \frac{B_{2k}}{2k(2k-1)x^{2k-1}} \quad \text{for } n \geqslant 1.$$

Then the function $x \mapsto (-1)^n f_n(x)$ *for* $n \in \mathbb{N}_0$ *is completely monotonic on* $(0, \infty)$.

A similar result is due to Theorem 2 of [13].

Theorem B. *Let*

$$g_0(x) = \ln\Gamma\Big(x + \frac{1}{2}\Big) - x\ln x + x - \frac{1}{2}\ln(2\pi),$$
$$g_n(x) = g_0(x) + \sum_{k=1}^{n} \frac{(1-2^{1-2k})B_{2k}}{2k(2k-1)x^{2k-1}} \quad \text{for } n \geqslant 1.$$

Then the function $x \mapsto (-1)^{n-1} g_n(x)$ *for* $n \in \mathbb{N}_0$ *is completely monotonic on* $(0, \infty)$.

The third result proved in [4] by Chen and Paris asserts that the function $x \mapsto (-1)^n h_n(x)$ for $n \in \mathbb{N}$ is completely monotonic on $(0, \infty)$, where

$$h_n(x) = \ln\frac{\Gamma(x+1)}{\Gamma(x+1/2)} - \frac{1}{2}\ln x - \sum_{k=1}^{n} \frac{(1-2^{-2k})B_{2k}}{k(2k-1)x^{2k-1}}. \tag{1.3}$$

This result can also follow from Theorems A and B due to $h_n(x) = f_n(x) + [-g_n(x)]$, see Remark 7 in [13].

The fourth result was established in [15] by Yang, Tian and Ha.

Theorem C. *Let* $p, q \in \mathbb{R}$ *with* $0 < r = p - q < 1$ *and* $\sigma = \frac{1}{2}(p+q-1)$, $s = \frac{1}{2}(1-r)$. *For* $n \in \mathbb{N}$, *let* $R_n(x; p, q)$ *be defined on* $(-\sigma, \infty)$ *by*

$$R_n(x; p, q) = \frac{\ln\Gamma(x+p) - \ln\Gamma(x+q)}{p-q} - \ln(x+\sigma) - \sum_{k=1}^{n} \frac{B_{2k+1}(s)}{rk(2k+1)(x+\sigma)^{2k}}.$$

Then the function $x \mapsto (-1)^n R_n(x; p, q)$ *is completely monotonic on* $(-\sigma, \infty)$.

Similar results can be found in [5], [6], [7], and recent papers [11], [16].

Let us consider the function $F_{p,q}$, for $p, q \in \mathbb{R}$ with $r = p - q \geqslant 0$ and $\sigma = \frac{1}{2}(p+q-1)$, defined on $(-q, \infty)$, by

$$F_{p,q}(x) = \frac{\psi(x+p) + \psi(x+q)}{2} - \ln(x+\sigma). \tag{1.4}$$

The first aim of this paper is to produce the asymptotic expansion of $F_{p,q}(x)$, which is contained in the following theorem.

Theorem 1.1. *Let $p, q \in \mathbb{R}$ with $r = p - q \geqslant 0$, $\sigma = \frac{1}{2}(p+q-1)$ and $s = \frac{1}{2}(1-r)$. Then*

$$F_{p,q}(x) \sim -\sum_{n=1}^{\infty} \frac{B_{2n}(s)}{2n(x+\sigma)^{2n}} \quad \text{as } x \to \infty. \tag{1.5}$$

The second aim of this paper is to determine the best parameters p and q such that $F_{p,q}(x)$ is completely monotonic on $(-\sigma, \infty)$, which is stated as follows.

Theorem 1.2. *Let $p, q \in \mathbb{R}$ with $r = p - q \geqslant 0$, $\sigma = \frac{1}{2}(p+q-1)$ and $s = \frac{1}{2}(1-r)$. Then $F_{p,q}(x)$ defined by* (1.4) *is completely monotonic on $(-\sigma, \infty)$ if and only if $0 \leqslant r \leqslant 1/\sqrt{3}$, while $-F_{p,q}(x)$ is completely monotonic on $(-\sigma, \infty)$ if and only if $r \geqslant 1$.*

The third aim of this paper is to investigate the complete monotonicity of the remainder in the asymptotic series (1.5). Our third result reads as follows.

Theorem 1.3. *Let $p, q \in \mathbb{R}$ with $r = p - q \geqslant 0$, $\sigma = \frac{1}{2}(p+q-1)$ and $s = \frac{1}{2}(1-r)$, and let*

$$\mathcal{D}_m(x; p, q) = \begin{cases} F_{p,q}(x) + \sum\limits_{k=1}^{m} \dfrac{B_{2k}(s)}{2k(x+\sigma)^{2k}} & \text{if } m \geqslant 1, \\ F_{p,q}(x) & \text{if } m = 0. \end{cases} \tag{1.6}$$

The following statements are valid:

(i) *For every $m \in \mathbb{N}_0$, the function $x \mapsto (-1)^m \mathcal{D}_m(x; p, q)$ is completely monotonic on $(-\sigma, \infty)$ if and only if $r \in [0, \frac{1}{2}]$.*

(ii) *For every $m \in \mathbb{N}_0$, the function $x \mapsto (-1)^{m+1} \mathcal{D}_m(x; p, q)$ is completely monotonic on $(-\sigma, \infty)$ if and only if $r = 1$.*

