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TWO-POINT OSCILLATORY SOLUTIONS TO SYSTEM WITH RELAY HYSTERESIS AND NONPERIODIC EXTERNAL DISTURBANCE

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Abstract. We study an n-dimensional system of ordinary differential equations with a constant matrix, a relay-type nonlinearity, and an external disturbance in the right-hand side. We consider a nonideal relay characteristic. The external disturbance is described by the product of an exponential function and a sine function with an initial phase as a parameter. We assume the matrix of the linear part and the vector at the relay characteristic such that, by a nonsingular transformation, the system is reduced to the form with the diagonal matrix and the vector being opposite to the unit vector. We establish a necessary and sufficient condition for the existence of two-point oscillatory solutions, i.e., the solutions with two fixed points on the hyperplanes of the relay switching in phase space. Also, we give the sufficient conditions under which such solutions do not exist. We provide a supporting example, which demonstrates how to apply the obtained results.

Keywords: ODE system; relay hysteresis; nonperiodic external disturbance; two-point oscillatory solution

MSC 2020: 34C15, 34C55, 93C15, 93C73

1. Introduction. Statement of problem

Hysteresis systems have been studied for long [5], [8], [23], [25], [26], and [38]. These systems are of interest for researchers to develop the automatic control theory as well as to extend the scope of application. At present, research in this field is actively continued (see, e.g., [2], [4], [6], [7], [10]–[12], [15]–[22], [24], [27], [28], [30], [33], [34], [37], and [39]–[45]). Scientists pay a special attention to dynamics of hysteresis systems with external disturbances.

Use of numerical methods in the research of such systems shows a variety of bounded and unbounded solutions. Besides, one can deal with a complicated de-

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composition of the system state space into the domains of the start points to which the qualitatively various solutions correspond. Also, one can face with a decomposition of the system parameter space into the domains of the qualitatively various dynamics of the system.

For this reason, in the paper we are engaged in a practical case and do research of a hysteresis system analytically to obtain the conditions for the existence of a two-point oscillatory solution—one of the possible solution types. We offer an approach to the analysis of dynamic behavior for the system with a relay hysteresis and an external disturbance. The existence and nonexistence of such oscillatory solutions are discussed. We also focus on consideration of how to select the system parameters such that there exist oscillatory solutions of this special type.

Consider the n-dimensional system of ordinary differential equations in the form

$$\dot{Y} = AY + BF(\sigma) + Kf(t), \quad \sigma = (C, Y). \tag{1.1}$$

Here Y is the state vector, $Y \in \mathbb{R}^n$. The matrix A of $(n \times n)$ order and the vectors B, K, C of $(n \times 1)$ order are real and constant. By (C, Y), we denote the scalar product of the vectors C and Y. The relay-type nonlinearity with the thresholds l_1, l_2 $(l_1 < l_2)$ and the outputs m_1, m_2 $(m_1 < m_2)$ is described by the relay characteristic $F(\sigma)$ (see [1]). The characteristic $F(\sigma(t))$ is piecewise constant for a continuous input $\sigma(t)$, $t \geqslant 0$, and is given as follows (see [31]): $F(\sigma) = m_1$ if $\sigma(t) \leqslant l_1$, $F(\sigma) = m_2$ if $\sigma(t) \geqslant l_2$, and $F(\sigma(t_1)) = F(\sigma(t_2))$ if $l_1 < \sigma(t) < l_2$ $(t_1 < t \leqslant t_2)$. This means [45] that $F(\sigma(t))$ assumes a constant value on the segment $[t_1, t_2]$ if either $F(\sigma(t_1)) = m_1$ and $\sigma(t) < l_2$ for $t \in [t_1, t_2]$ or $F(\sigma(t_1)) = m_2$ and $\sigma(t) > l_1$ for $t \in [t_1, t_2]$. Thus, in the case if $\sigma(0) \leqslant l_1$ or $\sigma(0) \geqslant l_2$, one of the outputs corresponds to $\sigma(t)$. If $l_1 < \sigma(0) < l_2$, then both the outputs correspond to $\sigma(t)$, therefore, it is necessary to choose $F(\sigma)$ first and then follow the positive spin in the plane $(\sigma, F(\sigma))$. The hysteresis loop described by the equations $\sigma = \sigma(t)$, $F = F(\sigma(t))$ is passed by counter-clockwise. We deal with the hysteresis model that is widely used in automatic control systems [35]. The vector C defines the feedback in the system. Research of relay feedback systems is a challenge [3], [13], and [46]. The function $f(t)$ describes an external disturbance on the system or a testing disturbance on the automatic system to be developed. Good engineering practice recommends a polynomial, an exponent, a sine, or a jump as disturbance. Such simplifications are often not quite adequate to disturbances in reality. That is why we study a more complicated model of the external disturbance

$$f(t) = \mathrm{e}^{\alpha t} \sin(\omega t + \varphi). \tag{1.2}$$

Here α, ω, and φ are real constants. We identify $\mathrm{e}^{\alpha t}$ with an amplitude, ω with a frequency, and φ with the phase angle allowing to define a deviation at time zero.

The function $f(t)$ in the form (1.2) describes a damped oscillatory process when $\alpha < 0$ and an increasing oscillatory process when $\alpha > 0$. In [32], the disturbance in the form (1.2) is applied to the Lavrent'ev problem for separated flows.

Mathematical model (1.1) with (1.2) can be utilized for studying the influence of the external disturbance on the work of automatic systems for vessels [9], [29]. As the automatic systems, one can consider the models of automatic steering and anti-rolling devices. Besides, model (1.1) with (1.2) is also possible to use for an analysis of a mutual impact of the mechanisms generating oscillations. For example, drawing on this model, one can explore the dynamics of the rotating shafts of metal-cutting machines, combustion engines, and turbines.

A regular rolling on the sea can be considered as a source of a harmonic oscillation described by the function $f(t)$ in the form (1.2) when $\alpha = 0$. Next we review examples of an irregular rolling on the sea. During the passage of a cyclone over the sea there is the sustained and strong wind that causes the heavy confused sea so-called a storm. The emergence of a storm is followed by the increasing wave induced oscillations, which can be described by the function $f(t)$ in the considered form with the increasing amplitude $\mathrm{e}^{\alpha t}$ when $\alpha > 0$. Abatement of the storm is followed by damped oscillations of waves, which can be described by $f(t)$ with poorly decreasing amplitude when $\alpha < 0$. By the function $f(t)$ in the form (1.2) with a strongly decreasing amplitude ($\alpha < 0$), one can describe, for instance, a shock sea wave induced by an explosion.

Mathematical model (1.1) with (1.2), like any mathematical model, is as a result of particular simplifications and assumptions that do not contradict to the physical nature of the described object. System (1.1) with the disturbance $f(t)$ in the form (1.2) can possess solutions with highly complicated structures. Therefore, it is very important to obtain the conditions such that system (1.1) can be integrated analytically and to establish the solutions that can be expected to occur. The aim of this paper is to solve the matters. The results are possible to use as fundamentals for the study of the more complicated mathematical model and also in the case of numerical experiments.

2. Approach to the problem

In phase space, the trajectory of any solution to system (1.1) is composed from the pieces of trajectories in virtue of the linear systems

$$\dot{Y} = AY + Bm_1 + Kf(t), \quad \dot{Y} = AY + Bm_2 + Kf(t). \tag{2.1}$$

On continuity, the joining of the trajectory pieces happens at the points on the hyperplanes $(C, Y) = l_1$ and $(C, Y) = l_2$ according to the fitting method.