(iii) *If $r \in [0, \frac{1}{2}]$ or $r = 1$, then for every $m \in \mathbb{N}_0$, the inequality*

$$|\mathcal{D}_m(x; p, q)| < \frac{|B_{2m+2}(s)|}{(2m+2)(x+\sigma)^{2m+2}} \quad \text{holds for } x > -\sigma. \tag{1.7}$$

2. Proof of Theorem 1.1

2.1. Watson's lemma. To prove Theorem 1.1, we need the following special case of Watson's lemma.

Lemma 2.1 ([8]). *Assume that the Laplace transform $\int_0^\infty f(t) \mathrm{e}^{-xt}\, \mathrm{d}t$ converges for all sufficiently large x, and $f(t)$ is infinitely differentiable in a neighborhood of the origin. Then*

$$\int_0^\infty f(t) \mathrm{e}^{-xt}\, \mathrm{d}t \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{x^{n+1}} \quad \text{as } x \to \infty. \tag{2.1}$$

2.2. Integral representation of $F_{p,q}(x)$.

Lemma 2.2. *Let $r = p - q \geqslant 0$ and $\sigma = \frac{1}{2}(p + q - 1)$. The function $F_{p,q}(x)$ defined by (1.4) has the integral representation*

$$F_{p,q}(x) = \int_0^\infty f(t)\mathrm{e}^{-(x+\sigma)t}\,\mathrm{d}t, \tag{2.2}$$

where

$$f(t) = \frac{1}{t} - \frac{\cosh(rt/2)}{2\sinh(t/2)}. \tag{2.3}$$

P r o o f. Using the integral representation of $\psi(z)$ (see equation (6.3.21) in [1])

$$\psi(z) = \int_0^\infty \Big(\frac{\mathrm{e}^{-t}}{t} - \frac{\mathrm{e}^{-zt}}{1-\mathrm{e}^{-t}}\Big)\,\mathrm{d}t \quad (\operatorname{Re} z > 0) \quad \text{and} \quad \ln z = \int_0^\infty \frac{\mathrm{e}^{-t} - \mathrm{e}^{-zt}}{t}\,\mathrm{d}t,$$

we have

$$\begin{aligned} F_{p,q}(t) &= \int_0^\infty \Big(\frac{\mathrm{e}^{-t}}{t} - \frac{1}{2}\frac{\mathrm{e}^{-pt}+\mathrm{e}^{-qt}}{1-\mathrm{e}^{-t}}\mathrm{e}^{-xt}\Big)\,\mathrm{d}t - \int_0^\infty \Big(\frac{\mathrm{e}^{-t}}{t} - \frac{\mathrm{e}^{-(x+\sigma)t}}{t}\Big)\,\mathrm{d}t \\ &= \int_0^\infty \Big(\frac{\mathrm{e}^{-(x+\sigma)t}}{t} - \frac{1}{2}\frac{\mathrm{e}^{-pt}+\mathrm{e}^{-qt}}{1-\mathrm{e}^{-t}}\mathrm{e}^{-xt}\Big)\,\mathrm{d}t = \int_0^\infty f(t)\mathrm{e}^{-(x+\sigma)t}\,\mathrm{d}t, \end{aligned}$$

where

$$f(t) = \frac{1}{t} - \frac{1}{2}\frac{\mathrm{e}^{-pt}+\mathrm{e}^{-qt}}{1-\mathrm{e}^{-t}}\mathrm{e}^{\sigma t}. \tag{2.4}$$

A simplification yields

$$f(t) = \frac{1}{t} - \frac{\cosh(rt/2)}{2\sinh(t/2)},$$

which implies (2.2), thereby completing the proof. □

2.3. Maclaurin series representation of $f(t)$.

Lemma 2.3. *Let $r \geqslant 0$ and $s = \frac{1}{2}(1 - r)$. The function $f(t)$ defined by (2.3) has the Maclaurin series representation*

$$f(t) = -\sum_{n=1}^{\infty} \frac{B_{2n}(s)}{(2n)!}t^{2n-1} \quad \text{for } |t| < 2\pi, \tag{2.5}$$

or equivalently,

$$\phi_r(t) := \frac{t\cosh(rt)}{\sinh t} = \sum_{n=0}^{\infty} \frac{2^{2n}B_{2n}(s)}{(2n)!}t^{2n} \quad \text{for } |t| < \pi. \tag{2.6}$$

P r o o f. Note that

$$1-p+\sigma=1-p+\frac{p+q-1}{2}=\frac{1-r}{2}=s,$$
$$1-q+\sigma=1-q+\frac{p+q-1}{2}=\frac{1+r}{2}=1-s.$$

From an arrangement of (2.4) we have that

$$f(t)=\frac{1}{t}-\frac{1}{2}\frac{\mathrm{e}^{-pt}+\mathrm{e}^{-qt}}{1-\mathrm{e}^{-t}}\mathrm{e}^{\sigma t}=\frac{1}{t}-\frac{1}{2t}\Big(\frac{t\mathrm{e}^{st}}{\mathrm{e}^{t}-1}+\frac{t\mathrm{e}^{(1-s)t}}{\mathrm{e}^{t}-1}\Big).$$

Using the definition of Bernoulli polynomial (1.2) and noting that $B_0(x)=1$ give

$$f(t)=\frac{1}{t}-\frac{1}{2t}\sum_{k=0}^{\infty}\frac{B_k(s)+B_k(1-s)}{k!}t^k=-\frac{1}{2}\sum_{k=0}^{\infty}\frac{B_{k+1}(s)+B_{k+1}(1-s)}{(k+1)!}t^k.$$

By (P1), that is, $B_n(1-x)=(-1)^nB_n(x)$, we see that

$$B_{k+1}(s)+B_{k+1}(1-s)=\begin{cases}0 & \text{if } k=2n,\\ 2B_{2n+2}(s) & \text{if } k=2n+1\end{cases}$$

which leads to

$$f(t)=-\sum_{n=0}^{\infty}\frac{B_{2n+2}(s)}{(2n+2)!}t^{2n+1}.$$

That is,

$$\frac{1}{t}-\frac{\cosh(rt/2)}{2\sinh(t/2)}=-\sum_{n=1}^{\infty}\frac{B_{2n}(s)}{(2n)!}t^{2n-1},$$

which, by arranging and replacing $\frac{1}{2}t$ by t, gives (2.6).

This completes the proof. □

2.4. Proof of Theorem 1.1. We are now in a position to prove Theorem 1.1.

P r o o f. By Lemma 2.3 we see that $f^{(2n)}(0)=0$ and

$$f^{(2n+1)}(0)=-(2n+1)!\,\frac{B_{2n+2}(s)}{(2n+2)!}=-\frac{B_{2n+2}(s)}{2n+2}.$$

It follows from Lemma 2.1 that

$$F_{p,q}(x)=\int_0^{\infty}f(t)\mathrm{e}^{-(x+\sigma)t}\,\mathrm{d}t\sim\sum_{n=0}^{\infty}\frac{f^{(2n+1)}(0)}{(x+\sigma)^{2n+2}}$$
$$=-\sum_{n=0}^{\infty}\frac{B_{2n+2}(s)}{(2n+2)(x+\sigma)^{2n+2}}\quad\text{as } x\to\infty,$$

which implies (1.5), and the proof is done. □

3. Proofs of Theorems 1.2 and 1.3

3.1. Complete monotonicity of the functions $\varphi_r(t)=\phi_r(\frac{1}{2}\sqrt{t})$ and $\varphi_r'(t)$.

Lemma 3.1. *For $r\geqslant 0$, let $\phi_r(t)$ be defined in (2.6). Then $t\mapsto\varphi_r(t)=\phi_r(\frac{1}{2}\sqrt{t})$ is completely monotonic on $(0,\infty)$ if and only if $r\in[0,\frac{1}{2}]$.*

Proof. *Necessity.* If $\varphi_r(t)$ is completely monotonic on $(0,\infty)$, then we have

$$\lim_{t\to 0}[(-1)^n\varphi_r^{(n)}(t)]\geqslant 0.$$

From (2.6) we see that $\varphi_r(t)$ has the Maclaurin series representation

$$\varphi_r(t)=\phi_r\Bigl(\frac{\sqrt{t}}{2}\Bigr)=\frac{\sqrt{t}\cosh(r\sqrt{t}/2)}{2\sinh(\sqrt{t}/2)}=\sum_{k=0}^{\infty}\frac{B_{2k}(s)}{(2k)!}t^k \tag{3.1}$$

for $0<t<4\pi^2$. Then

$$\lim_{t\to 0}[(-1)^n\varphi_r^{(n)}(t)]=\frac{n!}{(2n)!}(-1)^nB_{2n}(s)\geqslant 0$$

for every $n\in\mathbb{N}_0$. Obviously, $s=\frac{1}{2}(1-r)\leqslant\frac{1}{2}$ due to $r\geqslant 0$. From the inequality

$$B_2(s)=s^2-s+\frac{1}{6}\leqslant 0$$

we find that $s\in[\frac{1}{6}(3-\sqrt{3}),\frac{1}{2}]\subset[0,1]$. Further, from the inequality $B_{4n}(s)\geqslant 0$ we can find that $s\in[\frac{1}{4},\frac{1}{2}]$. It was listed in equation (23.1.18) of [1] that

$$B_{2n}(x)=\frac{(-1)^{n-1}2(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{\cos(2k\pi x)}{k^{2n}}\quad\text{for } n\in\mathbb{N}\text{ and } x\in[0,1].$$

Then

$$B_{4n}(s)=-\frac{2(4n)!}{(2\pi)^{4n}}\sum_{k=1}^{\infty}\frac{\cos(2k\pi s)}{k^{4n}}\quad\text{for all } n\in\mathbb{N}, \tag{3.2}$$

where $s\in[\frac{1}{6}(3-\sqrt{3}),\frac{1}{2}]\subset[0,1]$. Since the series $\sum_{k=1}^{\infty}k^{-4n}\cos(2k\pi s)$ is uniformly convergent, we have

$$\lim_{n\to\infty}\Bigl[\frac{(2\pi)^{4n}}{2(4n)!}B_{4n}(s)\Bigr]=-\lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{\cos(2k\pi s)}{k^{4n}}=-\cos(2\pi s)\geqslant 0,$$

which means that $s\in[\frac{1}{4},\frac{3}{4}]$. This, due to $s=\frac{1}{2}(1-r)$ with $r\geqslant 0$, shows that $r\in[0,\frac{1}{2}]$, which proves the necessity.