In the case that $f(t)$ is a periodic function (when $\alpha = 0$), we may look for the solutions to system (1.1) in the class of continuous periodic functions with switching points the number of which is multiple to two (see, e.g., [15], [21]). By a switching point we mean a system state such that $\sigma(t)$ reaches one of the thresholds and hence $F(\sigma(t))$ then changes the value of the output. The switching point belongs to the hyperplane $(C, Y) = l_i$, $i = 1, 2$, and hence coincides with the joining point. In n-dimensional phase space, the periodic solution to (1.1) corresponds to a closed bounded trajectory. In the expanded space (Y, t) of $(n+1)$ dimension, this periodic solution corresponds to an integral curve consisting of several integral curves in virtue of (2.1). These curves are repeated with period T_f, which we call the period of the forced oscillations of system (1.1) and is equal or multiple to the period of $f(t)$ (see [16]).

In the case that $f(t)$ is a nonperiodic function (when $\alpha \neq 0$), we may look for the oscillatory solutions in the class of continuous nonperiodic functions with 2ϱ joining points, $\varrho \in \mathbb{N}$. Here the representative point of the solution comes back to each joining point in the phase space of (1.1) in T_f, which we will call return time. Denote the joining points by return points.

Consider $\varrho = 1$. For the solution of (1.1), we use the Cauchy form

$$Y(t) = \mathrm{e}^{A(t-t_0)} Y_0 + \int_{t_0}^{t} \mathrm{e}^{-A(\tau - t)} (Bm_i + Kf(\tau)) \, \mathrm{d}\tau, \quad i = 1, 2,$$

where t_0 is the initial time, $Y_0 = Y(t_0)$.

By L_i we denote the hyperplane $(C, Y) = l_i$, $i = 1, 2$, in $\mathbb{R}^n$. Next we give the following definitions introduced in [45].

Definition 2.1. If the representative point belongs to the hyperplane L_i, $i = 1, 2$, at some time t', then we refer to the least time $t'' > t'$ at which the representative point belongs to the hyperplane L_{3-i} as the *time of first encounter* of the representative point with the hyperplane L_{3-i}.

Definition 2.2. A solution $Y(\cdot)$ is said to be *two-point oscillatory with time Δ of return to the hyperplane L_i, $i = 1, 2$,* if there exist points $Y^1 \in L_1$, $Y^2 \in L_2$ in phase space such that at the times of first encounter of the representative point of the solution with the hyperplane L_i, the representative point reaches the point Y^i, $i = 1, 2$, and the times themselves of first encounter with each of the hyperplanes are periodic with period Δ.

In the present paper we will say such a solution is *two-point oscillatory with return time T_f*. By t_1 we denote the time of first encounter.

Note that the two-point oscillatory solution can exist in system (1.1) with a periodic function $f(t)$ (see [43], [45]). The return points of the two-point oscillatory solution to (1.1) are defined as the switching points of the periodic solution to (1.1). We denote them by Y^i, $i = 1, 2$. Therefore Y^i has the identical property $Y^i = Y(t_0) = Y(t_0 + mT_f)$, $(C, Y^i) = l_i$, $i = 1, 2$, $m \in \mathbb{N}$.

Next we consider system (1.1) with $f(t)$ in the form (1.2) when $\alpha \neq 0$ and establish conditions for the existence of the two-point oscillatory solution to system (1.1) and its parameters. By the solution parameters we mean time of first encounter t_1, return time T_f, and return points Y^1, Y^2.

Assume that there exists a two-point oscillatory solution with return time T_f as well as return points Y^1, Y^2 per T_f, and besides, its behavior satisfies the following conditions. Let the representative point of the solution to (1.1) start its moving from point $Y^1 \in L_1$ at $t = 0$ and reach L_2 at point Y^2 at t_1 in virtue of (2.1) provided that $m_i = m_1$. Next, the representative point comes back to L_1 to the start point Y^1 at T_f in virtue of (2.1) provided that $m_i = m_2$, and so on m times, $m \in \mathbb{N}$: the representative point goes between points Y^1 and Y^2, falling alternately into each of them in T_f.

Thus, the solution is constructed by the following rule: the vector function $Y(t)$ is a solution to system (2.1) for $m_i = m_1$ on the interval $[(m-1)T_f, t_1 + (m-1)T_f)$ and to system (2.1) for $m_i = m_2$ on the interval $[t_1 + (m-1)T_f, mT_f)$ with return points Y^1, Y^2, which satisfy the condition $Y^1 = Y(0) = Y(mT_f)$, $Y^2 = Y(t_1) = Y(t_1 + mT_f)$, $m \in \mathbb{N}$, and $(C, Y^i) = l_i$, $i = 1, 2$.

Taking into account the rule above, for any $m \in \mathbb{N}$ we have

$$l_1 = (C, Y^1), \quad l_2 = (C, Y^2), \tag{2.2}$$

where

$$\begin{aligned} Y^2 &= \mathrm{e}^{At_1} Y^1 + \int_{(m-1)T_f}^{t_1+(m-1)T_f} \mathrm{e}^{A(t_1+(m-1)T_f-\tau)} (Bm_1 + Kf(\tau))\,\mathrm{d}\tau, \\ Y^1 &= \mathrm{e}^{A(T_f-t_1)} Y^2 + \int_{t_1+(m-1)T_f}^{mT_f} \mathrm{e}^{A(mT_f-\tau)} (Bm_2 + Kf(\tau))\,\mathrm{d}\tau. \end{aligned} \tag{2.3}$$

Thus, we obtain the transcendental system (2.2), (2.3) for t_1, T_f, Y^1 and Y^2. In general, we cannot solve this system analytically, but we have an opportunity to establish the conditions on the parameters of system (2.2), (2.3) under which this system has a solution. To have the conditions of its resolvability in the form of analytical expressions including the coefficients of (2.2), (2.3), and therefore the coefficients of (1.1), we use a transformation of system (1.1) that simplifies system (2.2) and (2.3) significantly.

Let the matrix A have simple, nonzero, and real eigenvalues λ_i $(i = \overline{1,n})$. Besides, let system (1.1) be completely controlled to the input $F(\sigma)$, i.e.,

$$\det(B\ AB \ldots A^{n-1}B) \neq 0.$$

Then, by the nonsingular transformation $Y = SX$, system (1.1) can be put in the following canonical form:

$$\dot{X} = A_0 X + B_0 F(\sigma) + K_0 f(t), \quad \sigma = (\Gamma, X), \tag{2.4}$$

where

$$A_0 = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \; B_0 = S^{-1}B = \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix}, \; K_0 = S^{-1}K = \begin{pmatrix} k_1^0 \\ \vdots \\ k_n^0 \end{pmatrix}, \; \Gamma = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}.$$

The coefficients γ_i $(i = \overline{1,n})$ are calculated by the formula

$$\gamma_i = \frac{1}{D'(\lambda_i)} \sum_{k=1}^{n} c_k N_k(\lambda_i), \tag{2.5}$$

where

$$D'(\lambda_i) = \frac{\mathrm{d}D(p)}{\mathrm{d}p}\Big|_{p=\lambda_i},$$

c_k are the elements of C, and $N_k(\lambda_i) = \sum\limits_{j=1}^{n} b_j D_{jk}(\lambda_i)$. Here b_j are the elements of B; λ_i are the roots of the characteristic equation $D(p) = \det(a_{kl} - \delta_{kl}p) = 0$; a_{kl} are the elements of A; δ_{kl} is the Kronecker symbol; $D_{jk}(p)$ is the algebraic cofactor of the element d_{jk} of $D(p)$.

The matrix S can have the form

$$S = \begin{pmatrix} \dfrac{N_1(\lambda_1)}{D'(\lambda_1)} & \cdots & \dfrac{N_1(\lambda_n)}{D'(\lambda_n)} \\ \vdots & \ddots & \vdots \\ \dfrac{N_n(\lambda_1)}{D'(\lambda_1)} & \cdots & \dfrac{N_n(\lambda_n)}{D'(\lambda_n)} \end{pmatrix}. \tag{2.6}$$

Next we assume that $(n-1)$ roots of the equation $D(p) = 0$ coincide with $(n-1)$ roots of the equation $\sum\limits_{k=1}^{n} c_k N_k(p) = 0$. Then $(n-1)$ values of γ_i defined by (2.5) are equal to zero. Thus, the n-order system is reduced to lower-order systems that can be integrated consecutively. In addition, we put $\gamma_s \neq 0$, $\gamma_j = 0$ $(j = \overline{1,n},\ j \neq s)$. This simplifies system (2.2).

The function $\sigma(t)=(\Gamma,X(t))$ is determined by the system

$$
\begin{cases} \sigma(t)=\gamma_s x_s, \\ \dot{x}_s=\lambda_s x_s - F(\sigma)+k_s^0 f(t). \end{cases} \tag{2.7}
$$

Now we write out the differential equation for $\sigma(t)$

$$
\dot{\sigma}(t)=\lambda_s\sigma(t)+\gamma_s(-F(\sigma(t))+k_s^0 f(t)). \tag{2.8}
$$

We consider the function $\sigma(t)$ from the class of continuous functions. By solving (2.8), we can find t_1, T_f. Then $x_s(t)=\sigma(t)/\gamma_s$.

The rest of the variables x_j ($j\neq s$) are defined by the first-order nonhomogeneous linear equations

$$
\dot{x}_j=\lambda_j x_j - F(\sigma)+k_j^0 f(t), \quad j\neq s. \tag{2.9}
$$

The solution to (2.7), (2.9) has the general form

$$
x_s(t)=\frac{\sigma(t)}{\gamma_s}, \quad x_j(t)=x_j(t_0)\mathrm{e}^{\lambda_j(t-t_0)}+\mathrm{e}^{\lambda_j t}\int_{t_0}^{t}(-F(\sigma(\tau))+k_j^0 f(\tau))\mathrm{e}^{-\lambda_j\tau}\,\mathrm{d}\tau. \tag{2.10}
$$

System (2.10) determines the point map of one hyperplane L_1 or L_2 into the other one. These hyperplanes are perpendicular to the axis x_s in the phase space of (2.4) because $\gamma_j=0$.

To construct the system for t_1, T_f, we write out the solution to (2.8) in the general form

$$
\sigma(t)=\sigma_0\mathrm{e}^{\lambda_s(t-t_0)}+\gamma_s\mathrm{e}^{\lambda_s t}\Big(-m_i\int_{t_0}^{t}\mathrm{e}^{-\lambda_s\tau}\,\mathrm{d}\tau+k_s^0\int_{t_0}^{t}\mathrm{e}^{-\lambda_s\tau}f(\tau)\,\mathrm{d}\tau\Big), \quad \sigma_0=\sigma(t_0),
$$

with the initial and boundary conditions: on the segment $[(m-1)T_f, t_1+(m-1)T_f]$ we have $m_i=m_1$, $t_0=(m-1)T_f$, $\sigma_0=l_1$, $\sigma(t_1+(m-1)T_f)=l_2$; on the segment $[t_1+(m-1)T_f, mT_f]$ we have $m_i=m_2$, $t_0=t_1+(m-1)T_f$, $\sigma_0=l_2$, $\sigma(mT_f)=l_1$.

The next purpose is to solve system (2.8), (2.9) with the initial and boundary conditions. Firstly, in Section 3, we consider this system for $m=1$, secondly, in Section 4, for any $m\in\mathbb{N}$.

Then on the segment $[0,T_f]$ when $m=1$, the system of transcendental equations for t_1, T_f takes the form

$$
\begin{aligned}
l_2&=\Big(l_1-\frac{\gamma_s m_1}{\lambda_s}\Big)\mathrm{e}^{\lambda_s t_1}+\frac{\gamma_s m_1}{\lambda_s}+\gamma_s k_s^0\int_0^{t_1}\mathrm{e}^{\lambda_s(t_1-\tau)}f(\tau)\,\mathrm{d}\tau,\\
l_1&=\Big(l_2-\frac{\gamma_s m_2}{\lambda_s}\Big)\mathrm{e}^{\lambda_s(T_f-t_1)}+\frac{\gamma_s m_2}{\lambda_s}+\gamma_s k_s^0\int_{t_1}^{T_f}\mathrm{e}^{\lambda_s(T_f-\tau)}f(\tau)\,\mathrm{d}\tau.
\end{aligned} \tag{2.11}
$$

Note that system (2.11) is equivalent to system (2.2) with $m=1$.

The points X^1, X^2 of system (2.4) and Y^1, Y^2 of (2.1) are in the relations $X^1 = S^{-1}Y^1$, $X^2 = S^{-1}Y^2$. The matrix $E - \mathrm{e}^{A_0 T_f}$ is nonsingular (E is the identity matrix). Therefore, for $m = 1$ the points X^1, X^2 are defined as follows:

$$
\begin{aligned}
X^1 &= (E - \mathrm{e}^{A_0 T_f})^{-1}\biggl(\int_0^{t_1} \mathrm{e}^{A_0(T_f-\tau)}(B_0 m_1 + K_0 f(\tau))\,\mathrm{d}\tau \\
&\quad + \int_{t_1}^{T_f} \mathrm{e}^{A_0(T_f-\tau)}(B_0 m_2 + K_0 f(\tau))\,\mathrm{d}\tau\biggr), \\
X^2 &= (E - \mathrm{e}^{A_0 T_f})^{-1}\biggl(\int_0^{t_1} \mathrm{e}^{A_0(t_1-\tau)}(B_0 m_1 + K_0 f(\tau))\,\mathrm{d}\tau \\
&\quad + \mathrm{e}^{A_0 t_1}\int_{t_1}^{T_f} \mathrm{e}^{A_0(T_f-\tau)}(B_0 m_2 + K_0 f(\tau))\,\mathrm{d}\tau\biggr).
\end{aligned} \tag{2.12}
$$

In Section 4, we show that X^1, X^2 are return points for any $m \in \mathbb{N}$.

3. Case $m = 1$

Consider system (2.11). After integration, we have

$$
\begin{aligned}
l_2 &= \Bigl(l_1 - \frac{\gamma_s m_1}{\lambda_s} - \frac{\gamma_s k_s^0((\alpha-\lambda_s)\sin\varphi - \omega\cos\varphi)}{(\alpha-\lambda_s)^2+\omega^2}\Bigr)\mathrm{e}^{\lambda_s t_1} \\
&\quad + \frac{\gamma_s k_s^0 \mathrm{e}^{\alpha t_1}((\alpha-\lambda_s)\sin(\omega t_1+\varphi) - \omega\cos(\omega t_1+\varphi))}{(\alpha-\lambda_s)^2+\omega^2} + \frac{\gamma_s m_1}{\lambda_s}, \\
l_1 &= \Bigl(l_2 - \frac{\gamma_s m_2}{\lambda_s} - \frac{\gamma_s k_s^0 \mathrm{e}^{\alpha t_1}((\alpha-\lambda_s)\sin(\omega t_1+\varphi) - \omega\cos(\omega t_1+\varphi))}{(\alpha-\lambda_s)^2+\omega^2}\Bigr)\mathrm{e}^{\lambda_s(T_f-t_1)} \\
&\quad + \frac{\gamma_s k_s^0 \mathrm{e}^{\alpha T_f}((\alpha-\lambda_s)\sin(\omega T_f+\varphi) - \omega\cos(\omega T_f+\varphi))}{(\alpha-\lambda_s)^2+\omega^2} + \frac{\gamma_s m_2}{\lambda_s}.
\end{aligned} \tag{3.1}
$$

The points $X^1 = (x_1^1, \dots, x_n^1)^\top$, $X^2 = (x_1^2, \dots, x_n^2)^\top$, which belong to the hyperplanes L_1, L_2, are determined by (2.12).