Sufficiency. Assume that $r \in [0, \frac{1}{2}]$. It was listed in equations (4.5.68), (4.5.69) of [1] that

$$\frac{\sinh z}{z} = \prod_{n=1}^{\infty}\Bigl(1 + \frac{z^2}{n^2\pi^2}\Bigr) \quad \text{and} \quad \cosh z = \prod_{n=1}^{\infty}\Bigl(1 + \frac{z^2}{(n-1/2)^2\pi^2}\Bigr)$$

for $z \in \mathbb{C}$. Logarithmic differentiation yields

$$\Bigl[\ln \frac{\sinh(\sqrt{t}/2)}{\sqrt{t}/2}\Bigr]' = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=1}^{\infty}\ln\Bigl(1 + \frac{t}{4n^2\pi^2}\Bigr) = \sum_{n=1}^{\infty}\frac{1}{4n^2\pi^2 + t}, \tag{3.3}$$

$$\Bigl[\ln\cosh\Bigl(\frac{r\sqrt{t}}{2}\Bigr)\Bigr]' = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=1}^{\infty}\ln\Bigl(1 + \frac{r^2 t}{4(n-1/2)^2\pi^2}\Bigr) = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2\pi^2/r^2 + t}.$$

We thus obtain that

$$\begin{aligned} -[\ln\varphi_r(t)]' &= -\frac{\mathrm{d}}{\mathrm{d}t}\Bigl[\ln\cosh\Bigl(\frac{r\sqrt{t}}{2}\Bigr) - \ln\frac{\sinh(\sqrt{t}/2)}{\sqrt{t}/2}\Bigr] \\ &= \sum_{n=1}^{\infty}\frac{1}{t + 4n^2\pi^2} - \sum_{n=1}^{\infty}\frac{1}{t + (2n-1)^2\pi^2/r^2} \\ &= \frac{\pi^2}{r^2}\sum_{n=1}^{\infty}\frac{(2nr + 2n - 1)((1-2r)n + n - 1)}{(t + 4n^2\pi^2)(t + (2n-1)^2\pi^2/r^2)}. \end{aligned}$$

Clearly, $(2nr + 2n - 1)((1-2r)n + n - 1) \geqslant 0$ for $r \in [0, \frac{1}{2}]$ and $n \in \mathbb{N}$. Since $t \mapsto (t + c)^{-1}$ $(c > 0)$ is completely monotonic on $(0, \infty)$, so is $t \mapsto -[\ln\varphi_r(t)]'$ on $(0, \infty)$. That is, $t \mapsto \phi_r(\frac{1}{2}\sqrt{t})$ is logarithmically completely monotonic on $(0, \infty)$.

As shown in [9], a (strictly) logarithmically completely monotonic function is also (strictly) completely monotonic. Therefore, the function $t \mapsto \phi_r(\frac{1}{2}\sqrt{t})$ is completely monotonic on $(0, \infty)$, which proves the sufficiency, and the proof is done. □

Lemma 3.2. *For* $r \geqslant 0$, *let* $\phi_r(t)$ *be defined in* (2.6). *Then* $t \mapsto \varphi_r'(t) = \mathrm{d}\phi_r(\frac{1}{2}\sqrt{t})/\mathrm{d}t$ *is completely monotonic on* $(0, \infty)$ *if and only if* $r = 1$.

P r o o f. *Necessity.* If $t \mapsto \varphi_r'(t)$ is completely monotonic on $(0, \infty)$ then $\varphi_r''(t) \leqslant 0$ for all $t > 0$. Since

$$\varphi_r(t) = \frac{\sqrt{t}\cosh(r\sqrt{t}/2)}{2\sinh(\sqrt{t}/2)} \sim \frac{1}{2}\sqrt{t}\mathrm{e}^{(r-1)\sqrt{t}/2} \quad \text{as } t \to \infty,$$

we have

$$\varphi_r''(t) \sim \Bigl[\frac{1}{2}\sqrt{t}\mathrm{e}^{(r-1)\sqrt{t}/2}\Bigr]'' = \frac{1}{32\sqrt{t}}\mathrm{e}^{(r-1)\sqrt{t}/2}\Bigl[(r-1)^2 + \frac{2(r-1)}{\sqrt{t}} - \frac{4}{t}\Bigr] \quad \text{as } t \to \infty,$$

which implies that

$$\lim_{t\to\infty}[\sqrt{t}\mathrm{e}^{(1-r)\sqrt{t}/2}\varphi_r''(t)] = \frac{1}{32}(r-1)^2.$$

This together with $\varphi_r''(t) \leqslant 0$ valid for all $t > 0$ means that $(r-1)^2 \leqslant 0$, and therefore, $r = 1$, which proves the necessity.

Sufficiency. Suppose that $r = 1$. We prove that $t \mapsto \varphi_r'(t)$ is completely monotonic on $(0, \infty)$. It follows from (3.3) that

$$\varphi_1(t) = \frac{\sqrt{t}\cosh(\sqrt{t}/2)}{2\sinh(\sqrt{t}/2)} = 1 + \sum_{n=1}^{\infty} \frac{2t}{4n^2\pi^2 + t},$$

$$\varphi_1'(t) = 8\pi^2 \sum_{n=1}^{\infty} \frac{n^2}{(4\pi^2 n^2 + t)^2},$$

and therefore, $t \mapsto \varphi_1'(t) = \mathrm{d}\phi_1(\frac{1}{2}\sqrt{t})/\mathrm{d}t$ is completely monotonic on $(0, \infty)$, which proves the sufficiency, thereby completing the proof. □

Remark 3.1. It is seen from Lemma 3.2 that the function $\varphi_r(t)$ given in (3.1) is a Bernstein function on $(0, \infty)$ if and only if $r = 1$. In other words, the function $\varphi_1(t) = \sqrt{t}\coth\sqrt{t}$ is a Bernstein function on $(0, \infty)$.

3.2. Inequalities for hyperbolic functions.

Lemma 3.3. *For $m \in \mathbb{N}_0$, $r \geqslant 0$ and $s = \frac{1}{2}(1-r)$, let*

$$I_m(t) = \frac{t\cosh(rt/2)}{2\sinh(t/2)} - \sum_{k=0}^{m} \frac{B_{2k}(s)}{(2k)!} t^{2k}. \tag{3.4}$$

Then $I_0(t) < (>) 0$ for all $t > 0$ if and only if $r \in [0, 1/\sqrt{3}]$ $(r \geqslant 1)$.