Let t_1, T_f exist and be found. Then $x_s^1 = l_1/\gamma_s$ and $x_s^2 = l_2/\gamma_s$. The rest of the coordinates x_j^1, x_j^2, $j = \overline{1,n}$, $j \neq s$, are defined as follows:

$$
\begin{aligned}
x_j^1 &= \frac{\mathrm{e}^{\lambda_j T_f}}{1-\mathrm{e}^{\lambda_j T_f}}\biggl(\int_0^{t_1} \frac{-m_1}{\mathrm{e}^{\lambda_j \tau}}\,\mathrm{d}\tau + \int_{t_1}^{T_f} \frac{-m_2}{\mathrm{e}^{\lambda_j \tau}}\,\mathrm{d}\tau + \int_0^{T_f} \frac{k_j^0 \mathrm{e}^{\alpha\tau}\sin(\omega\tau+\varphi)}{\mathrm{e}^{\lambda_j \tau}}\,\mathrm{d}\tau\biggr), \\
x_j^2 &= \frac{\mathrm{e}^{\lambda_j t_1}}{1-\mathrm{e}^{\lambda_j T_f}}\biggl(\int_0^{t_1} \frac{-m_1 + k_j^0 \mathrm{e}^{\alpha\tau}\sin(\omega\tau+\varphi)}{\mathrm{e}^{\lambda_j \tau}}\,\mathrm{d}\tau \\
&\quad + \int_{t_1}^{T_f} \frac{-m_2 + k_j^0 \mathrm{e}^{\alpha\tau}\sin(\omega\tau+\varphi)}{\mathrm{e}^{\lambda_j (T_f-\tau)}}\,\mathrm{d}\tau\biggr).
\end{aligned} \tag{3.2}
$$

Next, solving system (3.1), we look for t_1, T_f. Let $\lambda_s > \alpha$. System (3.1) takes the form

(3.3)
$$\begin{aligned}
l_2 = \Big(l_1 - \frac{\gamma_s m_1}{\lambda_s} + \frac{\gamma_s k_s^0 \sin(\varphi+\Delta\varphi)}{\sqrt{(\alpha-\lambda_s)^2+\omega^2}}\Big)\mathrm{e}^{\lambda_s t_1} - \frac{\gamma_s k_s^0 \mathrm{e}^{\alpha t_1}\sin(\omega t_1+\varphi+\Delta\varphi)}{\sqrt{(\alpha-\lambda_s)^2+\omega^2}} + \frac{\gamma_s m_1}{\lambda_s},\\
l_1 = \Big(l_2 - \frac{\gamma_s m_2}{\lambda_s} + \frac{\gamma_s k_s^0 \mathrm{e}^{\alpha t_1}\sin(\omega t_1+\varphi+\Delta\varphi)}{\sqrt{(\alpha-\lambda_s)^2+\omega^2}}\Big)\mathrm{e}^{\lambda_s (T_f - t_1)}\\
- \frac{\gamma_s k_s^0 \mathrm{e}^{\alpha T_f}\sin(\omega T_f+\varphi+\Delta\varphi)}{\sqrt{(\alpha-\lambda_s)^2+\omega^2}} + \frac{\gamma_s m_2}{\lambda_s},
\end{aligned}$$

where $\Delta\varphi = \arctan(\omega/(\lambda_s-\alpha))$.

From the first equation of (3.3) we express

$$\frac{\gamma_s k_s^0 \mathrm{e}^{\alpha t_1}\sin(\omega t_1+\varphi+\Delta\varphi)}{\sqrt{(\alpha-\lambda_s)^2+\omega^2}},$$

then substitute it into the second equation, and obtain

$$\begin{aligned}
(3.4)\quad &\frac{\gamma_s k_s^0 \mathrm{e}^{\alpha t_1}\sin(\omega t_1+\varphi+\Delta\varphi)}{\sqrt{(\alpha-\lambda_s)^2+\omega^2}}\\
&= \Big(l_1 - \frac{\gamma_s m_1}{\lambda_s} + \frac{\gamma_s k_s^0 \sin(\varphi+\Delta\varphi)}{\sqrt{(\alpha-\lambda_s)^2+\omega^2}}\Big)\mathrm{e}^{\lambda_s t_1} - l_2 + \frac{\gamma_s m_1}{\lambda_s},\\
&\frac{\gamma_s k_s^0 \mathrm{e}^{\alpha T_f}\sin(\omega T_f+\varphi+\Delta\varphi)}{\sqrt{(\alpha-\lambda_s)^2+\omega^2}}\\
&= \Big(\frac{\gamma_s(m_1-m_2)}{\lambda_s \mathrm{e}^{\lambda_s t_1}} + l_1 - \frac{\gamma_s m_1}{\lambda_s} + \frac{\gamma_s k_s^0 \sin(\varphi+\Delta\varphi)}{\sqrt{(\alpha-\lambda_s)^2+\omega^2}}\Big)\mathrm{e}^{\lambda_s T_f} - l_1 + \frac{\gamma_s m_2}{\lambda_s}.
\end{aligned}$$

First we look for the conditions on the parameters of the first equation of (3.4) such that there exist the solutions $t_1 \in (0, kT)$, where $k \in \mathbb{N}$, $T = 2\pi/\omega$, $\omega > 0$.

Put

$$\begin{aligned}
\psi_1(t) &= \frac{\gamma_s k_s^0 \mathrm{e}^{\alpha t}\sin(\omega t+\varphi+\Delta\varphi)}{\sqrt{(\alpha-\lambda_s)^2+\omega^2}},\\
\psi_2(t) &= \Big(l_1 - \frac{\gamma_s m_1}{\lambda_s} + \frac{\gamma_s k_s^0 \sin(\varphi+\Delta\varphi)}{\sqrt{(\alpha-\lambda_s)^2+\omega^2}}\Big)\mathrm{e}^{\lambda_s t} - l_2 + \frac{\gamma_s m_1}{\lambda_s}.
\end{aligned}$$

We have the following lemma.

Lemma 3.1. *Let $k_s^0 \neq 0$ and one of the following conditions is true: Either*

(1) *the inequality $\psi_1(kT) < \psi_2(kT)$, $k \in \mathbb{N}$, holds, or*
(2) *the inequality $\psi_1(t^k_{\min}) \leqslant \psi_2(t^k_{\min})$ holds, where $t^k_{\min}$ is a minimum point of the function $\psi_1(t)$ on the segment $[(k-1)T, kT]$, $k \in \mathbb{N}$.*

Then the first equation of (3.4) has at least one solution $t_1 \in (0, kT)$.

P r o o f. Let us study the first equation of system (3.4). For $\psi_1(t_1)$, we have the left side of (3.4) and for $\psi_2(t_1)$ its right side. Consider the plots of the functions $y = \psi_1(t)$, $y = \psi_2(t)$ in the plane (t, y). The crossing of these plots gives a solution t_1 of the equation $\psi_1(t) = \psi_2(t)$. By assumption, $l_1 < l_2$. Then $\psi_2(0) = \psi_1(0) - (l_2 - l_1)$. As a consequence, $\psi_1(0) > \psi_2(0)$.

Let condition (1) of Lemma 3.1 hold, i.e., $\psi_1(kT) < \psi_2(kT)$, $k \in \mathbb{N}$. Obviously, then there exists at least one point of intersection on the interval $(0, kT)$. Hence, there exists a solution $t_1 \in (0, kT)$.

The function $\psi_1(t)$ has a minimum point $t^k_{\min}$ on the segment $[(k-1)T, kT]$. By assumption, $k^0_s \neq 0$. Hence $\psi_1(t)$ is not equal to the identical zero. If condition (2) of Lemma 3.1 holds, then the plots of functions $y = \psi_1(t)$ and $y = \psi_2(t)$ cross on the interval $(0, kT)$ at least once. Thus, the assertion of Lemma 3.1 is also true. Lemma 3.1 is proved. □

Let the solution $t_1 > 0$ exist and be found. We substitute t_1 in the second equation of (3.4). Next we establish the conditions under which the second equation has the solution T_f such that $T_f > t_1$ provided that t_1 is a parameter. Put

$$\psi_3(t) = \Big(\frac{\gamma_s(m_1 - m_2)}{\lambda_s \mathrm{e}^{\lambda_s t_1}} + l_1 - \frac{\gamma_s m_1}{\lambda_s} + \frac{\gamma_s k^0_s \sin(\varphi + \Delta\varphi)}{\sqrt{(\alpha - \lambda_s)^2 + \omega^2}}\Big)\mathrm{e}^{\lambda_s t} - l_1 + \frac{\gamma_s m_2}{\lambda_s},$$

where t_1 is the solution of the first equation of (3.4). The following theorem takes place.

Theorem 3.1. *Let the assumption and one of the conditions of Lemma* 3.1 *be fulfilled,* t_1 *be the smallest positive solution of the first equation of* (3.4). *Also, let the inequality* $\psi_1(kT) \geqslant \psi_3(kT)$, $k \in \mathbb{N}$, *hold. Then system* (3.4) *has at least one pair of the solutions* t_1, T_f *such that* $0 < t_1 < T_f \leqslant kT$.

P r o o f. Lemma 3.1 states that the number of the solutions t_1 can be more than one and it is finite, so far as $t_1 \in (0, kT)$. By t_1 we mean the time of first encounter with the hyperplane L_2. According to Definition 2.2, time t_1 has to take the smallest value. Therefore, the desirable solution t_1 of the first equation of (3.4) is the smallest and hence unique.

Let t_1 be found. Then $\psi_1(t_1) = \psi_2(t_1)$. Next we will prove that $\psi_3(t_1) > \psi_2(t_1)$. Indeed,

$$\psi_3(t_1) = \frac{\gamma_s m_1}{\lambda_s} + \Big(l_1 - \frac{\gamma_s m_1}{\lambda_s} + \frac{\gamma_s k^0_s}{\sqrt{(\alpha - \lambda_s)^2 + \omega^2}} \sin(\varphi + \Delta\varphi)\Big)\mathrm{e}^{\lambda_s t_1} - l_1,$$

$$\psi_2(t_1) = \frac{\gamma_s m_1}{\lambda_s} + \Big(l_1 - \frac{\gamma_s m_1}{\lambda_s} + \frac{\gamma_s k^0_s}{\sqrt{(\alpha - \lambda_s)^2 + \omega^2}} \sin(\varphi + \Delta\varphi)\Big)\mathrm{e}^{\lambda_s t_1} - l_2.$$

Subtracting these equalities, we obtain $\psi_3(t_1) - \psi_2(t_1) = l_2 - l_1$. Since $l_1 < l_2$, we come to the true inequality $\psi_3(t_1) > \psi_2(t_1)$. Taking into account that $\psi_1(t_1) = \psi_2(t_1)$, we have $\psi_3(t_1) > \psi_1(t_1)$.

The second equation of (3.4) takes the form $\psi_1(T_f) = \psi_3(T_f)$. Thus, we look for T_f satisfying the equation $\psi_1(t) = \psi_3(t)$ and such that $T_f > t_1$. If $\psi_1(kT) \geqslant \psi_3(kT)$, then there exists at least one intersection of the plots of the functions $y = \psi_1(t)$ and $y = \psi_3(t)$ on the interval $(t_1, kT]$. Consequently, there exists at least one T_f such that $t_1 < T_f \leqslant kT$. From this the assertion of Theorem 3.1 follows. Theorem 3.1 is proved. $\square$

Thus, Theorem 3.1 gives the sufficient conditions for the existence of at least one pair of the solutions t_1, T_f to system (3.4). Put

$$H_1 = l_1 - \frac{\gamma_s m_1}{\lambda_s} + \frac{\gamma_s k_s^0 \sin(\varphi + \Delta\varphi)}{\sqrt{(\alpha - \lambda_s)^2 + \omega^2}}, \quad H_2 = H_1 + \frac{\gamma_s(m_1 - m_2)}{\lambda_s \mathrm{e}^{\lambda_s t_1}}.$$

The theorem below gives the conditions on the parameters of system (3.4) such that the solutions t_1, T_f do not exist.

Theorem 3.2. *Let $k_s^0 \neq 0$ and the following two cases be possible.*

Case 1. Let the following conditions hold: Either

(C_{11}) *for $\lambda_s < 0$ the inequality $H_1 > 0$ is true, and for $\lambda_s > 0$, $\alpha < 0$ the inequality $H_1 < 0$ is true;*

(C_{12}) *$\psi_1(t^1_{\min}) \geqslant \psi_2(0)$, where $t^1_{\min}$ is a minimum point of the function $\psi_1(t)$ on the segment $[0, T]$; or*

(C_{13}) *one of the conditions of Lemma 3.1 is satisfied;*

(C_{14}) *for $\lambda_s < 0$ the inequality $H_2 < 0$ is true, and for $\lambda_s > 0$, $\alpha < 0$ the inequality $H_2 > 0$ is true;*

(C_{15}) *$\psi_3(t_1) \geqslant \psi_1(t^q_{\max})$, where $t^q_{\max}$ is a maximum point of $\psi_1(t)$ on the interval $((q-1)T, qT]$ such that $t^q_{\max} > t_1$, where $q \leqslant k$, $q, k \in \mathbb{N}$.*

Then system (3.4) has no solutions t_1, T_f such that $0 < t_1 < T_f$.

Case 2. Let the following conditions hold: Either

(C_{21}) *for $\lambda_s > 0$, $\alpha > 0$ the inequality $H_1 < 0$ is true;*

(C_{22}) *$\psi_1(t^k_{\min}) \geqslant \psi_2(0)$, where $t^k_{\min}$ is a minimum point of $\psi_1(t)$ on the segment $[(k-1)T, kT]$, $k \in \mathbb{N}$; or*

(C_{23}) *one of the conditions of Lemma 3.1 is satisfied;*

(C_{24}) *for $\lambda_s > 0$, $\alpha > 0$ the inequality $H_2 > 0$ is true;*

(C_{25}) *$\psi_3(t_1) \geqslant \psi_1(t^k_{\max})$, where $t^k_{\max}$ is a maximum point of $\psi_1(t)$ on the interval $((k-1)T, kT]$ such that $t^k_{\max} > t_1$, $k \in \mathbb{N}$.*

Then system (3.4) has no solutions t_1, T_f such that $0 < t_1 < T_f \leqslant kT$.

P r o o f. We put $\lambda_s < 0$ and $H_1 > 0$. The function $y = \psi_2(t)$ decreases for $t > 0$. Since system (3.4) is written out when $\lambda_s > \alpha$, we have $\alpha < 0$. The function $y = \psi_1(t)$ has the decreasing amplitude $|\gamma_s k_s^0|((\alpha - \lambda_s)^2 + \omega^2)^{-1/2} e^{\alpha t}$. We remind that $\psi_1(0) > \psi_2(0)$. The first equation of (3.4) has no solution t_1 if the inequality $\psi_1(t^1_{\min}) \geqslant \psi_2(0)$ is met.

Let $\lambda_s > 0$ and $H_1 < 0$. The function $y = \psi_2(t)$ decreases for $t > 0$ as well. If $\psi_1(t^1_{\min}) \geqslant \psi_2(0)$, then the first equation of (3.4) has no solution $t_1 > 0$ when $\alpha < 0$. Let $\alpha > 0$. Then $y = \psi_1(t)$ has the increasing amplitude. There exists no solution t_1 on the interval $(0, kT]$ if $\psi_1(t^k_{\min}) \geqslant \psi_2(0)$.

Assume that one of the conditions of Lemma 3.1 holds. Then there exists a positive solution t_1. We recall that $\psi_3(t_1) > \psi_1(t_1)$.

Let $\lambda_s < 0$; then $\alpha < 0$. In addition, we suppose $H_2 < 0$ and $\psi_3(t_1) \geqslant \psi_1(t^q_{\max})$. Then the second equation of (3.4) has no solution T_f such that $T_f > t_1$.