P r o o f. We have

$$I_0(t) = \frac{t\cosh(rt/2)}{2\sinh(t/2)} - 1 = \frac{(t/2)\cosh(rt/2) - \sinh(t/2)}{\sinh(t/2)} = \frac{u(t/2)}{\sinh(t/2)},$$

where $u(t) = t\cosh(rt) - \sinh t$.

(i) If $I_0(t) < 0$ for all $t > 0$, then

$$\lim_{t\to 0} \frac{u(t)}{t^3} = \frac{1}{2}\Big(r^2 - \frac{1}{3}\Big) \leqslant 0,$$

which implies that $r \in [0, 1/\sqrt{3}]$.

Suppose that $r \in [0, 1/\sqrt{3}]$. Expanding in power series and using an elementary inequality that $3^n - (2n+1) \geqslant 0$ for $n \in \mathbb{N}_0$ yield

$$u(t) = \sum_{n=0}^{\infty} \frac{(2n+1)r^{2n} - 1}{(2n+1)!} t^{2n+1} \leqslant -\sum_{n=0}^{\infty} \frac{3^n - (2n+1)}{3^n(2n+1)!} t^{2n+1} < 0$$

for $t > 0$.

(ii) If $I_0(t) > 0$ for all $t > 0$, then $\lim_{t\to\infty} I_0(t) \geqslant 0$. Suppose that $0 \leqslant r < 1$. Then $\lim_{t\to\infty} I_0(t) = -1$, which yields a contradiction. Hence, $r \geqslant 1$. If $r \geqslant 1$ then

$$u(t) = \sum_{n=0}^{\infty} \frac{(2n+1)r^{2n} - 1}{(2n+1)!} t^{2n+1} \geqslant \sum_{n=0}^{\infty} \frac{2n}{(2n+1)!} t^{2n+1} > 0$$

for $t > 0$. This completes the proof. □

Remark 3.2. Lemma 3.3 implies that, for $\alpha, \beta \geqslant 0$, the double inequality

$$\cosh(\beta t) < \frac{\sinh t}{t} < \cosh(\alpha t)$$

for $t > 0$ if and only if $\alpha \geqslant 1$ and $\beta \leqslant 1/\sqrt{3}$. A stronger double inequality can be seen in Theorem 1 of [14].

Using Lemmas 3.1 and 3.2, we can easily prove Lemmas 3.4 and 3.5, which are crucial to prove Theorem 1.3.

Lemma 3.4. *Let $s = \frac{1}{2}(1-r)$ with $r \geqslant 0$ and let $I_m(t)$ be defined by (3.4). The inequality $(-1)^{m+1} I_m(t) > 0$ for all $t > 0$ and every $m \in \mathbb{N}_0$ holds if and only if $r \in [0, \frac{1}{2}]$.*

Proof. *Sufficiency.* Suppose that $r \in [0, \frac{1}{2}]$. By the Maclaurin series representation (3.1) we see that

$$\varphi_r(t) = \phi_r\left(\frac{\sqrt{t}}{2}\right) = \frac{\sqrt{t}\cosh(r\sqrt{t}/2)}{2\sinh(\sqrt{t}/2)} = \sum_{k=0}^{\infty} \frac{\varphi_r^{(k)}(0)}{k!} t^k \tag{3.5}$$

for $0 < t < 4\pi^2$, where

$$\frac{\varphi_r^{(k)}(0)}{k!} = \frac{B_{2k}(s)}{(2k)!} \quad \text{for } k \in \mathbb{N}_0.$$

Now using the Lagrange remainder formula we have

$$I_m(\sqrt{t}) = \varphi_r(t) - \sum_{k=0}^{m} \frac{\varphi_r^{(k)}(0)}{k!} t^k = \int_0^t \varphi_r^{(m+1)}(x) \frac{(t-x)^m}{m!}\,\mathrm{d}x, \tag{3.6}$$

and then using Lemma 3.1 gives

$$(-1)^{m+1} I_m(\sqrt{t}) = \int_0^t (-1)^{m+1} \varphi_r^{(m+1)}(x) \frac{(t-x)^m}{m!}\,\mathrm{d}x > 0$$

for $t > 0$. Hence, $(-1)^{m+1} I_m(\sqrt{t}) > 0$ for $t > 0$, and therefore, $(-1)^{m+1} I_m(t) > 0$ for $t > 0$, which proves the sufficiency.

Necessity. Assume that the inequality $(-1)^{m+1}I_m(t) > 0$ holds for all $t > 0$ and every $m \in \mathbb{N}_0$. We prove that $r \in [0, \frac{1}{2}]$. Firstly, by Lemma 3.3, $(-1)^{m+1}I_m(t) > 0$ for $t > 0$ and $m = 0$ implies that $r \in [0, 1/\sqrt{3}]$, or equivalently, $s = \frac{1}{2}(1-r) \in [\frac{1}{6}(3-\sqrt{3}), \frac{1}{2}] \subset [0,1]$. Secondly, $(-1)^{m+1}I_m(t) > 0$ for $t > 0$ and $m = 2n-1$ implies that

$$\lim_{t\to 0} \frac{I_{2n-1}(\sqrt{t})}{t^{2n}} = \frac{\varphi_r^{(2n)}(0)}{(2n)!} = \frac{B_{4n}(s)}{(4n)!} \geqslant 0$$

for all $n \in \mathbb{N}$ and $s \in [\frac{1}{6}(3-\sqrt{3}), \frac{1}{2}]$. From the proof of Lemma 3.1 we see that this means that $s \in [\frac{1}{4}, \frac{3}{4}]$, which, due to $s = \frac{1}{2}(1-r)$ with $r \geqslant 0$, indicates that $r \in [0, \frac{1}{2}]$, which proves the necessity, thereby completing the proof. □