Now let $\lambda_s > 0$ and $H_2 > 0$. If $\alpha < 0$, then the second equation of (3.4) has no solution T_f such that $T_f > t_1$. If for $\alpha > 0$ the inequality $\psi_3(t_1) \geqslant \psi_1(t^k_{\max})$ is met, then there exists no solution T_f such that $t_1 < T_f \leqslant kT$. Theorem 3.2 is proved. □

4. Main results

Once a pair of the solutions t_1, T_f is obtained, return points Y^1, Y^2 of system (2.1) with $m = 1$ can be calculated. We recall that t_1, T_f are determined by Theorem 3.1, which gives the necessary condition for the existence of two-point oscillatory solutions. So, the conditions for system (3.4) to have the solutions t_1, T_f provide no guarantees for the existence of a closed trajectory with return points Y^1, Y^2 (or the so-called unimodal trajectory [36]).

Therefore, the two-point oscillatory solution exists if and only if the points Y^1, Y^2 calculated for t_1, T_f are possible to be connected by the arches of the integrated curves in virtue of system (2.1) for any $m \in \mathbb{N}$.

Taking this into account, we prove the following theorem, which gives a necessary and sufficient condition for the existence of a two-point oscillatory solution.

Theorem 4.1. *Let the function $f(t)$ of system* (1.1) *be given by* (1.2) *with $\alpha \neq 0$. Let system* (1.1) *be reduced to the canonical form* (2.7), (2.9) *by a nonsingular transformation $Y = SX$ provided that $\gamma_s \neq 0$, $\gamma_j = 0$, $j = \overline{1, n}$, $j \neq s$. System* (1.1) *has a two-point oscillatory solution; moreover, it is unique, with the parameters t_1, T_f, Y^1, Y^2, where $Y^i = SX^i$, $i = 1, 2$, and X^i is defined by* (3.2) *if and only if the following conditions hold:*

(1) *system* (2.1) *meets the conditions of Theorem* 3.1*;*
(2) $t_1 = \pi k/\omega$, $T_f = \pi h/\omega$, *where* $k, h \in \mathbb{N}$, $k < h$, *and, in addition,* T_f *is the smallest solution of the second equation of system* (3.4)*;*
(3) $\varphi + \Delta\varphi = \pi q$, $q \in \mathbb{Z}$, *and* $k_j^0 = 0$.

Proof. The proof is based on equivalent statements. So, we use one-directional proof instead of two directions.

System (1.1) has a unique two-point oscillatory solution with the parameters t_1, T_f, Y^1, Y^2 if and only if system (2.7), (2.9) has a unique two-point oscillatory solution with the parameters t_1, T_f, X^1, X^2 provided $\gamma_s \neq 0$, $\gamma_j = 0$, $j = \overline{1,n}$, $j \neq s$. According to Definition 2.2, this means the following equivalent conditions:

(i) $X(0) = X(mT_f) = X^1 \in L_1$, $X(t_1) = X(t_1 + mT_f) = X^2 \in L_2$, $m \in \mathbb{N}$, and the pair t_1, T_f is unique;
(ii) (a) $\sigma(0) = \sigma(mT_f) = l_1$, $\sigma(t_1) = \sigma(t_1 + mT_f) = l_2$, $x_s^i = l_i/\gamma_s$, $i = 1, 2$,
(b) $x_j(0) = x_j(mT_f) = x_j^1$, $x_j(t_1) = x_j(t_1 + mT_f) = x_j^2$, $m \in \mathbb{N}$, and the pair t_1, T_f is unique.

We need to prove that condition (ii) holds if and only if conditions (1)–(3) of Theorem 4.1 hold.

By condition (1) of Theorem 4.1, system (3.4) has a pair of the solutions t_1, T_f such that $0 < t_1 < T_f \leqslant kT$, and besides, t_1 is the smallest solution of the first equation of system (3.4). By condition (2) of Theorem 4.1, T_f is the smallest solution of the second equation. This means that t_1, T_f is a unique pair of solutions to (3.4) and hence is a unique pair of the parameters of a two-point oscillatory solution.

Now consider condition (ii) (a). On the first segment $[0, t_1]$, we use the first equation of (3.3)

$$l_2 = \Big(l_1 - \frac{\gamma_s m_1}{\lambda_s} + \frac{\gamma_s k_s^0 \sin(\varphi + \Delta\varphi)}{\sqrt{(\alpha - \lambda_s)^2 + \omega^2}}\Big)\mathrm{e}^{\lambda_s t_1} - \frac{\gamma_s k_s^0 \mathrm{e}^{\alpha t_1} \sin(\omega t_1 + \varphi + \Delta\varphi)}{\sqrt{(\alpha - \lambda_s)^2 + \omega^2}} + \frac{\gamma_s m_1}{\lambda_s}.$$

Next, on the segments in the form $[mT_f, mT_f + t_1]$, $m \in \mathbb{N}$, we have

$$\begin{aligned} l_2 = \Big(l_1 - \frac{\gamma_s m_1}{\lambda_s} &+ \frac{\gamma_s k_s^0 \mathrm{e}^{\alpha m T_f} \sin(\omega m T_f + \varphi + \Delta\varphi)}{\sqrt{(\alpha - \lambda_s)^2 + \omega^2}}\Big)\mathrm{e}^{\lambda_s t_1} \\ &- \frac{\gamma_s k_s^0 \mathrm{e}^{\alpha(mT_f + t_1)} \sin(\omega(mT_f + t_1) + \varphi + \Delta\varphi)}{\sqrt{(\alpha - \lambda_s)^2 + \omega^2}} + \frac{\gamma_s m_1}{\lambda_s}. \end{aligned}$$

On the second segment $[t_1, T_f]$, we use the second equation of (3.3)

$$\begin{aligned} l_1 = \Big(l_2 - \frac{\gamma_s m_2}{\lambda_s} &+ \frac{\gamma_s k_s^0 \mathrm{e}^{\alpha t_1} \sin(\omega t_1 + \varphi + \Delta\varphi)}{\sqrt{(\alpha - \lambda_s)^2 + \omega^2}}\Big)\mathrm{e}^{\lambda_s t_2} \\ &- \frac{k_s^0 \gamma_s \mathrm{e}^{\alpha T_f} \sin(\omega T_f + \varphi + \Delta\varphi)}{\sqrt{(\alpha - \lambda_s)^2 + \omega^2}} + \frac{\gamma_s m_2}{\lambda_s}, \quad \text{where } t_2 = T_f - t_1. \end{aligned}$$

By analogy, on the segments in the form $[mT_f + t_1, (m+1)T_f]$, we have

$$l_1 = \Big(l_2 - \frac{\gamma_s m_2}{\lambda_s} + \frac{\gamma_s k_s^0 \mathrm{e}^{\alpha(mT_f+t_1)} \sin(\omega(mT_f+t_1)+\varphi+\Delta\varphi)}{\sqrt{(\alpha-\lambda_s)^2+\omega^2}}\Big)\mathrm{e}^{\lambda_s t_2} \\ - \frac{k_s^0\gamma_s \mathrm{e}^{\alpha(m+1)T_f}\sin(\omega(m+1)T_f+\varphi+\Delta\varphi)}{\sqrt{(\alpha-\lambda_s)^2+\omega^2}} + \frac{\gamma_s m_2}{\lambda_s}.$$

Comparing the right sides of the equalities that have equal left sides, we see the following: a necessary and sufficient condition for the system of equalities above to be true provided $\gamma_s k_s^0 \neq 0$ is that the equalities below take place

$$\begin{aligned} \sin(\omega t_1 + \varphi + \Delta\varphi) &= \mathrm{e}^{\alpha(mT_f+t_1)} \sin(\omega(mT_f+t_1)+\varphi+\Delta\varphi), \\ \sin(\varphi+\Delta\varphi) &= \mathrm{e}^{\alpha m T_f}\sin(\omega m T_f + \varphi + \Delta\varphi) \quad \forall\, m \in \mathbb{N}. \end{aligned} \tag{4.1}$$

For $\alpha \neq 0$, equalities (4.1) are satisfied if and only if $\varphi + \Delta\varphi = \pi q$, $q \in \mathbb{Z}$, and $t_1 = \pi k/\omega$, $T_f = \pi h/\omega$, where $k, h \in \mathbb{N}$. Since $t_1 < T_f$, we have $k < h$.