Lemma 3.5. *Let $s = \frac{1}{2}(1-r)$ with $r \geqslant 0$ and let $I_m(t)$ be defined by* (3.4). *The inequality $(-1)^m I_m(t) > 0$ for all $t > 0$ and every $m \in \mathbb{N}_0$ holds if and only if $r = 1$.*

P r o o f. *Sufficiency.* Suppose that $r = 1$. In Lemma 3.2 we have shown that $t \mapsto \varphi_r'(t) = \mathrm{d}\phi_r(\frac{1}{2}\sqrt{t})/\mathrm{d}t$ is completely monotonic on $(0,\infty)$ if and only if $r = 1$. Then $(-1)^m\varphi_1^{(m+1)}(t) > 0$ for $t > 0$ and $m \in \mathbb{N}_0$. Using the Lagrange remainder formula (3.6) we conclude that

$$(-1)^m I_m(\sqrt{t}) = \int_0^t (-1)^m \varphi_1^{(m+1)}(x) \frac{(t-x)^m}{m!}\,\mathrm{d}x > 0$$

for $t > 0$ and $m \in \mathbb{N}_0$, and so $(-1)^m I_m(t) > 0$ for $t > 0$ and $m \in \mathbb{N}_0$, which proves the sufficiency.

Necessity. Assume that the inequality $(-1)^m I_m(t) > 0$ holds for all $t > 0$ and every $m \in \mathbb{N}_0$. We prove that $r = 1$. Firstly, by Lemma 3.3, $(-1)^m I_m(t) > 0$ for $t > 0$ and $m = 0$ implies that $r \geqslant 1$. Secondly, $(-1)^m I_m(t) > 0$ for $t > 0$ and $m = 1$ implies that

$$I_1(t) = \frac{t\cosh(rt/2)}{2\sinh(t/2)} - 1 - \frac{1}{2}t^2\Big(s^2 - s + \frac{1}{6}\Big) \leqslant 0$$

for all $t > 0$. Obviously, if $r > 1$ then $\lim_{t\to\infty} I_1(t) = \infty$, which yields a contradiction. Therefore, $r = 1$, which proves the necessity, thereby completing the proof. □

3.3. Proof of Theorem 1.2. With the aid of Lemma 3.3 we can prove Theorem 1.2.

P r o o f. Using the integral representation (2.2) we have

$$F_{p,q}(x) = \int_0^\infty \Big[\frac{1}{t} - \frac{\cosh(rt/2)}{2\sinh(t/2)}\Big]\mathrm{e}^{-(x+\sigma)t}\,\mathrm{d}t = -\int_0^\infty I_0(t)\frac{\mathrm{e}^{-(x+\sigma)t}}{t}\,\mathrm{d}t.$$

By the Bernstein Theorem, $\pm F_{p,q}(x)$ is completely monotonic on $(-\sigma,\infty)$ if and only if $I_0(t) \leqslant (\geqslant) 0$ for all $t > 0$. As proved in Lemma 3.3, $I_0(t) < (>) 0$ for all $t > 0$ if and only if $r \in [0, 1/\sqrt{3}]$ $(r \geqslant 1)$. The required complete monotonicity follows. □

3.4. Proof of Theorem 1.3. Finally, we use Lemmas 3.4 and 3.5 to prove Theorem 1.3.

P r o o f. Using the integral representation (2.2) and

$$\frac{1}{x^n} = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}\mathrm{e}^{-xt}\,\mathrm{d}t,$$

we immediately get

$$\begin{aligned}
\mathcal{D}_m(x;p,q) &= \int_0^\infty f(t)\mathrm{e}^{-(x+\sigma)t}\,\mathrm{d}t + \sum_{k=1}^m \frac{B_{2k}(s)}{2k}\frac{1}{(2k-1)!}\int_0^\infty t^{2k-1}\mathrm{e}^{-(x+\sigma)t}\,\mathrm{d}t\\
&= \int_0^\infty \biggl[\frac{1}{t} - \frac{\cosh(rt/2)}{2\sinh(t/2)} + \sum_{k=1}^m \frac{B_{2k}(s)}{(2k)!}t^{2k-1}\biggr]\mathrm{e}^{-(x+\sigma)t}\,\mathrm{d}t\\
&= -\int_0^\infty \biggl[\frac{t\cosh(rt/2)}{2\sinh(t/2)} - \sum_{k=0}^m \frac{B_{2k}(s)}{(2k)!}t^{2k}\biggr]\frac{\mathrm{e}^{-(x+\sigma)t}}{t}\,\mathrm{d}t\\
&= -\int_0^\infty I_m(t)\frac{\mathrm{e}^{-(x+\sigma t}}{t}\,\mathrm{d}t.
\end{aligned}$$

Then

$$\begin{aligned}
(-1)^m\mathcal{D}_m(x;p,q) &= \int_0^\infty (-1)^{m+1}I_m(t)\frac{\mathrm{e}^{-(x+\sigma)t}}{t}\,\mathrm{d}t,\\
(-1)^{m+1}\mathcal{D}_m(x;p,q) &= \int_0^\infty (-1)^{m}I_m(t)\frac{\mathrm{e}^{-(x+\sigma)t}}{t}\,\mathrm{d}t.
\end{aligned}$$

Using the Bernstein Theorem, Lemmas 3.4 and 3.5, the first and second statements follow immediately.