Next consider condition (ii) (b). Using (2.10), we have

$$x_j(t) = x_j(0)\mathrm{e}^{\lambda_j t} + \mathrm{e}^{\lambda_j t}\int_0^t \mathrm{e}^{-\lambda_j\tau}(-m_1 + k_j^0\mathrm{e}^{\alpha\tau}\sin(\omega\tau+\varphi))\,\mathrm{d}\tau \quad \forall\, t \in [0, t_1),$$

$$x_j(t) = x_j(t_1)\mathrm{e}^{\lambda_j (t-t_1)} + \mathrm{e}^{\lambda_j t}\int_{t_1}^t \mathrm{e}^{-\lambda_j\tau}(-m_2 + k_j^0\mathrm{e}^{\alpha\tau}\sin(\omega\tau+\varphi))\,\mathrm{d}\tau \quad \forall\, t \in [t_1, T_f).$$

Taking the denotations in condition (ii) (b) into account, upon integration, we come to

$$x_j^2 = x_j^1\mathrm{e}^{\lambda_j t_1} + \frac{m_1}{\lambda_j}(1-\mathrm{e}^{\lambda_j t_1}) + C_1, \quad x_j^1 = x_j^2\mathrm{e}^{\lambda_j t_2} + \frac{m_2}{\lambda_j}(1-\mathrm{e}^{\lambda_j t_2}) + C_2,$$

where

$$C_1 = \frac{\mathrm{e}^{\alpha t_1}\sin(\omega t_1+\varphi+\Delta\varphi_j) - \mathrm{e}^{\lambda_j t_1}\sin(\varphi+\Delta\varphi_j)}{\sqrt{(\alpha-\lambda_j)^2+\omega^2}}k_j^0,$$

$$C_2 = \frac{\mathrm{e}^{\alpha T_f}\sin(\omega T_f+\varphi+\Delta\varphi_j) - \mathrm{e}^{\alpha t_1+\lambda_j t_2}\sin(\omega t_1+\varphi+\Delta\varphi_j)}{\sqrt{(\alpha-\lambda_j)^2+\omega^2}}\,k_j^0,$$

$$\Delta\varphi_j = \arctan\frac{\omega}{\lambda_j - \alpha}.$$

To continue this on the next segments, we change the integration limits and obtain the following: on the segments in the form $[mT_f, mT_f + t_1]$

$$x_j^2 = x_j^1\mathrm{e}^{\lambda_j t_1} + \frac{m_1}{\lambda_j}(1-\mathrm{e}^{\lambda_j t_1}) + C_{2m+1},$$

on the segments in the form $[mT_f + t_1, (m+1)T_f]$

$$x_j^1 = x_j^2\mathrm{e}^{\lambda_j t_2} + \frac{m_2}{\lambda_j}(1-\mathrm{e}^{\lambda_j t_2}) + C_{2m+2},$$

where

$$
\begin{aligned}
C_{2m+1} = k_j^0 \Big(& \frac{\mathrm{e}^{\alpha(t_1+mT_f)} \sin(\omega(t_1+mT_f)+\varphi+\Delta\varphi_j)}{\sqrt{(\alpha-\lambda_j)^2+\omega^2}} \\
& - \frac{\mathrm{e}^{\alpha m T_f+\lambda_j t_1} \sin(\omega m T_f+\varphi+\Delta\varphi_j)}{\sqrt{(\alpha-\lambda_j)^2+\omega^2}}\Big), \\
C_{2m+2} = k_j^0 \Big(& \frac{\mathrm{e}^{\alpha(m+1)T_f} \sin(\omega(m+1)T_f+\varphi+\Delta\varphi_j)}{\sqrt{(\alpha-\lambda_j)^2+\omega^2}} \\
& - \frac{\mathrm{e}^{\alpha(t_1+mT_f)+\lambda_j t_2} \sin(\omega(t_1+mT_f)+\varphi+\Delta\varphi_j)}{\sqrt{(\alpha-\lambda_j)^2+\omega^2}}\Big), \quad m\in\mathbb{N}.
\end{aligned}
$$

Condition (ii) (b) holds if and only if

$$C_1 = C_2 = C_{2m+1} = C_{2m+2} = 0. \tag{4.2}$$

Notice that the conditions $\varphi+\Delta\varphi = \pi q$ and $\varphi+\Delta\varphi_j = \pi p$, $p\in\mathbb{Z}$, cannot be met simultaneously since $\lambda_j \neq \lambda_s$ by assumption. Therefore, a necessary and sufficient conditions for equalities (4.2) are met provided $\varphi+\Delta\varphi_j \neq \pi p$ is that $k_j^0 = 0$.

Thus, condition (ii) holds if and only if conditions (1)–(3) of Theorem 4.1 hold. Consequently, the representative point of the solution returns m times to the point $X^i \in L_i$, $i=1,2$, per time T_f. This means that X^1, X^2 are return points.

Next we find return points X^1, X^2 by formula (3.2), which for $k_j^0=0$ takes the following form: $x_s^i = l_i/\gamma_s$, $i=1,2$,

$$
\begin{aligned}
x_j^1 &= (1-\mathrm{e}^{\lambda_j T_f})^{-1}\lambda_j^{\;-1}(m_1\mathrm{e}^{\lambda_j t_2}(1-\mathrm{e}^{\lambda_j t_1})+m_2(1-\mathrm{e}^{\lambda_j t_2})),\\
x_j^2 &= (1-\mathrm{e}^{\lambda_j T_f})^{-1}\lambda_j^{\;-1}(m_2\mathrm{e}^{\lambda_j t_1}(1-\mathrm{e}^{\lambda_j t_2})+m_1(1-\mathrm{e}^{\lambda_j t_1})).
\end{aligned}
$$

Then return points Y^1, Y^2 are derived from the relations $Y^1 = SX^1$, $Y^2 = SX^2$. Here S is a transformation matrix that can be defined, for example, by (2.6). Thus, system (1.1) has a unique two-point oscillatory solution with the parameters t_1, T_f, Y^1, Y^2. Theorem 4.1 is completely proved. □

Remark 4.1. System (1.1) can be reduced to the canonical form (2.4) with the vector $B_0 = (-1,\dots,-1)^\top$ by a nonsingular transformation with a matrix that is different from S defined by (2.6) (see [14], [20]).

Remark 4.2. If $\lambda_j < 0$, $j=\overline{1,n}$, $j\neq s$, and the function $f(t)$ is bounded for any $t \geqslant 0$, i.e., with $\alpha \leqslant 0$, then a two-point oscillatory solution to (1.1) is stable. This follows from the analysis of stability given in [41] and [42] for the periodic solutions with two switching points, because such a type of periodic solutions is in the class of two-point oscillatory solutions.

R e m a r k 4.3. If $\alpha < 0$, then there exists a $\tau > 0$ such that for any $t \geqslant \tau$ the approximate equality $f(t) \approx 0$ takes place. Therefore the behavior of system (2.7), (2.9) with $f(t) \neq 0$ per time τ is like the behavior of this system with $f(t)$ being identical zero. The autonomous system has the virtual centers of stability, which are derived from $\dot{X} = 0$ as follows: $X_i = A_0^{-1} B_0 m_i$, $i = 1, 2$. For the representative point of the solution to reach the hyperplane L_i, $i = 1, 2$, when $f(t) \approx 0$, the parameters l_1, l_2, m_1, and m_2 have to satisfy the following condition for the virtual centers of stability: $(\Gamma, A_0^{-1} B_0 m_2) > -l_1$, $(\Gamma, A_0^{-1} B_0 m_1) < -l_2$. These inequalities mean that X_1 and X_2 are out of the ambiguity zone of $\sigma(t)$ in phase space.