Finally, we prove the inequality (1.7). In the case of $r \in [0, \frac{1}{2}]$, if m is even, then from the inequalities $\mathcal{D}_m(x;p,q) > 0$, $\mathcal{D}_{m+1}(x;p,q) < 0$ for $x > -\sigma$ and $r \in [0, \frac{1}{2}]$, and the relation

$$\mathcal{D}_{m+1}(x;p,q) = \mathcal{D}_m(x;p,q) + \frac{B_{2m+2}(s)}{(2m+2)(x+\sigma)^{2m+2}},$$

we have

$$0 < \mathcal{D}_m(x;p,q) < -\frac{B_{2m+2}(s)}{(2m+2)(x+\sigma)^{2m+2}} \quad \text{for } x > -\sigma; \tag{3.7}$$

if m is odd, then from the inequalities $\mathcal{D}_m(x;p,q) < 0$, $\mathcal{D}_{m+1}(x;p,q) > 0$ we have

$$-\frac{B_{2m+2}(s)}{(2m+2)(x+\sigma)^{2m+2}} < \mathcal{D}_m(x;p,q) < 0 \quad \text{for } x > -\sigma. \tag{3.8}$$

Inequalities (3.7) and (3.8) imply (1.7). In a similar way, the inequality (1.7) also holds for $t > 0$ and $m \in \mathbb{N}_0$ when $r = 1$. The limit relation

$$\lim_{x\to\infty}[(x+\sigma)^{2m+2}|\mathcal{D}_m(x;p,q)|] = \frac{|B_{2m+2}(s)|}{2m+2}$$

means that the upper bound given in (1.7) is sharp, thereby completing the proof. □

4. Concluding remarks

In this paper, we established the asymptotic expansion

$$\begin{aligned} F_{p,q}(x) &= \frac{\psi(x+p)+\psi(x+q)}{2} - \ln(x+\sigma) \\ &\sim -\sum_{k=1}^{m} \frac{B_{2k}(s)}{(2k)(x+\sigma)^{2k}} + \mathcal{D}_m(x;p,q) \quad \text{as } x \to \infty, \end{aligned}$$

where $\sigma = \frac{1}{2}(p+q-1)$, $s = \frac{1}{2}(1-r)$ with $r = p-q \geqslant 0$. Further, we proved that the functions $x \mapsto (-1)^m \mathcal{D}_m(x;p,q)$ and $x \mapsto (-1)^{m+1}\mathcal{D}_m(x;p,q)$ for every $m \in \mathbb{N}_0$ are completely monotonic on $(-\sigma,\infty)$ if and only if $r \in [0,\frac{1}{2}]$ and $r = 1$, respectively. This yields a sharp error bound when $r \in [0,\frac{1}{2}]$ or $r = 1$, that is,

$$|\mathcal{D}_m(x;p,q)| < \frac{|B_{2m+2}(s)|}{(2m+2)(x+\sigma)^{2m+2}}$$

for $x > -\sigma$. It should be noted that the proofs of our main results are difficult and mainly depend on Lemmas 3.1 and 3.2 which are very challenging.

Finally, taking certain special values $r \in [0,\frac{1}{2}]$ and $r = 1$, we would obtain several interesting consequences.

Case 1: $(p,q) = (1,0)$. Then $r = 1$, $\sigma = s = 0$, and then

$$\mathcal{D}_m(x;1,0) = \psi(x) - \ln x + \frac{1}{2x} + \sum_{k=1}^{m} \frac{B_{2k}}{2kx^{2k}}. \tag{4.1}$$

According to our Theorem 1.3 we have:

Corollary 4.1. *The function* $x \mapsto (-1)^{m+1}\mathcal{D}_m(x;1,0)$ *is completely monotonic on* $(0,\infty)$ *for* $m \in \mathbb{N}_0$.

Remark 4.1. Let us return to Theorem A, we easily find that $f_n'(x) = \mathcal{D}_n(x;1,0)$. Since $\lim_{x\to\infty} f_n(x) = 0$, we conclude that $x \mapsto (-1)^n f_n(x)$ is completely monotonic on $(0,\infty)$. This gives a concise proof of Alzer's result in Theorem 8 of [2]. In other words, Theorem A and Corollary 4.1 are equivalent.

Case 2: $(p,q)=(\frac{1}{2},\frac{1}{2})$. Then $r=\sigma=0$, $s=\frac{1}{2}$, and then

$$\mathcal{D}_m\Big(x;\frac{1}{2},\frac{1}{2}\Big)=\psi\Big(x+\frac{1}{2}\Big)-\ln x+\sum_{k=1}^{m}\frac{B_{2k}(1/2)}{2kx^{2k}}.$$

By Theorem 1.3, we have:

Corollary 4.2. *The function* $x\mapsto(-1)^m\mathcal{D}_m(x;\frac{1}{2},\frac{1}{2})$ *is completely monotonic on* $(0,\infty)$ *for* $m\in\mathbb{N}_0$.