Now, based on Theorem 3.2, we give the sufficient conditions for system (1.1) to have no two-point oscillatory solution.

Theorem 4.2. *Let system* (1.1) *with* $f(t)$ *in the form* (1.2) *when* $\alpha \neq 0$ *be reduced to the canonical form* (2.7), (2.9) *by a nonsingular transformation provided that* $\gamma_s \neq 0$, $\gamma_j = 0$, $j = \overline{1, n}$, $j \neq s$. *Then*

(1) *there is no two-point oscillatory solution if the parameters of system* (3.4) *satisfy Case* 1 *of Theorem* 3.2*;*
(2) *there is no two-point oscillatory solution with return time* T_f *such that* $T_f < kT$, $k \in \mathbb{N}$, *if the parameters of system* (3.4) *satisfy Case* 2 *of Theorem* 3.2.

P r o o f. This theorem directly follows from the fact that the construction of system (3.4) is based on the assumption of the existence of two-point oscillatory solutions to system (1.1). Theorem 3.2 does not provide a necessary condition for the existence of such solutions. □

5. Example

Now we give an example to illustrate how the results of Theorem 4.1 are applied to a two-dimensional system.

Put $A = \begin{pmatrix} -4 & -1 \\ 2 & -1 \end{pmatrix}$. The eigenvalues of A are $\lambda_1 = -2$, $\lambda_2 = -3$. We take $B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ satisfying the condition of complete controllability. In other words, we exclude the vector B such that $\det(B\ AB) = 0$. From the last equality, we obtain $2b_1^2 + 3b_1 b_2 + b_2^2 = 0$. This equality defines the straight lines $b_1 = -0.5 b_2$, $b_1 = -b_2$, which meet at the point $(0, 0)$ in the plane $(b_1 O b_2)$. Therefore, we do not consider the vector B lying on these straight lines. For example, let $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Let us construct the transformation matrix S by (2.6). We first find $N_1(\lambda_1) = b_1 D_{11}(\lambda_1) + b_2 D_{21}(\lambda_1) = 1 \cdot 1 + 1 \cdot 1 = 2$. By analogy, we obtain $N_1(\lambda_2) = 3$, $N_2(\lambda_1) = -4$, and $N_2(\lambda_2) = -3$. Here λ_1, λ_2 are the roots of the equation

$$D(\lambda) = \lambda^2 + 5\lambda + 6 = 0.$$

Next we have $D'(\lambda) = 2\lambda + 5$. It now follows that $D'(\lambda_1) = 1$ and $D'(\lambda_2) = -1$. The result is $S = \begin{pmatrix} 2 & -3 \\ -4 & 3 \end{pmatrix}$, where $\det S = -6 \neq 0$. Next we calculate the inverse of the matrix S. So, $S^{-1} = \begin{pmatrix} -1/2 & -1/2 \\ -2/3 & -1/3 \end{pmatrix}$. Then $B_0 = S^{-1}B = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$.

To obtain γ_i $(i = 1, 2)$ by (2.5) such that $\gamma_1 \neq 0$, $\gamma_2 = 0$, we need to choose the feedback vector $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. We put, for instance, $C = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. Then $\gamma_1 = 2$, $\gamma_2 = 0$. We take $K = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ and find $K_0 = S^{-1}K = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} k_1^0 \\ k_2^0 \end{pmatrix}$. We have $k_1^0 = 1 \neq 0$, $k_2^0 = 0$.

Now we verify the conditions of Theorem 4.1. Put $\omega = 2$; then $T = 2\pi/\omega = \pi$. Let $\lambda_s = \lambda_1 = -2$. Suppose $\alpha = -4$. The inequality $\lambda_s > \alpha$ is met. Then $\Delta\varphi = \pi/4$. Set $\varphi + \Delta\varphi = 0$ for $q = 0$. Then $\varphi = -\pi/4$. We have $k_s^0 = k_1^0 = 1 \neq 0$; so, $k_s^0 \neq 0$.

Set $l_1 = -1$ and $m_1 = -2$. The minimum point of $\psi_1(t) = \mathrm{e}^{-4t}\sin(2t)/\sqrt{2}$ on the segment $[0, T] = [0, \pi]$ is $t_{\min}^1 \approx 1.8$. Here $k = 1$.

Put $t_1 = \pi/2$. Next we calculate l_2, using the first equation of (3.3) as a formula. So, $l_2 \approx 1.87$. Then we come to $\psi_2(t) \approx -3\mathrm{e}^{-2t} + 0.13$. condition (2) of Lemma 3.1 is true because $-2.3 \cdot 10^{-4} < 4.8 \cdot 10^{-2}$. Hence, there exists a solution $t_1 \in (0, \pi)$. Solving the first equation of (3.4) with respect to t_1 when $l_2 \approx 1.87$, we obtain that it has a unique solution $t_1 = \pi/2$ and no other ones. Therefore, $t_1 = \pi/2$ is the smallest positive solution.

Put $T_f = \pi$. Next we calculate m_2 by the formula

$$m_2 = \frac{\lambda_s}{\gamma_s}(\mathrm{e}^{\lambda_s(T_f - t_1)} - 1)^{-1}\Big(\Big(\frac{\gamma_s m_1}{\lambda_s \mathrm{e}^{\lambda_s t_1}} + l_1 - \frac{\gamma_s m_1}{\lambda_s}\Big)\mathrm{e}^{\lambda_s T_f} - l_1\Big)$$

obtained from the second equation of (3.4). Calculations yield $m_2 \approx 1.13$. Then we have $\psi_3(t) \approx 69.42\mathrm{e}^{-2t} - 0.13$. The equality $\psi_1(T) \approx \psi_3(T)$ is met. The conditions of Theorem 3.1 hold. Hence, system (3.4) has at least one pair of the solutions t_1, T_f such that $0 < t_1 < T_f = \pi$. Solving the second equation of (3.4) with respect to T_f when $m_2 \approx 1.13$, we obtain that it has one solution $T_f = \pi$ and no other ones. Therefore the pair $t_1 = \pi/2$ and $T_f = \pi$ is unique. By assumption, $l_1 < l_2$, $m_1 < m_2$. The values $l_1 = -1$, $l_2 \approx 1.87$, $m_1 = -2$, and $m_2 \approx 1.13$ meet this assumption.

Next let us verify that the obtained values of parameters satisfy the condition for the virtual centers of stability according to Remark 4.3. We have $\Gamma = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $A_0 = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$, $A_0^{-1} = \begin{pmatrix} -1/2 & 0 \\ 0 & -1/3 \end{pmatrix}$, and $A_0^{-1}B_0 = \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix}$. In this case, we come to the following: $m_2 > -l_1$ and $m_1 < -l_2$. The direct substitution gives the true inequalities $1.13 > 1$ and $-2 < -1.87$.

Thus, system (1.1) with the values of parameters above has a unique two-point oscillatory solution with return time $T_f = \pi$.

6. Conclusion

System (1.1) with disturbance in the form (1.2) has been shown to have a two-point oscillatory solution with return time being commensurable to time of first encounter. We have obtained the necessary and sufficient condition for the existence of this solution, and besides, we have proved that it is unique. In addition, the sufficient conditions under which the solutions do not exist have been established. We have provided a supporting example, which demonstrates how to apply the results of Theorem 4.1. For the two-dimensional system in the example, the values of relay characteristic parameters have been found for given values of the solution parameters.

We have considered system (2.4) with the feedback vector in a special form when the only one element is nonzero. In the future, this form can become complicated with two or more nonzero elements.

To our mind, the proposed approach and the obtained results can be used later on as a basis for analytical researches of system in the form (1.1).

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