Remark 4.2. Let us return to Theorem B. Differentiation gives

$$g_n'(x)=\psi\Big(x+\frac{1}{2}\Big)-\ln x-\sum_{k=1}^{n}\frac{(1-2^{1-2k})B_{2k}}{2kx^{2k}}.$$

In view of (P5), that is, $B_n(\frac{1}{2})=-(1-2^{1-n})B_n$, we see that $g_n'(x)=\mathcal{D}_n(x;\frac{1}{2},\frac{1}{2})$, and then, $-[(-1)^{n-1}g_n(x)]'=(-1)^n\mathcal{D}_n(x;\frac{1}{2},\frac{1}{2})$. Since $\lim\limits_{x\to\infty}g_n(x)=0$, we find that $x\mapsto(-1)^{n-1}g_n(x)$ is completely monotonic on $(0,\infty)$ for $n\in\mathbb{N}_0$. This gives a new proof Yang's result in Theorem 2 of [13]. In other words, Theorem B and Corollary 4.2 are equivalent.

Case 3: $(p,q)=(1,\frac{1}{2})$. Then $r=\frac{1}{2}$, $\sigma=s=\frac{1}{4}$, and

$$\mathcal{D}_m\Big(x;1,\frac{1}{2}\Big)=\frac{1}{2}\Big[\psi(x+1)+\psi\Big(x+\frac{1}{2}\Big)\Big]-\ln\Big(x+\frac{1}{4}\Big)+\sum_{k=1}^{m}\frac{B_{2k}(1/4)}{2k(x+1/4)^{2k}}.$$

By Theorem 1.3, we have:

Corollary 4.3. *The function* $x\mapsto(-1)^m\mathcal{D}_m(x;1,\frac{1}{2})$ *is completely monotonic on* $(-\frac{1}{4},\infty)$ *for* $m\in\mathbb{N}_0$.

Finally, a problem arises from Theorem 1.3.

Problem 4.1. If $m\in\mathbb{N}$ instead of $m\in\mathbb{N}_0$ in Theorem 1.3, then what are the conditions for which the functions $(-1)^m\mathcal{D}_m(x;p,q)$ and $(-1)^{m+1}\mathcal{D}_m(x;p,q)$ are completely monotonic with respect to x on $(-\sigma,\infty)$?

References

[1] *M. Abramowitz, I. A. Stegun (eds.)*: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. National Bureau of Standards, Applied Mathematics Series 55. John Wiley, New York, 1972. zbl MR

[2] *H. Alzer*: On some inequalities for the gamma and psi functions. Math. Comput. *66* (1997), 373–389. zbl MR doi

[3] *R. D. Atanassov, U. V. Tsoukrovski*: Some properties of a class of logarithmically completely monotonic functions. C. R. Acad. Bulg. Sci. *41* (1988), 21–23. zbl MR

[4] *C.-P. Chen, R. B. Paris*: Inequalities, asymptotic expansions and completely monotonic functions related to the gamma function. Appl. Math. Comput. *250* (2015), 514–529. zbl MR doi

[5] *J. L. Fields*: The uniform asymptotic expansion of a ratio of Gamma functions. Constructive Theory of Functions. Publishing House of the Bulgarian Academy of Sciences, Sofia, 1970, pp. 171–176. zbl MR

[6] *C. L. Frenzen*: Error bounds for asymptotic expansions of the ratio of two gamma functions. SIAM J. Math. Anal. *18* (1987), 890–896. zbl MR doi

[7] *Y. L. Luke*: On the ratio of two gamma functions. Jñānābha *9-10* (1980), 143–148. zbl MR

[8] *F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark (eds.)*: NIST Handbook of Mathematical Functions. Cambridge University Press, Cambridge, 2010. zbl MR doi

[9] *F. Qi, C.-P. Chen*: A complete monotonicity property of the gamma function. J. Math. Anal. Appl. *296* (2004), 603–607. zbl MR doi

[10] *R. L. Schilling, R. Song, Z. Vondraček*: Bernstein functions: Theory and Applications. de Gruyter Studies in Mathematics 37. Walter de Gruyter, Berlin, 2010. zbl MR doi

[11] *J.-F. Tian, Z. Yang*: Asymptotic expansions of Gurland's ratio and sharp bounds for their remainders. J. Math. Anal. Appl. *493* (2021), Article ID 124545, 19 pages. zbl MR doi

[12] *D. V. Widder*: The Laplace Transform. Princeton Mathematical Series 6. Princeton University Press, Princeton, 1941. zbl MR

[13] *Z.-H. Yang*: Approximations for certain hyperbolic functions by partial sums of their Taylor series and completely monotonic functions related to gamma function. J. Math. Anal. Appl. *441* (2016), 549–564. zbl MR doi

[14] *Z.-H. Yang, Y.-M. Chu*: Jordan type inequalities for hyperbolic functions and their applications. J. Funct. Spaces *2015* (2015), Article ID 370979, 4 pages. zbl MR doi

[15] *Z.-H. Yang, J.-F. Tian, M.-H. Ha*: A new asymptotic expansion of a ratio of two gamma functions and complete monotonicity for its remainder. Proc. Am. Math. Soc. *148* (2020), 2163–2178. zbl MR doi

[16] *Z. Yang, J.-F. Tian*: Complete monotonicity of the remainder of the asymptotic series for the ratio of two gamma functions. J. Math. Anal. Appl. *517* (2023), Article ID 126649, 15 pages. zbl MR doi

Authors' addresses: Zhen-Hang Yang, State Grid Zhejiang Electric Power Company Research Institute, Hangzhou, Zhejiang 310014, P. R. China, e-mail: `yzhkm@163.com`; Jing-Feng Tian (corresponding author), Hebei Key Laboratory of Physics and Energy Technology, Department of Mathematics and Physics, North China Electric Power University, Baoding, Hebei 071003, P. R. China, e-mail: `tianjf@ncepu.edu.cn`.